Problem B1 (80 pts). Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton. Define the relations $\approx$ and $\sim$ on $\Sigma^*$ as follows:

$$x \approx y \text{ if and only if, for all } p \in Q, \quad \delta^*(p, x) \in F \iff \delta^*(p, y) \in F;$$

and

$$x \sim y \text{ if and only if, for all } p \in Q, \quad \delta^*(p, x) = \delta^*(p, y).$$

(a) Show that $\approx$ is a left-invariant equivalence relation and that $\sim$ is an equivalence relation that is both left and right invariant. (A relation $R$ on $\Sigma^*$ is left invariant iff $uwRv$ implies that $wuRv$ for all $w \in \Sigma^*$, and $R$ is left and right invariant iff $uwRv$ implies that $xuyRxvy$ for all $x,y \in \Sigma^*$.)

(b) Let $n$ be the number of states in $Q$ (the set of states of $D$). Show that $\approx$ has at most $2^n$ equivalence classes and that $\sim$ has at most $n^n$ equivalence classes.

Hint. In the case of $\approx$, consider the function $f: \Sigma^* \to 2^Q$ given by

$$f(u) = \{ p \in Q \mid \delta^*(p, u) \in F \}, \quad u \in \Sigma^*,$$

and show that $x \approx y$ iff $f(x) = f(y)$. In the case of $\sim$, let $Q^Q$ be the set of all functions from $Q$ to $Q$ and consider the function $g: \Sigma^* \to Q^Q$ defined such that $g(u)$ is the function given by

$$g(u)(p) = \delta^*(p, u), \quad u \in \Sigma^*, \quad p \in Q,$$

and show that $x \sim y$ iff $g(x) = g(y)$.

(c) Given any language $L \subseteq \Sigma^*$, define the relations $\lambda_L$ and $\mu_L$ on $\Sigma^*$ as follows:

$$u \lambda_L v \text{ iff, for all } z \in \Sigma^*, \quad zu \in L \iff zv \in L,$$
and

\[ u \mu_L v \text{ iff, for all } x, y \in \Sigma^*, \quad xuy \in L \text{ iff } xvy \in L. \]

Prove that \( \lambda_L \) is left-invariant, and that \( \mu_L \) is left and right-invariant. Prove that if \( L \) is regular, then both \( \lambda_L \) and \( \mu_L \) have a finite number of equivalence classes.

*Hint:* Show that the number of classes of \( \lambda_L \) is at most the number of classes of \( \approx \), and that the number of classes of \( \mu_L \) is at most the number of classes of \( \sim \).

**Problem B2 (80 pts).** This problem illustrates the power of the congruence version of Myhill-Nerode.

Let \( L \) be any regular language over some alphabet \( \Sigma \). Define the languages

\[
L^\infty = \bigcup_{k \geq 1} \{ w^k \mid w \in L \},
\]

\[
L^{1/\infty} = \{ w \mid w^k \in L, \text{ for all } k \geq 1 \}, \quad \text{and}
\]

\[
\sqrt{L} = \{ w \mid w^k \in L, \text{ for some } k \geq 1 \}.
\]

Also, for any natural number \( k \geq 1 \), let

\[
L^{(k)} = \{ w^k \mid w \in L \},
\]

and

\[
L^{(1/k)} = \{ w \mid w^k \in L \}.
\]

(a) Prove that \( L^{(1/3)} \) is regular. What about \( L^{(3)} \)?

(b) Let \( k \geq 1 \) be any natural number. Prove that there are only finitely many languages of the form \( L^{(1/k)} = \{ w \mid w^k \in L \} \) and that they are all regular. (In fact, if \( L \) is accepted by a DFA with \( n \) states, there are at most \( 2^{(n^2)} \) languages of the form \( L^{(1/k)} \).

(c) Is \( L^{1/\infty} \) regular or not? Is \( \sqrt{L} \) regular or not? What about \( L^{\infty} \)?

**TOTAL:** 160 points