

Announcements

- Homework 3 due in **one week**
- Quiz due **tomorrow**
- List of project datasets to be released today
 - **Computer vision:** Object classification/detection
 - **NLP:** Sentiment analysis
 - **Tabular:** Time series forecasting, recommendation systems

Lecture 10: Learning Ensembles

CIS 4190/5190

Spring 2023

Ensemble Design Decisions

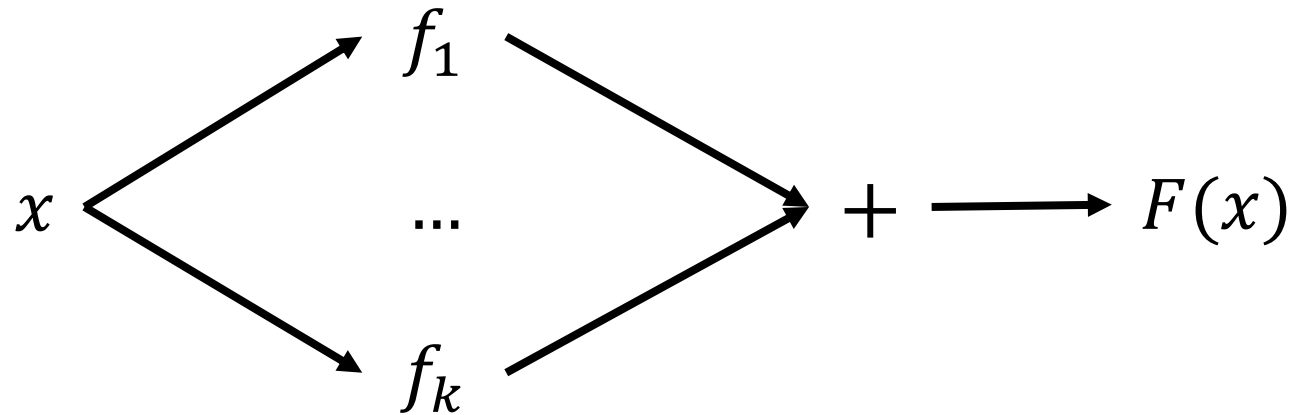
- How to learn the base models?
- How to combine the learned base models?

Ensemble Design Decisions

- How to learn the base models?
- **How to combine the learned base models?**

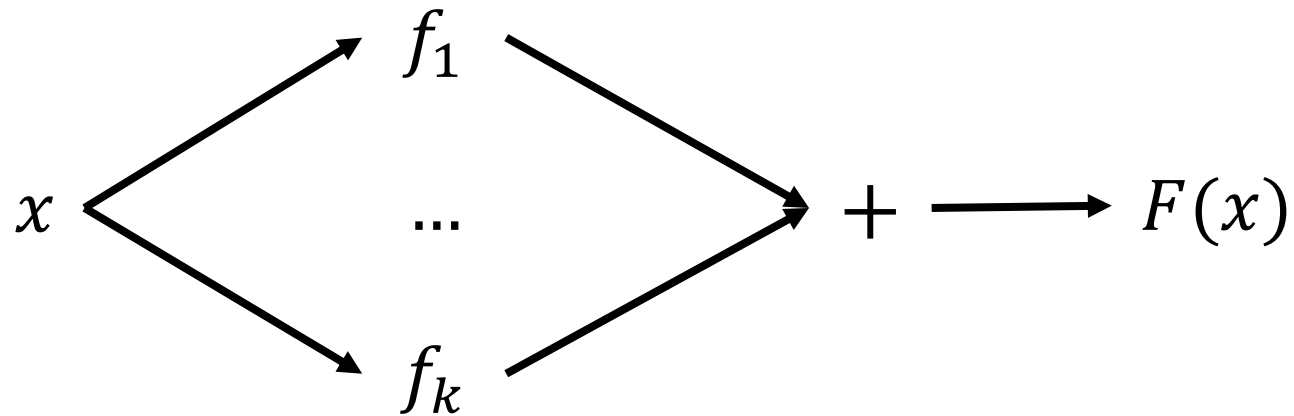
Combining Learned Base Models

- **Regression:** Average predictions $F(x) = \frac{1}{k} \sum_{i=1}^k f_i(x)$
 - Works well if the base models have similar performance



