CIS 500

Software Foundations

Fall 2003

1 October

Types

Formal Methods in Computer Science

lightweight formal methods

Type Systems

big topic in PL very successful in practice very active "enabling technology" for all sorts of other things, e.g. language-based security the "skeleton" around which modern programming languages are often designed

Plan

- For today, we'll go back to the simple language of arithmetic and boolean expressions and show how to give it a (very simple) type system
- On October 15th (after the midterm and fall break), we'll develop a simple type system for the lambda-calculus
- We'll spend a good part of the rest of the semester adding features to this type system

Outline

1. begin with a set of terms, a set of values, and an evaluation relation
2. define a set of **types** classifying values according to their "shapes"
3. define a **typing relation** `t : T` that classifies terms according to the shape of the values that result from evaluating them
4. check that the typing relation is **sound** in the sense that,
   (a) if `t : T` and $\texttt{t} \longrightarrow^* \texttt{v}$, then `v : T`
   (b) if `t : T`, then evaluation of `t` will not get stuck

(N.b.: we actually state #4a in a slightly more general way...)

Arithmetic Expressions - Syntax

`t ::=`		terms
	`true`	constant true
	`false`	constant false
	`if t then t else t`	conditional
	`0`	constant zero
	`succ t`	successor
	`pred t`	predecessor
	`iszero t`	zero test
`v ::=`		values
	`true`	true value
	`false`	false value
	`nv`	numeric value
`nv ::=`		numeric values
	`0`	zero value
	`succ nv`	successor value

Evaluation Rules

$$\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2 \qquad \text{(E-IfTrue)}$$

$$\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3 \qquad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \qquad \text{(E-If)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{succ } t_1 \longrightarrow \texttt{succ } t_1'} \qquad \text{(E-Succ)}$$

$$\texttt{pred } 0 \longrightarrow 0 \qquad \text{(E-PredZero)}$$

$$\texttt{pred (succ } nv_1\texttt{)} \longrightarrow nv_1 \qquad \text{(E-PredSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{pred } t_1 \longrightarrow \texttt{pred } t_1'} \qquad \text{(E-Pred)}$$

$$\texttt{iszero } 0 \longrightarrow \texttt{true} \qquad \text{(E-IszeroZero)}$$

$$\texttt{iszero (succ } nv_1\texttt{)} \longrightarrow \texttt{false} \qquad \text{(E-IszeroSucc)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{iszero } t_1 \longrightarrow \texttt{iszero } t_1'} \qquad \text{(E-IsZero)}$$

Types

In this language, values have two possible “shapes”: they are either booleans or numbers.

T ::=		types
	Bool	type of booleans
	Nat	type of numbers

Typing Rules

$$\texttt{true} : \texttt{Bool} \qquad \text{(T-True)}$$

$$\texttt{false} : \texttt{Bool} \qquad \text{(T-False)}$$

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$$\texttt{0} : \texttt{Nat} \qquad \text{(T-Zero)}$$

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$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{succ } \texttt{t}_1 : \texttt{Nat}} \qquad \text{(T-Succ)}$$

$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{pred } \texttt{t}_1 : \texttt{Nat}} \qquad \text{(T-Pred)}$$

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$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{pred } \texttt{t}_1 : \texttt{Nat}} \qquad \text{(T-Pred)}$$

$$\frac{\texttt{t}_1 : \texttt{Nat}}{\texttt{iszero } \texttt{t}_1 : \texttt{Bool}} \qquad \text{(T-IsZero)}$$

Imprecision of Typing

Like other static program analyses, type systems are generally imprecise: they do not predict exactly what kind of value will be returned by every program, but just a conservative (safe) approximation.

$$\frac{\texttt{t}_1 : \texttt{Bool} \quad \texttt{t}_2 : \texttt{T} \quad \texttt{t}_3 : \texttt{T}}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 : \texttt{T}} \qquad \text{(T-If)}$$

Using this rule, we cannot assign a type to

```
if true then 0 else false
```

even though this term will certainly evaluate to a number.

Properties of the Typing Relation

Type Safety

The safety (or soundness) of this type system can be expressed by two properties:

1. Progress: A well-typed term is not stuck

   If $\mathtt{t} : \mathtt{T}$, then either $\mathtt{t}$ is a value or else $\mathtt{t} \longrightarrow \mathtt{t}'$ for some $\mathtt{t}'$.

2. Preservation: Types are preserved by one-step evaluation

   If $\mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t}'$, then $\mathtt{t}' : \mathtt{T}$.

