Lemma 16.1.1 *Given a derivation $\mathcal{D} :: \sigma <: \tau$ using the subtyping rules including* S-RcdDepth*,* S-RcdWidth*, and* S-RcdPerm *(but not* S-Rcd*), then it is possible to construct a derivation using only* S-Rcd *(and not* S-RcdDepth*,* S-RcdWidth*, or* S-RcdPerm*), and vice versa.*

Proof. By straightforward induction on the subtyping derivations. For the forward direction:

- **Case**:

$$\mathcal{D} = \frac{}{\{l_i : \tau_i^{i\in 1\ldots n+k}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}}\textsf{S-RcdWidth}$$

  Because $\{l_i^{i\in 1\ldots n}\}$ is a prefix of $\{l_i^{i\in 1\ldots n+k}\}$ we can conclude it is the case that $\{l_i^{i\in 1\ldots n}\} \subseteq \{l_i^{i\in 1\ldots n+k}\}$. Furthermore for $i \in 1\ldots n$ we can conclude there exist derivations $\mathcal{E}_i :: \tau_i <: \tau_i$ by using S-Refl. Therefore, using S-Rcd on each $\mathcal{E}_i$ we can construct a derivation $\mathcal{E} :: \{l_i : \tau_i^{i\in 1\ldots n+k}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}$.

- **Case**:

$$\mathcal{D} = \frac{\dfrac{\mathcal{D}_i}{\sigma_i <: \tau_i} \quad \text{for each } i}{\{l_i : \sigma_i^{i\in 1\ldots n}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}}\textsf{S-RcdDepth}$$

  By appealing to the induction hypothesis for each $\mathcal{D}_i$ we can produce derivations $\mathcal{E}_i :: \sigma_i <: \tau_i$ that do not contain uses of S-RcdWidth, S-RcdDepth, or S-RcdPerm. The set of labels $\{l_i^{i\in 1\ldots n}\}$ is the same for both records, so it is trivial to conclude it is the case that $\{l_i^{i\in 1\ldots n}\} \subseteq \{l_i^{i\in 1\ldots n}\}$. Finally, using S-Rcd on each $\mathcal{E}_i$ we can construct a derivation witnessing $\mathcal{E} :: \{l_i : \sigma_i^{i\in 1\ldots n}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}$.

- **Case**:

$$\mathcal{D} = \frac{\{k_j : \sigma_j^{j\in 1\ldots n}\} \text{ is a permutation of } \{l_i : \tau_i^{i\in 1\ldots n}\}}{\{k_j : \sigma_j^{j\in 1\ldots n}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}}\textsf{S-RcdPerm}$$

  By the definition of permutation, we know that the sets of labels $\{k_j^{j\in 1\ldots n}\}$ and $\{l_i^{i\in 1\ldots n}\}$ are identical with a bijection between them, licensing us to know $\{l_i^{i\in 1\ldots n}\} \subseteq \{k_j^{j\in 1\ldots n}\}$. Furthermore, because we know that for every σ_j it is identical to one τ_i where $k_j = l_i$, by S-Refl we can conclude there exist derivations $\mathcal{E}_{ji} :: \sigma_j <: \tau_i$ for each pair. Consequently, we can conclude $\mathcal{E} :: \{k_j : \tau_j^{j\in 1\ldots n}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}$ by using S-Rcd on each $\mathcal{E}_{ji}$.

- **Cases**: The remaining cases follow by straightforward appeals to the induction hypothesis on the premises followed by application of the subtyping rule.

For the reverse direction:

- **Case**:

$$\mathcal{E} = \frac{\{l_i^{i\in 1\ldots n}\} \subseteq \{k_j^{j\in 1\ldots m}\} \qquad k_j = l_i \Rightarrow \dfrac{\mathcal{E}_{ji}}{\sigma_j <: \tau_i}}{\{k_j : \sigma_j^{j\in 1\ldots m}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}}\textsf{S-Rcd}$$

  By $\{l_i^{i\in 1\ldots n}\} \subseteq \{k_j^{j\in 1\ldots m}\}$ we know that for every l_i there must exist some k_j such that $k_j = l_i$. For each of these l_i, by appealing to the induction hypothesis on each $\mathcal{E}_{ji}$ we get $\mathcal{D}_{ji} :: \sigma_j <: \tau_i$ such that each $\mathcal{D}_{ji}$ does not contain a use of S-Rcd. Next we define a permutation $f : \{1\ldots m\} \to \{1\ldots m\}$ such that

$$f(x) = \begin{cases} x \mapsto y & \text{if } k_x = l_y \\ x \mapsto y \text{ st } y > n \land \neg\exists x'.x' \neq x \land f(x') = y & \text{otherwise} \end{cases}$$

  Using this permutation with S-RcdPerm we can conclude $\mathcal{E}_1 :: \{k_j : \sigma_j^{j\in 1\ldots m}\} <: \{q_v : \rho_v^{v\in 1\ldots m}\}$ where $k_j = q_{f(j)}$ and $\sigma_j = \rho_{f(j)}$. Next by using S-RcdWidth we have that $\mathcal{E}_2 :: \{q_v : \rho_v^{v\in 1\ldots m}\} <: \{q_v : \rho_v^{v\in 1\ldots n}\}$.

Furthermore, using the identities described above we know for each $\mathcal{D}_{ji} :: \sigma_j <: \tau_i$ we also have a derivation $\mathcal{D}'_{f(j)i} :: \rho_{f(j)} <: \tau_i$. Given that $\mathcal{D}_{ji} :: \sigma_j <: \tau_i$ for all $k_j = l_i$ and we defined $f(j) = i$, each $\mathcal{D}'_{f(j)i}$ is simply $\mathcal{D}''_{ii} :: \rho_i <: \tau_i$. So by S-RcdDepth on each $\mathcal{D}''_{ii}$ we have $\mathcal{E}_3 :: \{q_v : \rho_v^{v\in 1\ldots n}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}$. Finally we tie $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ together three with two uses of S-Trans giving us the desired result $\mathcal{E} :: \{k_j : \sigma_j^{j\in 1\ldots m}\} <: \{l_i : \tau_i^{i\in 1\ldots n}\}$.

- **Cases**: The remaining cases follow by straightforward appeals to the induction hypothesis on the premises followed by application of the subtyping rule.

□