CIS 500

Software Foundations

Fall 2004

6 October

Midterm 1 is next Wednesday

- Today's lecture will not be covered by the midterm.
- Next Monday, review class.
- Old exams and review questions on webpage.
- No recitation sections next week.
- New office hours next week, watch newsgroup for details.

Plans

Where we've been:

- Inductive definitions
  - abstract syntax
  - inference rules
- Proofs by structural induction
- Operational semantics
- The lambda-calculus
- Typing rules and type soundness

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Where we've been:

- Inductive definitions
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Where we're going:

- "Simple types" for the lambda-calculus
- Formalizing more features of real-world languages (records, datatypes, references, exceptions, etc.)
- Subtyping
- Objects

The Simply Typed Lambda-Calculus

Lambda-calculus with booleans

`t` ::=		*terms*
	`x`	*variable*
	`λx.t`	*abstraction*
	`t t`	*application*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
`v` ::=		*values*
	`λx.t`	*abstraction value*
	`true`	*true value*
	`false`	*false value*

"Simple Types"

`T`	::=	*types*
	`Bool`	*type of booleans*
	`T`$\rightarrow$`T`	*types of functions*

Typing rules

$$\texttt{true} : \texttt{Bool} \qquad \text{(T-True)}$$

$$\texttt{false} : \texttt{Bool} \qquad \text{(T-False)}$$

$$\frac{t_1 : \texttt{Bool} \qquad t_2 : T \qquad t_3 : T}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T} \qquad \text{(T-If)}$$

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$$\frac{}{x : T} \qquad \text{(T-Var)}$$

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$$\frac{\texttt{x:T} \in \Gamma}{\Gamma \vdash \texttt{x} : \texttt{T}} \qquad \text{(T-Var)}$$

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$$\frac{\Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x{:}T_1 . t_2 : T_1 \to T_2} \qquad \text{(T-Abs)}$$

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$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \quad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \qquad \text{(T-App)}$$

Typing Derivations

What derivations justify the following typing statements?

- $\vdash$ `(λx:Bool.x) true : Bool`
- `f:Bool→Bool` $\vdash$ `f (if false then true else false) : Bool`
- `f:Bool→Bool` $\vdash$ `λx:Bool. f (if x then false else x) : Bool→Bool`

Properties of $\boldsymbol{\lambda}_{\rightarrow}$

As before, the fundamental property of the type system we have just defined is soundness with respect to the operational semantics.

Properties of $\lambda_{\to}$

As before, the fundamental property of the type system we have just defined is soundness with respect to the operational semantics.

1. Progress: A closed, well-typed term is not stuck

   If $\vdash \mathtt{t} : \mathtt{T}$, then either $\mathtt{t}$ is a value or else $\mathtt{t} \longrightarrow \mathtt{t'}$ for some $\mathtt{t'}$.

2. Preservation: Types are preserved by one-step evaluation

   If $\Gamma \vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \longrightarrow \mathtt{t'}$, then $\Gamma \vdash \mathtt{t'} : \mathtt{T}$.

Proving progress

Same steps as before...

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- inversion lemma for typing relation
- canonical forms lemma
- progress theorem

Typing rules again (for reference)

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$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \quad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \qquad \text{(T-App)}$$

Inversion

Lemma:

1. If $\Gamma \vdash \texttt{true} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
2. If $\Gamma \vdash \texttt{false} : \texttt{R}$, then $\texttt{R} = \texttt{Bool}$.
3. If $\Gamma \vdash \texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : \texttt{R}$, then $\Gamma \vdash t_1 : \texttt{Bool}$ and $\Gamma \vdash t_2, t_3 : \texttt{R}$.

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5. If $\Gamma \vdash \lambda \texttt{x:}T_1 . t_2 : \texttt{R}$, then $\texttt{R} = T_1 \to R_2$ for some R_2 with $\Gamma, \texttt{x:}T_1 \vdash t_2 : R_2$.

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6. If $\Gamma \vdash t_1\ t_2 : \texttt{R}$, then there is some type T_{11} such that $\Gamma \vdash t_1 : T_{11} \to \texttt{R}$ and $\Gamma \vdash t_2 : T_{11}$.

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Canonical Forms

Lemma:

1. If $\texttt{v}$ is a value of type $\texttt{Bool}$, then $\texttt{v}$ is either $\texttt{true}$ or $\texttt{false}$.
2. If $\texttt{v}$ is a value of type $\texttt{T}_1 \to \texttt{T}_2$, then $\texttt{v}$ has the form $\lambda\texttt{x:T}_1\texttt{.t}_2$.

Progress

Theorem: Suppose `t` is a closed, well-typed term (that is, $\vdash$ `t : T` for some `T`). Then either `t` is a value or else there is some `t'` with `t` $\longrightarrow$ `t'`.