Combining Learned Base Models

- **Classification:** Majority vote $F(x) = 1 \left(\sum_{i=1}^k f_i(x) \geq \frac{k}{2} \right)$ (for binary)
 - Can also average probabilities for classification



Combining Learned Base Models

- Can use weighted average:

$$F(x) = \sum_{i=1}^k \beta_i \cdot f_i(x)$$

- Can fit weights using linear regression on second training set
- More generally, can fit a second layer model

$$F(x) = g_{\beta}(f_1(x), \dots, f_k(x))$$

Combining Learned Base Models

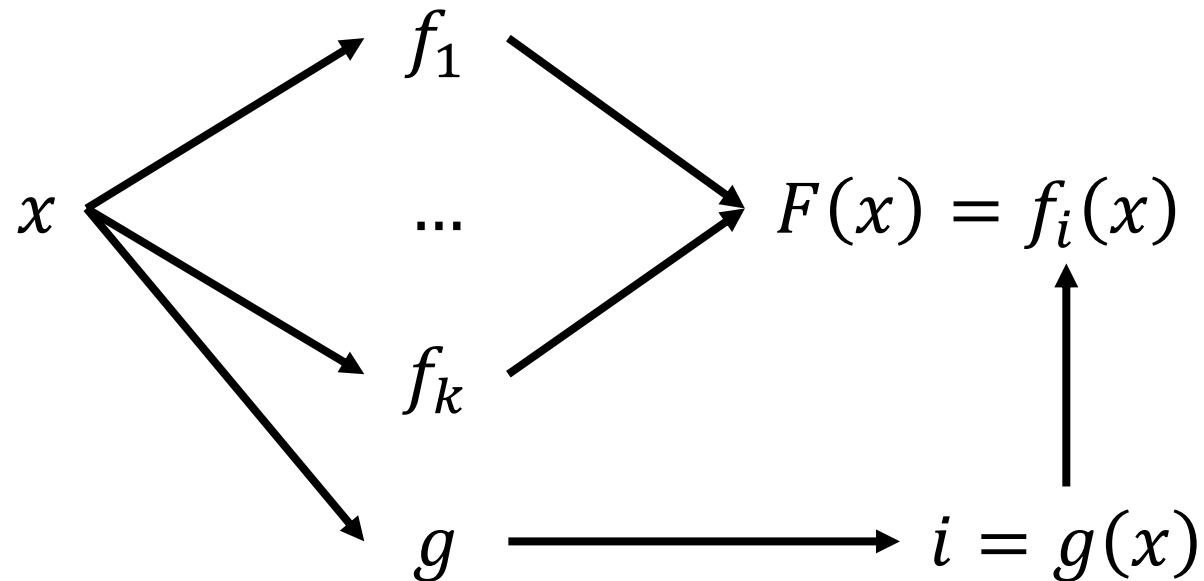
- Second model as “mixture of experts”:

$$F(x) = \sum_{i=1}^k g(x)_i \cdot f_i(x)$$

- Second stage model predicts weights over “experts” $f_i(x)$

Combining Learned Base Models

- Second model as “mixture of experts”:
 - **Special case:** $g(x)$ is one-hot
 - **Advantage:** Only need to run $g(x)$ and $f_{g(x)}(x)$



Ensemble Design Decisions

- How to learn the base models?
- How to combine the learned base models?

