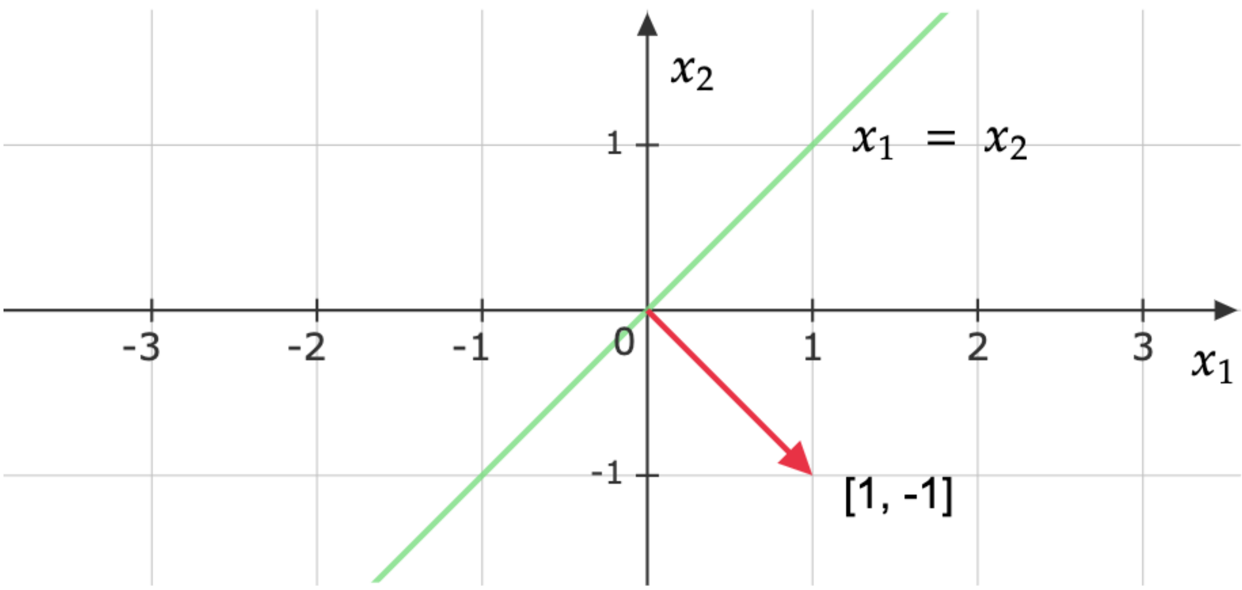
CIS 419/519

Primer - Geometry

In this primer, we will review some geometry concepts in linear algebra fashion. The key concepts are:

- Line
- Distance

Line



In 2D space, we usually represent a line(green) using its equation.

Equation:

$$x_1 = x_2, 0 = x_1 - x_2$$

Alternatively, we could represent a line using a point through which the line passes, and the line's direction(a unit vector):

Vector Equation:

$$L(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, t \in \mathbb{R}$$

Using the vector equation, we could represent points on the line by selecting appropriate t values. For exmaple, point [1, 1] could be represented as

$$L(\sqrt{2}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We could also represent the same line using its ***normal vector***(red). The is the general way to define a hyperplane in space.

Using Normal Vector:

$$0 = [1, -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here the vector [1, -1] is perpendicular to the direction of the line.

Hyperplane and Normal Vectors

Lines can also be called hyperplanes in 2D space. In geometry, a hyperplane is a subspace whose dimension is one less than that of its ambient space. For example a 2-dimensional plane in 3D space. In general a hyper plane in a n-dimensional space could be defined as:

$$0 = \theta^T x + b$$

Where $w \in \mathbb{R}^n$ is the normal vector of the hyperplane. The normal vector, often simply called the "normal," to a hyperplane is a vector which is perpendicular to the hyperplane at a given point. In the example above, $\theta = [1, -1]$ is a normal vector of the green hyperplane.

