Problem B1 (70 pts). Let $\Sigma = \{a_1, \ldots, a_k\}$ be any alphabet. Given a string $w \in \Sigma^*$, its reversal $w^R$ is defined inductively as follows: $\epsilon^R = \epsilon$, and $(ua)^R = au^R$, where $a \in \Sigma$ and $u \in \Sigma^*$.

A palindrome is a string $w$ such that $w = w^R$. Here are some examples of palindromes:

- eye
- racecar
- never odd or even
- god saw I was dog
- campus motto bottoms up mac
- do geese see god

If $k = 1$, every string is a palindrome. Therefore we assume that $k \geq 2$.

We would like to give a formula giving the number $p_n$ of all palindromes $w$ of length $|w| = n \geq 0$ over the alphabet $\Sigma = \{a_1, \ldots, a_k\}$ with $k$ letters.

1. Prove that a palindrome $w \in \Sigma^*$ is either the empty string $w = \epsilon$, or $w = a$ with $a \in \Sigma$, or $w = auua$ where $u$ is a palindrome of length $n - 2$ where $n = |w| \geq 2$ and $a \in \Sigma$ is some letter.

2. Prove that $p_0 = 1$, $p_1 = k$, and

$$p_{n+2} = kp_n, \text{ for all } n \geq 0.$$  

Give a formula for $p_n$. Distinguish between the cases where $n = 2m$ ($n$ is even) and $n = 2m + 1$ ($n$ is odd). You must prove the correctness of your formulae (use induction).

Do not give formulae in terms of $n/2$ when $n$ is even or $(n - 1)/2$ when $n$ odd. Please give formulae for $p_{2m}$ and $p_{2m+1}$ in terms of $m$.  


(3) Prove that the number \( P_n \) of all palindromes \( w \) of length \( \leq n \) (which means that \( 0 \leq |w| \leq n \)) over the alphabet \( \Sigma = \{a_1, \ldots, a_k\} \) with \( k \) letters is given by

\[
P_{2m} = \frac{2k^{m+1} - k - 1}{k - 1} \\
P_{2m+1} = \frac{k^{m+2} + k^{m+1} - k - 1}{k - 1}
\]

for any natural number \( m \in \mathbb{N} \). Prove that the number \( Q_n \) of all non-palindromes \( w \) of length \( \leq n \) over the alphabet \( \Sigma = \{a_1, \ldots, a_k\} \) is given by

\[
Q_{2m} = \frac{k^{2m+1} - 2k^{m+1} + k}{k - 1} \\
Q_{2m+1} = \frac{k^{2m+2} - k^{m+2} - k^{m+1} + k}{k - 1}
\]

for any natural number \( m \in \mathbb{N} \).

Hint. Figure out the total number of strings of length \( \leq n \) over an alphabet of size \( k \geq 2 \).

(4) If \( k = 2 \), prove that if \( m \geq 2 \), then \( P_{2m}/Q_{2m} < 1 \) and \( P_{2m+1}/Q_{2m+1} < 1 \), so there are more non-palindromes than palindromes. What is 536 870 909 (in relation to palindromes)? Show that

\[
\frac{536 870 909}{2^{55} - 1} \approx 2^{-26} \approx 1.4901 \times 10^{-8}.
\]

What the interpretation of the above ratio as a probability?

Problem B2 (30 pts). Let \( \Sigma \) be any alphabet. For any string \( w \in \Sigma^* \) recall that \( w^n \) is defined inductively as follows:

\[
w^0 = \epsilon \\
w^{n+1} = w^nw, \quad n \in \mathbb{N}.
\]

Prove the following property: for any two strings \( u, v \in \Sigma^* \), \( uv = vu \) iff there is some \( w \in \Sigma^* \) such that \( u = w^m \) and \( v = w^n \), for some \( m, n \geq 0 \).

Hint. In the “hard” direction, consider the subcases

(1) \( |u| = |v| \),

(2) \( |u| < |v| \) and

(3) \( |u| > |v| \)

and use an induction on \( |u| + |v| \).

TOTAL: 100 points.