Problem B1 (40 pts). The Fibonacci sequence, $u_n$, is given by
\[
\begin{align*}
  u_0 &= 1 \\
  u_1 &= 1 \\
  u_{n+2} &= u_{n+1} + u_n, \quad n \geq 0.
\end{align*}
\]
So, the Fibonacci sequence begins with
\[1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\]

(a) Prove that
\[u_n \geq \left(\frac{\sqrt{5} + 1}{2}\right)^{n-1}, \quad n \geq 1.
\]

(b) Prove that the language over \{a\} given by
\[L = \{a^{u_n} \mid n \geq 0\}\]
is not regular.

*Hint*. Use (a) and the Myhill-Nerode Theorem.

Problem B2 (120 pts). Which of the following languages are regular? Justify each answer.

1. $L_1 = \{wcw \mid w \in \{a, b\}^\ast\}$. (here $\Sigma = \{a, b, c\}$).
2. $L_2 = \{xy \mid x, y \in \{a, b\}^\ast \text{ and } |x| = |y|\}$. (here $\Sigma = \{a, b\}$)
3. $L_3 = \{a^n \mid n \text{ is a prime number}\}$. (here $\Sigma = \{a\}$).
4. $L_4 = \{a^m b^n \mid \gcd(m, n) = 23\}$. (here $\Sigma = \{a, b\}$).
(5) Consider the language

\[ L_5 = \{a^{4n+3} \mid 4n + 3 \text{ is prime}\}. \]

Assuming that \( L_5 \) is infinite, prove that \( L_5 \) is not regular.

(6) Let \( F_n = 2^{2^n} + 1 \), for any integer \( n \geq 0 \), and let

\[ L_6 = \{a^{F_n} \mid n \geq 0\}. \]

Here \( \Sigma = \{a\} \).

**Extra Credit (from 10 up to 100 pts).** Find explicitly what \( F_0, F_1, F_2, F_3 \) are, and check that they are prime. What about \( F_4 \)?

Is the language

\[ L_7 = \{a^{F_n} \mid n \geq 0, \ F_n \text{ is prime}\} \]

regular?

**Extra Credit (20 pts).** Prove that there are infinitely many primes of the form \( 4n + 3 \).

The list of such primes begins with

\[ 3, 7, 11, 19, 23, 31, 43, \cdots \]

Say we already have \( n + 1 \) of these primes, denoted by

\[ 3, p_1, p_2, \cdots, p_n, \]

where \( p_i > 3 \). Consider the number

\[ m = 4p_1p_2\cdots p_n + 3. \]

If \( m = q_1 \cdots q_k \) is a prime factorization of \( m \), prove that \( q_j > 3 \) for \( j = 1, \ldots k \) and that no \( q_j \) is equal to any of the \( p_i \)'s. Prove that one of the \( q_j \)'s must be of the form \( 4s + 3 \), which shows that there is a prime of the form \( 4s + 3 \) greater than any of the previous primes of the same form.

**Problem B3 (80 pts).** The purpose of this problem is to get a fast algorithm for testing state equivalence in a DFA. Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a deterministic finite automaton. Recall that state equivalence is the equivalence relation \( \equiv \) on \( Q \), defined such that,

\[ p \equiv q \iff \forall z \in \Sigma^* (\delta^*(p, z) \in F \iff \delta^*(q, z) \in F), \]

and that \( i\text{-equivalence} \) is the equivalence relation \( \equiv_i \) on \( Q \), defined such that,

\[ p \equiv_i q \iff \forall z \in \Sigma^*, |z| \leq i (\delta^*(p, z) \in F \iff \delta^*(q, z) \in F). \]
A relation \( S \subseteq Q \times Q \) is a forward closure iff it is an equivalence relation and whenever \((p, q) \in S\), then \((\delta(p, a), \delta(q, a)) \in S\), for all \( a \in \Sigma \).

We say that a forward closure \( S \) is good iff whenever \((p, q) \in S\), then \(\text{good}(p, q)\), where \(\text{good}(p, q)\) holds iff either both \( p, q \in F \), or both \( p, q \notin F \).

Given any relation \( R \subseteq Q \times Q \), recall that the smallest equivalence relation \( R \approx \) containing \( R \) is the relation \((R \cup R^{-1})^*\) (where \( R^{-1} = \{(q, p) \mid (p, q) \in R\}\), and \((R \cup R^{-1})^*\) is the reflexive and transitive closure of \((R \cup R^{-1})\)). We define the sequence of relations \( R_i \subseteq Q \times Q \) as follows:

\[
R_0 = R_{\approx} \\
R_{i+1} = (R_i \cup \{(\delta(p, a), \delta(q, a)) \mid (p, q) \in R_i, \ a \in \Sigma\})_{\approx}.
\]

(1) Prove that \( R_{i_0 + 1} = R_{i_0} \) for some least \( i_0 \). Prove that \( R_{i_0} \) is the smallest forward closure containing \( R \).

Hint. First, prove that

\[
R_i \subseteq R_{i+1}
\]

for all \( i \geq 0 \). Next, prove that \( R_{i_0} \) is forward closed.

If \( \sim \) is any forward closure containing \( R \), prove by induction that

\[
R_i \subseteq \sim
\]

for all \( i \geq 0 \).

We denote the smallest forward closure \( R_{i_0} \) containing \( R \) as \( R^\dagger \), and call it the forward closure of \( R \).

(2) Prove that \( p \equiv q \) iff the forward closure \( R^\dagger \) of the relation \( R = \{(p, q)\} \) is good.

Hint. First, prove that if \( R^\dagger \) is good, then

\[
R^\dagger \subseteq \equiv.
\]

For this, prove by induction that

\[
R^\dagger \subseteq \equiv_i
\]

for all \( i \geq 0 \).

Then, prove that if \( p \equiv q \), then

\[
R^\dagger \subseteq \equiv.
\]

For this, prove that \( \equiv \) is an equivalence relation containing \( R = \{(p, q)\} \) and that \( \equiv \) is forward closed.

**TOTAL: 240 + 30+ points**