Distance

Distance between two points

In 2D space, the **Euclidean distance** between two points $P_A = (x_A, y_A)$ and $P_B = (x_B, y_B)$ can be calculated as:

$$d = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

This is also the ***L2 norm*** of the vector between this two points:

$$d = ||P_A - P_B||_2 = ||(x_A - x_B, y_A - y_B)||_2$$

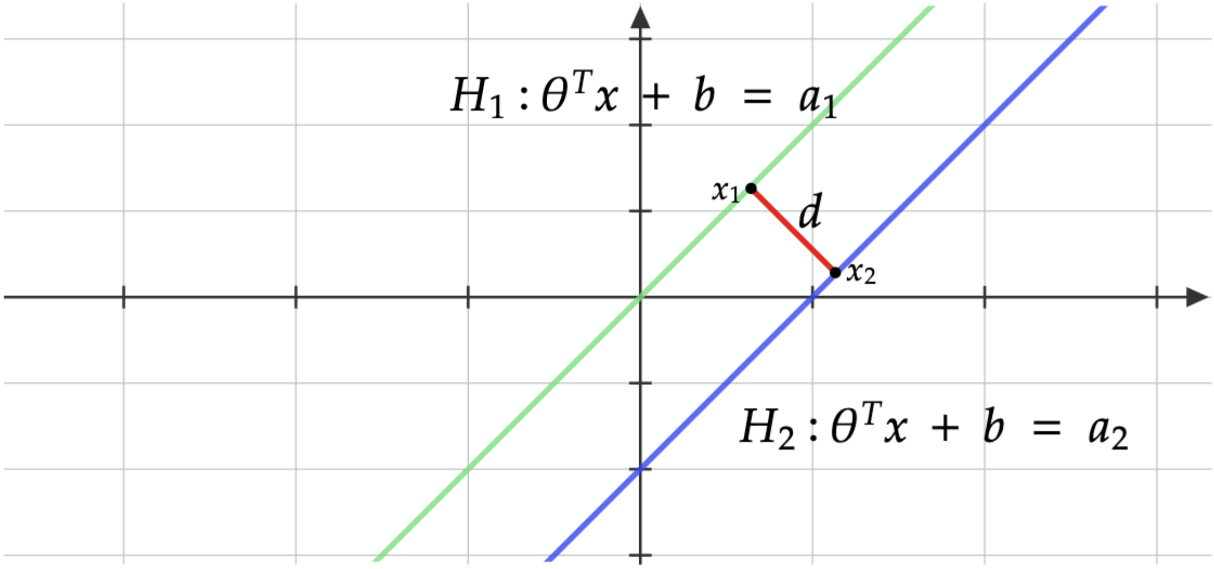
Similarly, in 3D space (points represented as $P = [x_1, x_2, x_3]$), the Euclidean distance between two points P_A and P_B is :

$$d = ||(x_{1A} - x_{1B}, x_{2A} - x_{2B}, x_{3A} - x_{3B})||_2$$

This could be easily generalized to n-dimensional space where $P_A, P_B \in \mathbb{R}^n$:

$$d = ||(x_{1A} - x_{1B}, \ldots, x_{nA} - x_{nB})||_2$$

Distance between two parallel hyperplanes



Suppose we have two hyperplanes in 2D:

$$H_1 : \theta^T x + b = a_1$$

$$H_2 : \theta^T x + b = a_2$$

The distance between this two hyperplanes can be calculated as follow:

Let x_1 be any point on H_1 such that:

$$\theta^T x_1 + b = a_1$$

Consider the line L that passes through x_1 in the direction of the normal vector θ and intersects H_2 at point x_2. An equation for L is given by

$$L(t) = x_1 + t\frac{\theta}{||\theta||}$$

where $t \in \mathbb{R}$ is a scaling factor(Vector Equation of a line). As x_2 is also on L, we can represent x_2 as:

$$x_2 = x_1 + d\frac{\theta}{||\theta||}$$

where d is the distance between x_1 and x_2. Note that the distance between these two points is also the distance between the hyperplanes as L is perpendicular to both H_1 and H_2. Plug this into the expression of H_2 we get:

$$\theta^T(x_1 + d\frac{\theta}{||\theta||}) + b = a_2 \iff (\theta^T x_1 + b) + d\frac{\theta^T\theta}{||\theta||} = a_2$$

Note here $\theta^T x_1 + b = a_1$ (Equation for H_1), and therefore we have:

$$a_1 + d||\theta|| = a_2 \iff d = \frac{|a_1 - a_2|}{||\theta||}$$