

## **Representing Numbers**

We have seen how certain terms in the lambda-calculus can be used to represent natural numbers.

```
c_0 = \lambda s. \lambda z. z

c_1 = \lambda s. \lambda z. s z

c_2 = \lambda s. \lambda z. s (s z)

c_3 = \lambda s. \lambda z. s (s (s z))
```

Other lambda-terms represent common operations on numbers:

```
scc = \lambdan. \lambdas. \lambdaz. s (n s z)
```

In what sense can we say this representation is "correct"?

In particular, on what basis can we argue that scc on church numerals corresponds to ordinary successor on numbers?

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#### The naive approach

#### One possibility:

For each n, the term scc  $c_n$  evaluates to  $c_{n+1}$ .

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The naive approach... doesn't work

One possibility:

For each n, the term scc  $c_n$  evaluates to  $c_{n+1}$ .

Unfortunately, this is false.

E.g.:

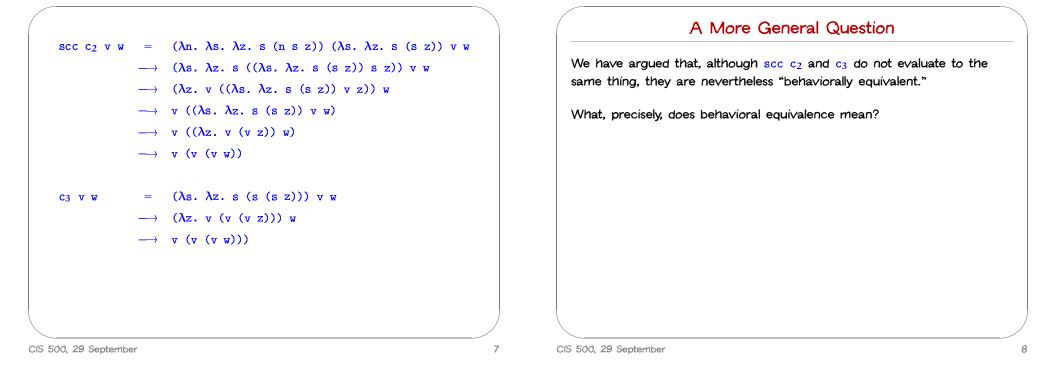
$$scc c_2 = (\lambda n. \lambda s. \lambda z. s (n s z)) (\lambda s. \lambda z. s (s z))$$
$$\longrightarrow \lambda s. \lambda z. s ((\lambda s. \lambda z. s (s z)) s z)$$
$$\neq \lambda s. \lambda z. s (s (s z))$$
$$= c_3$$

## A better approach

Recall the intuition behind the church numeral representation:

- $\blacklozenge$  a number n is represented as a term that "does something n times to something else"
- $\blacklozenge$  scc takes a term that "does something n times to something else" and returns a term that "does something n+1 times to something else"

I.e., what we really care about is that  $scc c_2$  behaves the same as  $c_3$  when applied to two arguments.



# Intuition

Roughly,

terms s and t are behaviorally equivalent

should mean:

there is no "test" that distinguishes s and t — i.e., no way to use them in the same context and obtain different results.

#### Some test cases

tru =  $\lambda t. \lambda f. t$ tru' =  $\lambda t. \lambda f. (\lambda x.x) t$ fls =  $\lambda t. \lambda f. f$ omega =  $(\lambda x. x x) (\lambda x. x x)$ poisonpill =  $\lambda x.$  omega placebo =  $\lambda x.$  tru  $Y_f = (\lambda x. f (x x)) (\lambda x. f (x x))$ 

Which of these are behaviorally equivalent?

### Observational equivalence

As a first step toward defining behavioral equivalence, we can use the notion of normalizability to define a simple way of testing terms.

Two terms s and t are said to be observationally equivalent if either both are normalizable (i.e., they reach a normal form after a finite number of evaluation steps) or both are divergent.

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# Observational equivalence

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Aside:

Is observational equivalence a decidable property?

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# Examples

• omega and tru are not observationally equivalent

Aside:

- Is observational equivalence a decidable property?
- Does this mean the definition is ill-formed?

running it on our abstract machine.

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#### Examples

 $\blacklozenge$  omega and tru are not observationally equivalent

• tru and fls are observationally equivalent

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## Behavioral Equivalence

This primitive notion of observation now gives us a way of "testing" terms for behavioral equivalence

Terms s and t are said to be behaviorally equivalent if, for every finite sequence of values  $v_1, v_2, \ldots, v_n$ , the applications

s v1 v2  $\cdots$  vn

t v1 v2 ... vn

are observationally equivalent.

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and

# Examples

These terms are behaviorally equivalent:

tru =  $\lambda t. \lambda f. t$ tru' =  $\lambda t. \lambda f. (\lambda x.x) t$ 

#### So are these:

```
\begin{array}{l} \text{omega = } (\lambda \text{x. x x}) \ (\lambda \text{x. x x}) \\ \text{Y}_{\text{f}} = (\lambda \text{x. f} \ (\text{x x})) \ (\lambda \text{x. f} \ (\text{x x})) \end{array}
```

These are not behaviorally equivalent (to each other, or to any of the terms above):

```
fls = \lambda t. \lambda f. f
poisonpill = \lambda x. omega
placebo = \lambda x. tru
```

Formalizing the Lambda-Calculus

(From	TAPL c	hapter	6, on tr	le board	I)	

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