Proof: By induction

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Consider the case for application, where $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$ with $\vdash \mathtt{t}_1 : \mathtt{T}_{11} \to \mathtt{T}_{12}$ and $\vdash \mathtt{t}_2 : \mathtt{T}_{11}$.

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Progress

Theorem: Suppose t is a closed, well-typed term (that is, $\vdash t : T$ for some T). Then either t is a value or else there is some t' with $t \longrightarrow t'$.

Proof: By induction on typing derivations. The cases for boolean constants and conditions are the same as before. The variable case is trivial (because t is closed). The abstraction case is immediate, since abstractions are values.

Consider the case for application, where $t = t_1\ t_2$ with $\vdash t_1 : T_{11} \to T_{12}$ and $\vdash t_2 : T_{11}$. By the induction hypothesis, either t_1 is a value or else it can make a step of evaluation, and likewise t_2. If t_1 can take a step, then rule E-App1 applies to t. If t_1 is a value and t_2 can take a step, then rule E-App2 applies. Finally, if both t_1 and t_2 are values, then the canonical forms lemma tells us that t_1 has the form $\lambda x{:}T_{11}.t_{12}$, and so rule E-AppAbs applies to t.

Proving Preservation

Theorem: If $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

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Case T-App: Given $t = t_1\ t_2$

$\Gamma \vdash t_1 : T_{11} \to T_{12}$

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$T = T_{12}$

Show $\Gamma \vdash t' : T_{12}$

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By the inversion lemma for evaluation, there are three subcases...

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Subcase: $t_1 = \lambda x{:}T_{11}.\ t_{12}$

t_2 a value v_2

$t' = [x \mapsto v_2]t_{12}$

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Theorem: If $\Gamma \vdash t : T$ and $t \longrightarrow t'$, then $\Gamma \vdash t' : T$.

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Show $\Gamma \vdash t' : T_{12}$

By the inversion lemma for evaluation, there are three subcases...

Subcase: $t_1 = \lambda x{:}T_{11}.\ t_{12}$

t_2 a value v_2

$t' = [x \mapsto v_2]t_{12}$

Uh oh.

The "Substitution Lemma"

Lemma: Types are preserved under substitition.

If $\Gamma, \texttt{x:S} \vdash \texttt{t} : \texttt{T}$ and $\Gamma \vdash \texttt{s} : \texttt{S}$, then $\Gamma \vdash [\texttt{x} \mapsto \texttt{s}]\texttt{t} : \texttt{T}$.

The "Substitution Lemma"

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If $\Gamma, \mathtt{x{:}S} \vdash \mathtt{t} : \mathtt{T}$ and $\Gamma \vdash \mathtt{s} : \mathtt{S}$, then $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

Proof: ...

On to real programming languages...

The `Unit` type

`t`	::=	...	*terms*
		`unit`	*constant* `unit`
`v`	::=	...	*values*
		`unit`	*constant* ***`unit`***
`T`	::=	...	*types*
		`Unit`	*unit type*

New typing rules

$$\boxed{\Gamma \vdash \texttt{t} : \texttt{T}}$$

$$\Gamma \vdash \texttt{unit} : \texttt{Unit} \qquad \text{(T-Unit)}$$

Sequencing

`t`	::=	...	*terms*
		$\texttt{t}_1\texttt{;t}_2$	

Sequencing

$\mathtt{t} \quad ::= \quad \ldots$ *terms*

$\quad\quad\quad \mathtt{t}_1\,;\mathtt{t}_2$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\,;\mathtt{t}_2 \longrightarrow \mathtt{t}_1'\,;\mathtt{t}_2} \qquad \text{(E-Seq)}$$

$$\mathtt{unit};\mathtt{t}_2 \longrightarrow \mathtt{t}_2 \qquad \text{(E-SeqNext)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Unit} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \mathtt{t}_1\,;\mathtt{t}_2 : \mathtt{T}_2} \qquad \text{(T-Seq)}$$

Derived forms

- Syntatic sugar
- Internal language vs. external (surface) language

Sequencing as a derived form

$$\texttt{t}_1\texttt{;t}_2 \quad \stackrel{\textbf{def}}{=} \quad (\lambda\texttt{x:Unit.t}_2)\ \texttt{t}_1$$

where $\texttt{x} \notin FV(\texttt{t}_2)$

Equivalence of the two definitions

[board]

Ascription

New syntactic forms

$$\begin{array}{lcll} \mathtt{t} & ::= & \ldots & \textit{terms} \\ & & \mathtt{t\ as\ T} & \textit{ascription} \end{array}$$