Ensemble Design Decisions

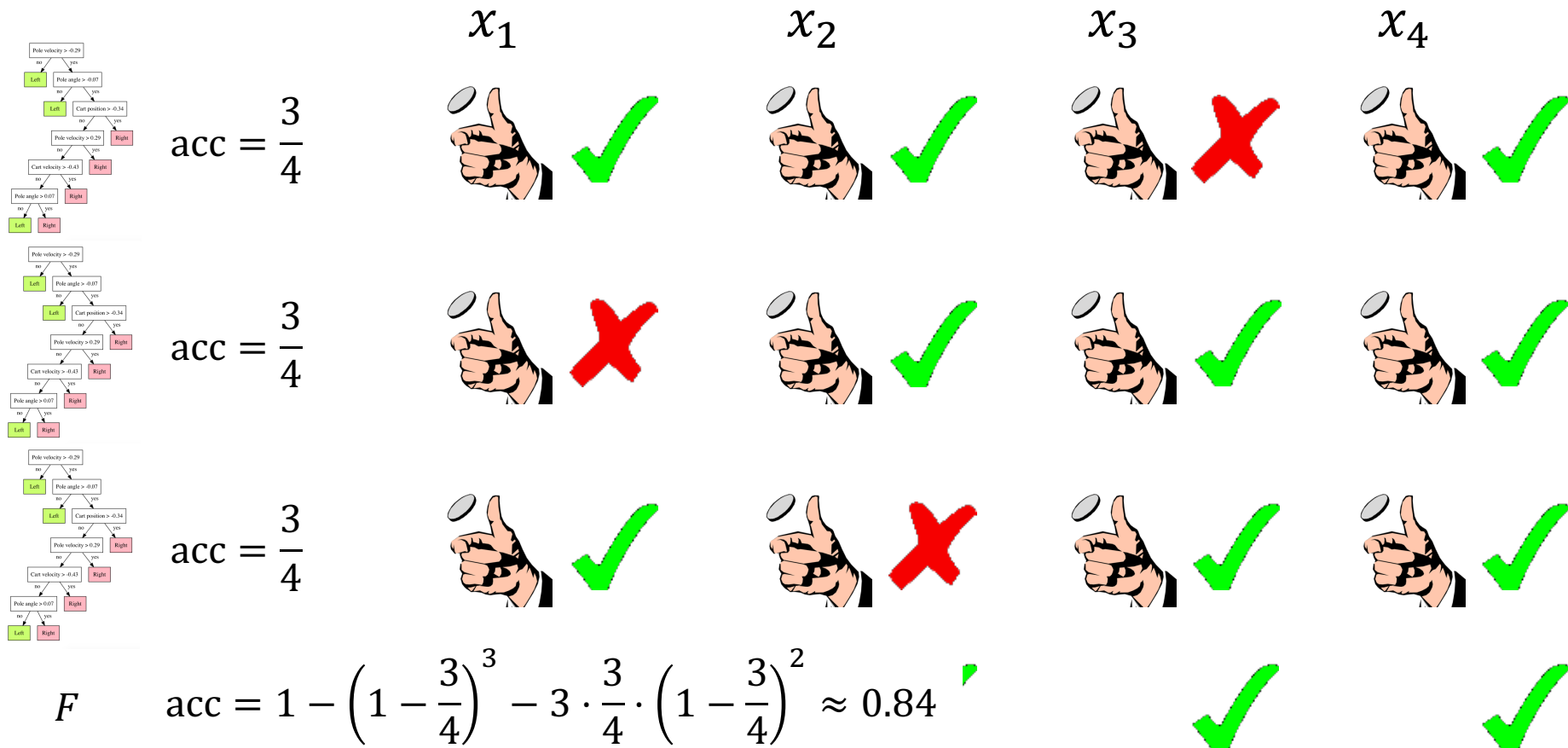
- **How to learn the base models?**
- How to combine the learned base models?