Typing Derivations

Every pair $(\mathtt{t}, \mathtt{T})$ in the typing relation can be justified by a derivation tree built from instances of the inference rules.

$$\dfrac{\dfrac{\dfrac{}{0 : \mathtt{Nat}}\ \textsc{T-Zero}}{\mathtt{iszero}\ 0 : \mathtt{Bool}}\ \textsc{T-IsZero} \qquad \dfrac{}{0 : \mathtt{Nat}}\ \textsc{T-Zero} \qquad \dfrac{\dfrac{}{0 : \mathtt{Nat}}\ \textsc{T-Zero}}{\mathtt{pred}\ 0 : \mathtt{Nat}}\ \textsc{T-Pred}}{\mathtt{if\ iszero\ 0\ then\ 0\ else\ pred\ 0} : \mathtt{Nat}}\ \textsc{T-If}$$

Proofs of properties about the typing relation often proceed by induction on typing derivations.

Inversion

Lemma:

1. If $\mathtt{true} : \mathtt{R}$, then $\mathtt{R} = \mathtt{Bool}$.
2. If $\mathtt{false} : \mathtt{R}$, then $\mathtt{R} = \mathtt{Bool}$.
3. If $\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 : \mathtt{R}$, then $\mathtt{t}_1 : \mathtt{Bool}$, $\mathtt{t}_2 : \mathtt{R}$, and $\mathtt{t}_3 : \mathtt{R}$.
4. If $0 : \mathtt{R}$, then $\mathtt{R} = \mathtt{Nat}$.
5. If $\mathtt{succ}\ \mathtt{t}_1 : \mathtt{R}$, then $\mathtt{R} = \mathtt{Nat}$ and $\mathtt{t}_1 : \mathtt{Nat}$.
6. If $\mathtt{pred}\ \mathtt{t}_1 : \mathtt{R}$, then $\mathtt{R} = \mathtt{Nat}$ and $\mathtt{t}_1 : \mathtt{Nat}$.
7. If $\mathtt{iszero}\ \mathtt{t}_1 : \mathtt{R}$, then $\mathtt{R} = \mathtt{Bool}$ and $\mathtt{t}_1 : \mathtt{Nat}$.

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Proof: ...

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Proof: ...

This leads directly to a recursive algorithm for calculating the type of a term...

Typechecking Algorithm

```
typeof(t) = if t = true then Bool
            else if t = false then Bool
            else if t = if t1 then t2 else t3 then
              let T1 = typeof(t1) in
              let T2 = typeof(t2) in
              let T3 = typeof(t3) in
              if T1 = Bool and T2=T3 then T2
              else "not typable"
            else if t = 0 then Nat
            else if t = succ t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Nat else "not typable"
            else if t = pred t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Nat else "not typable"
            else if t = iszero t1 then
              let T1 = typeof(t1) in
              if T1 = Nat then Bool else "not typable"
```


Canonical Forms

Lemma:

1. If $\texttt{v}$ is a value of type $\texttt{Bool}$, then $\texttt{v}$ is either $\texttt{true}$ or $\texttt{false}$.
2. If $\texttt{v}$ is a value of type $\texttt{Nat}$, then $\texttt{v}$ is a numeric value

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Theorem: Suppose t is a well-typed term (that is, $t : T$ for some T). Then either t is a value or else there is some t' with $t \longrightarrow t'$.

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Theorem: Suppose $\mathtt{t}$ is a well-typed term (that is, $\mathtt{t : T}$ for some $\mathtt{T}$). Then either $\mathtt{t}$ is a value or else there is some $\mathtt{t'}$ with $\mathtt{t} \longrightarrow \mathtt{t'}$.

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The T-True, T-False, and T-Zero cases are immediate, since $\mathtt{t}$ in these cases is a value.

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Case T-If: $\mathtt{t = if\ t_1\ then\ t_2\ else\ t_3}$

$\mathtt{t_1 : Bool} \quad \mathtt{t_2 : T} \quad \mathtt{t_3 : T}$

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$\mathtt{t_1 : Bool} \quad \mathtt{t_2 : T} \quad \mathtt{t_3 : T}$

By the induction hypothesis, either $\mathtt{t_1}$ is a value or else there is some $\mathtt{t_1'}$ such that $\mathtt{t_1} \longrightarrow \mathtt{t_1'}$. If $\mathtt{t_1}$ is a value, then the canonical forms lemma tells us that it must be either $\mathtt{true}$ or $\mathtt{false}$, in which case either E-IfTrue or E-IfFalse applies to $\mathtt{t}$. On the other hand, if $\mathtt{t_1} \longrightarrow \mathtt{t_1'}$, then, by E-If, $\mathtt{t} \longrightarrow \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}$.

Preservation

Theorem: If $\mathtt{t : T}$ and $\mathtt{t} \longrightarrow \mathtt{t'}$, then $\mathtt{t' : T}$.

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Theorem: If $t : T$ and $t \longrightarrow t'$, then $t' : T$.

Proof: ...