Mostly a review of induction.

**Problem B1 (50 pts).** Let \( S : \mathbb{N} \to \mathbb{N} \) be the function given by

\[
S(n) = n + 1, \quad \text{for all } n \in \mathbb{N}.
\]

Define the function \( \text{add} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) recursively as follows for all \( m, n \in \mathbb{N} \):

\[
\begin{align*}
\text{add}(m, 0) &= m, \\
\text{add}(m, S(n)) &= S(\text{add}(m, n)).
\end{align*}
\]

(A) \( \text{add}(m, 0) = m \)

(B) \( \text{add}(m, S(n)) = S(\text{add}(m, n)) \)

(1) Prove that

\[
\text{add}(\text{add}(m, n), p) = \text{add}(m, \text{add}(n, p))
\]

for all \( m, n, p \in \mathbb{N} \). In other words, \( \text{add} \) is associative.

*Hint.* Use induction on \( p \). It turns out that \( \text{add}(m, n) = m + n \), where + is the usual addition of natural numbers but *you can’t use this fact!*

(2) We would like to prove that

\[
\text{add}(m, n) = \text{add}(n, m), \quad \text{for all } m, n \in \mathbb{N}
\]

but this is a little tricky. First prove

(2a)

\[
\text{add}(0, n) = n, \quad \text{for all } n \in \mathbb{N}.
\]

Also prove

(2b)

\[
\text{add}(S(m), n) = S(\text{add}(m, n)), \quad \text{for all } m, n \in \mathbb{N}.
\]

*Hint.* Use induction on \( n \).

Finally, prove that

\[
\text{add}(m, n) = \text{add}(n, m), \quad \text{for all } m, n \in \mathbb{N}.
\]
In other words, \( add \) is commutative.

*Hint.* Use induction on \( m \).

**Problem B2 (10 pts).** Let \( \Sigma \) be any alphabet. For any string \( w \in \Sigma^* \) recall that \( w^n \) is defined inductively as follows:

\[
\begin{align*}
w^0 &= \epsilon \\
w^{n+1} &= w^n w, \quad n \in \mathbb{N}.
\end{align*}
\]

For any string \( w \in \Sigma^* \) and any natural numbers \( m, n \in \mathbb{N} \), prove that

\[ w^m w^n = w^{m+n}. \]

*Hint:* Use induction on \( n \).

**Problem B3 (40 pts).** Let \( \Sigma \) be any alphabet. Given a string \( w \in \Sigma^* \), its reversal \( w^R \) is defined inductively as follows: \( \epsilon^R = \epsilon \), and \( (ua)^R = au^R \), where \( a \in \Sigma \) and \( u \in \Sigma^* \).

1. Prove that \( (uv)^R = v^Ru^R \), for all \( u, v \in \Sigma^* \).
2. Prove that \( (w^R)^R = w \), for all \( w \in \Sigma^* \).

**TOTAL:** 100 points.