Learning Base Models

- Successful ensembles require **diversity**
 - Different model families
 - Different training data
 - Different features
 - Different hyperparameters
- **Intuition:** Models should make **independent** mistakes

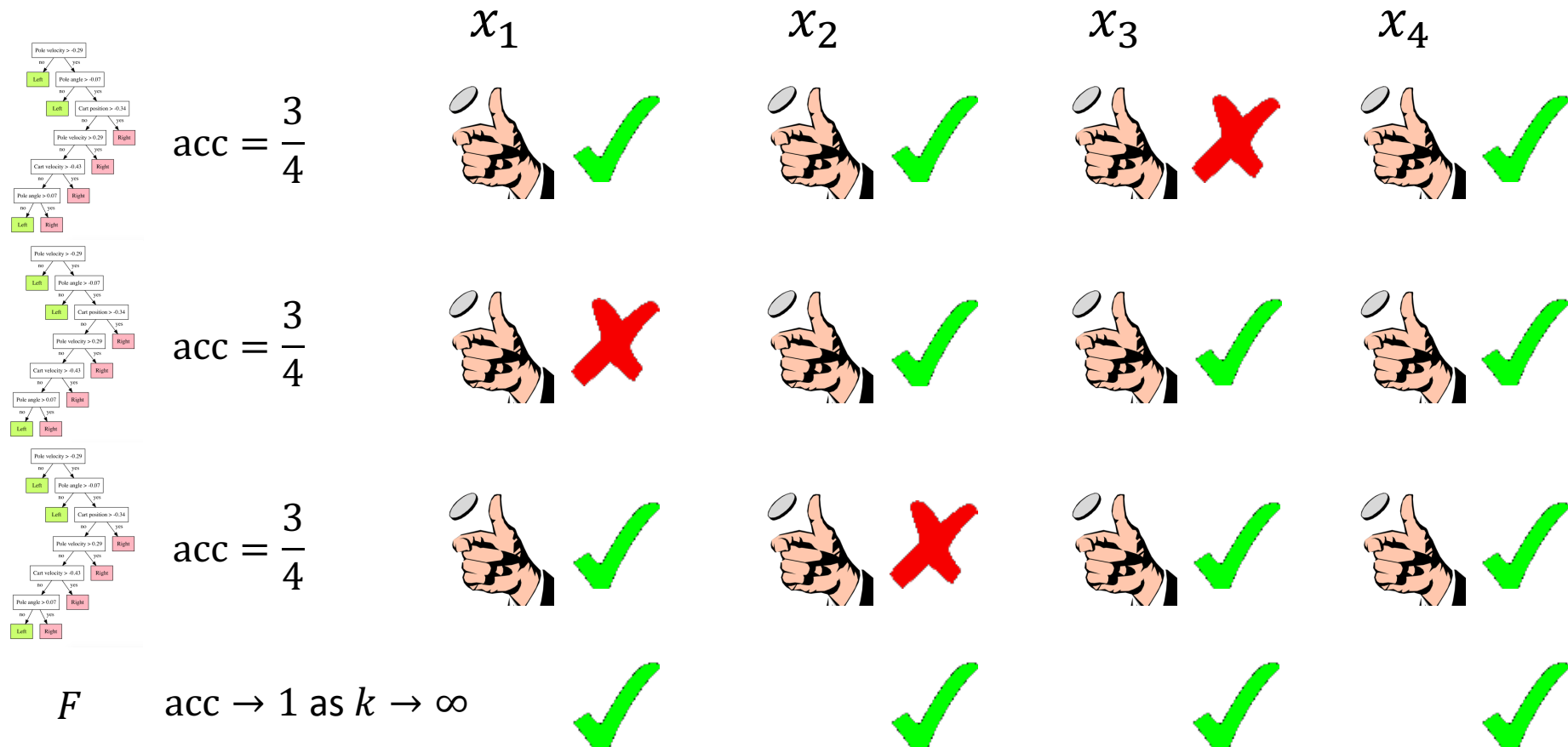
Learning Base Models

- **Intuition:** Models should make **independent** mistakes



Learning Base Models

- **Intuition:** Models should make **independent** mistakes



Learning Base Models

- Ensemble can be built from different learning algorithms
 - **Example:** Decision tree, logistic regression, kNN, ...
- What if we want an ensemble of decision trees?
 - **Issue:** Decision tree learning algorithm is deterministic
 - **Solution:** Randomize the learning algorithm (may sacrifice performance)!
- Randomize decisions inside learning algorithm
 - **Example:** Randomize splits weighted (somehow) by information gain
 - **Issue:** Very specific to the algorithm
 - **Solution:** Randomize input to learning algorithm (i.e., training data)!