New evaluation rules

$\boxed{\mathtt{t} \longrightarrow \mathtt{t'}}$

$$\mathtt{v_1\ as\ T} \longrightarrow \mathtt{v_1} \qquad \text{(E-Ascribe)}$$

$$\frac{\mathtt{t_1} \longrightarrow \mathtt{t_1'}}{\mathtt{t_1\ as\ T} \longrightarrow \mathtt{t_1'\ as\ T}} \qquad \text{(E-Ascribe1)}$$

New typing rules

$\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \mathtt{t_1} : \mathtt{T}}{\Gamma \vdash \mathtt{t_1\ as\ T} : \mathtt{T}} \qquad \text{(T-Ascribe)}$$

Ascription as a derived form

$$\texttt{t as T} \stackrel{\textbf{def}}{=} \texttt{(}\lambda\texttt{x:T. x) t}$$

Let-bindings

New syntactic forms

$\texttt{t} \quad ::= \quad \ldots$ *terms*

$\texttt{let x=t in t}$ *let binding*

New evaluation rules

$\boxed{\texttt{t} \longrightarrow \texttt{t}'}$

$$\texttt{let x=v}_1 \texttt{ in t}_2 \longrightarrow [\texttt{x} \mapsto \texttt{v}_1]\texttt{t}_2 \qquad \text{(E-LetV)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{let x=t}_1 \texttt{ in t}_2 \longrightarrow \texttt{let x=t}_1' \texttt{ in t}_2} \qquad \text{(E-Let)}$$

New typing rules

$\boxed{\Gamma \vdash \texttt{t} : \texttt{T}}$

$$\frac{\Gamma \vdash \texttt{t}_1 : \texttt{T}_1 \qquad \Gamma, \texttt{x:T}_1 \vdash \texttt{t}_2 : \texttt{T}_2}{\Gamma \vdash \texttt{let x=t}_1 \texttt{ in t}_2 : \texttt{T}_2} \qquad \text{(T-Let)}$$

Pairs

`t`	::=	...	*terms*
		`{t,t}`	*pair*
		`t.1`	*first projection*
		`t.2`	*second projection*
`v`	::=	...	*values*
		`{v,v}`	*pair value*
`T`	::=	...	*types*
		$\mathtt{T}_1 \times \mathtt{T}_2$	*product type*

Evaluation rules for pairs

$$\{v_1, v_2\}.1 \longrightarrow v_1 \qquad \text{(E-PairBeta1)}$$

$$\{v_1, v_2\}.2 \longrightarrow v_2 \qquad \text{(E-PairBeta2)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.1 \longrightarrow t_1'.1} \qquad \text{(E-Proj1)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.2 \longrightarrow t_1'.2} \qquad \text{(E-Proj2)}$$

$$\frac{t_1 \longrightarrow t_1'}{\{t_1, t_2\} \longrightarrow \{t_1', t_2\}} \qquad \text{(E-Pair1)}$$

$$\frac{t_2 \longrightarrow t_2'}{\{v_1, t_2\} \longrightarrow \{v_1, t_2'\}} \qquad \text{(E-Pair2)}$$

Typing rules for pairs

$$\frac{\Gamma \vdash t_1 : T_1 \qquad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \{t_1, t_2\} : T_1 \times T_2} \quad \text{(T-Pair)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_1.1 : T_{11}} \quad \text{(T-Proj1)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \times T_{12}}{\Gamma \vdash t_1.2 : T_{12}} \quad \text{(T-Proj2)}$$

Tuples

$\mathtt{t}$	::=	...	*terms*
		$\{\mathtt{t}_i \ ^{i \in 1..n}\}$	*tuple*
		$\mathtt{t.i}$	*projection*
$\mathtt{v}$	::=	...	*values*
		$\{\mathtt{v}_i \ ^{i \in 1..n}\}$	*tuple value*
$\mathtt{T}$	::=	...	*types*
		$\{\mathtt{T}_i \ ^{i \in 1..n}\}$	*tuple type*

Evaluation rules for tuples

$$\{v_i \ ^{i \in 1..n}\}.j \longrightarrow v_j \qquad \text{(E-ProjTuple)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.i \longrightarrow t_1'.i} \qquad \text{(E-Proj)}$$

$$\frac{t_j \longrightarrow t_j'}{\{v_i \ ^{i \in 1..j-1}, t_j, t_k \ ^{k \in j+1..n}\} \longrightarrow \{v_i \ ^{i \in 1..j-1}, t_j', t_k \ ^{k \in j+1..n}\}} \qquad \text{(E-Tuple)}$$

Typing rules for tuples

$$\frac{\text{for each } \mathtt{i} \quad \Gamma \vdash \mathtt{t_i} : \mathtt{T_i}}{\Gamma \vdash \{\mathtt{t_i}^{\ \mathtt{i}\in 1..n}\} : \{\mathtt{T_i}^{\ \mathtt{i}\in 1..n}\}} \qquad \text{(T-Tuple)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \{\mathtt{T_i}^{\ \mathtt{i}\in 1..n}\}}{\Gamma \vdash \mathtt{t_1}.\mathtt{j} : \mathtt{T_j}} \qquad \text{(T-Proj)}$$

Records

$\mathtt{t}$	::=	...	*terms*
		$\{\mathtt{l}_i=\mathtt{t}_i \ ^{i\in 1..n}\}$	*record*
		$\mathtt{t.l}$	*projection*
$\mathtt{v}$	::=	...	*values*
		$\{\mathtt{l}_i=\mathtt{v}_i \ ^{i\in 1..n}\}$	*record value*
$\mathtt{T}$	::=	...	*types*
		$\{\mathtt{l}_i:\mathtt{T}_i \ ^{i\in 1..n}\}$	*type of records*

Evaluation rules for records

$$\{l_i{=}v_i \ ^{i\in 1..n}\}.l_j \longrightarrow v_j \qquad \text{(E-ProjRcd)}$$

$$\frac{t_1 \longrightarrow t_1'}{t_1.l \longrightarrow t_1'.l} \qquad \text{(E-Proj)}$$

$$\frac{t_j \longrightarrow t_j'}{\begin{array}{l}\{l_i{=}v_i \ ^{i\in 1..j-1}, l_j{=}t_j, l_k{=}t_k \ ^{k\in j+1..n}\} \\ \longrightarrow \{l_i{=}v_i \ ^{i\in 1..j-1}, l_j{=}t_j', l_k{=}t_k \ ^{k\in j+1..n}\}\end{array}} \qquad \text{(E-Rcd)}$$

Typing rules for records

$$\frac{\text{for each } \mathtt{i} \quad \Gamma \vdash \mathtt{t_i} : \mathtt{T_i}}{\Gamma \vdash \{\mathtt{l_i}=\mathtt{t_i}^{\ i\in 1..n}\} : \{\mathtt{l_i}:\mathtt{T_i}^{\ i\in 1..n}\}} \quad \text{(T-Rcd)}$$

$$\frac{\Gamma \vdash \mathtt{t_1} : \{\mathtt{l_i}:\mathtt{T_i}^{\ i\in 1..n}\}}{\Gamma \vdash \mathtt{t_1}.\mathtt{l_j} : \mathtt{T_j}} \quad \text{(T-Proj)}$$

Discussion

Intro vs. elim forms

An introduction form for a given type gives us a way of constructing elements of this type.

An elimination form for a type gives us a way of using elements of this type.

What typing rules are introduction forms? What are elimination forms?

The Curry-Howard Correspondence

In constructive logics, a proof of **P** must provide evidence for **P**.

- "law of the excluded middle" — $\mathbf{P} \vee \neg\mathbf{P}$ — not recognized.

A proof of $\mathbf{P} \wedge \mathbf{Q}$ is a pair of evidence for **P** and evidence for **Q**.

A proof of $\mathbf{P} \supset \mathbf{Q}$ is a procedure for transforming evidence for **P** into evidence for **Q**.

Propositions as Types

Logic	Programming languages
propositions	types
proposition $\mathbf{P} \supset \mathbf{Q}$	type `P`$\rightarrow$`Q`
proposition $\mathbf{P} \wedge \mathbf{Q}$	type `P` $\times$ `Q`
proof of proposition $\mathbf{P}$	term `t` of type `P`
proposition $\mathbf{P}$ is provable	type `P` is inhabited (by some term)

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	evaluation

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proof of proposition $\mathbf{P}$	term `t` of type `P`
proposition $\mathbf{P}$ is provable	type `P` is inhabited (by some term)
proof simplification (a.k.a. "cut elimination")	evaluation

Erasure

$$
\begin{array}{lcl}
\mathit{erase}(\mathtt{x}) & = & \mathtt{x} \\
\mathit{erase}(\lambda \mathtt{x{:}T_1}\,.\ \mathtt{t_2}) & = & \lambda \mathtt{x}.\ \mathit{erase}(\mathtt{t_2}) \\
\mathit{erase}(\mathtt{t_1}\ \mathtt{t_2}) & = & \mathit{erase}(\mathtt{t_1})\ \mathit{erase}(\mathtt{t_2})
\end{array}
$$

Typability

An untyped λ-term $\mathtt{m}$ is said to be typable if there is some term $\mathtt{t}$ in the simply typed lambda-calculus, some type $\mathtt{T}$, and some context Γ such that $\mathit{erase}(\mathtt{t}) = \mathtt{m}$ and $\Gamma \vdash \mathtt{t} : \mathtt{T}$.

Cf. type reconstruction in OCaml.