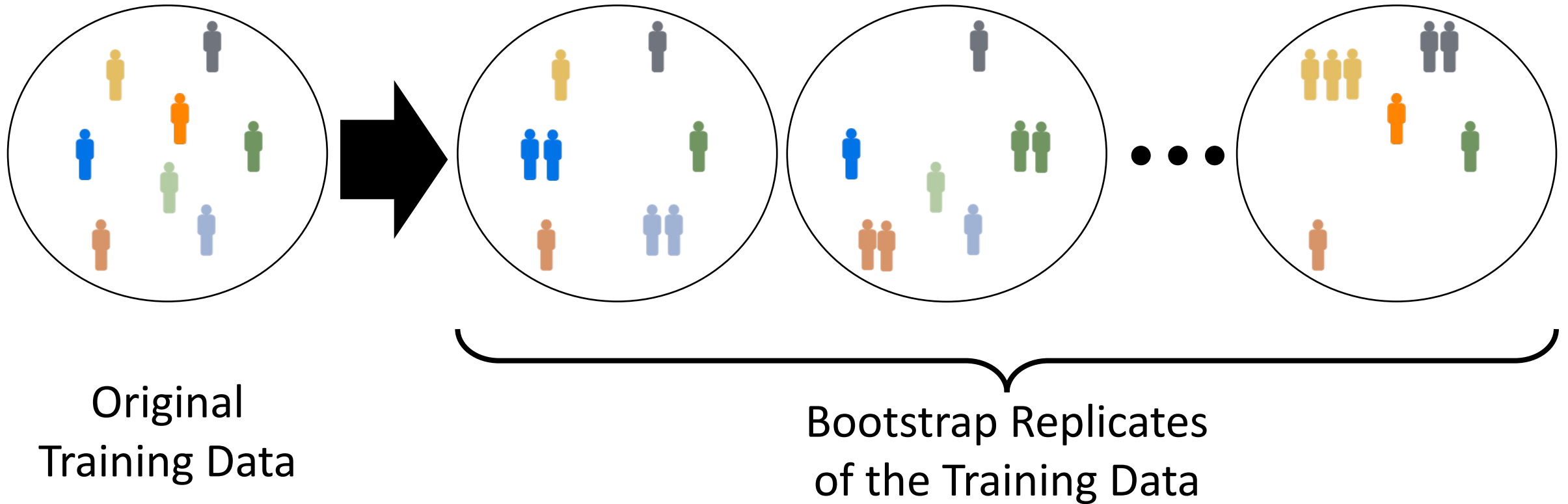
Randomizing Learning Algorithms

- **Bagging:** Randomize training data (“Bootstrap Aggregating”)
 - **Random examples:** Subsample examples $\{(x, y)\}$ (obtain $X \in \mathbb{R}^{n' \times d}$)
 - **Random features:** Subsample features x_j (obtain $X \in \mathbb{R}^{n \times d'}$)
- Meta-strategy that can build ensembles from arbitrary base learners
- Can be thought of as a form of regularization

Bootstrap

- Subsample examples $\{(x, y)\}$ **with replacement** (obtain $X \in \mathbb{R}^{n \times d}$)
- Excludes $\left(1 - \frac{1}{n}\right)^n$ of the training examples
 - Separately in each of the replicates
 - As $n \rightarrow \infty$, excludes $\rightarrow \frac{1}{e} \approx 36.8\%$ examples
- Has good statistical properties

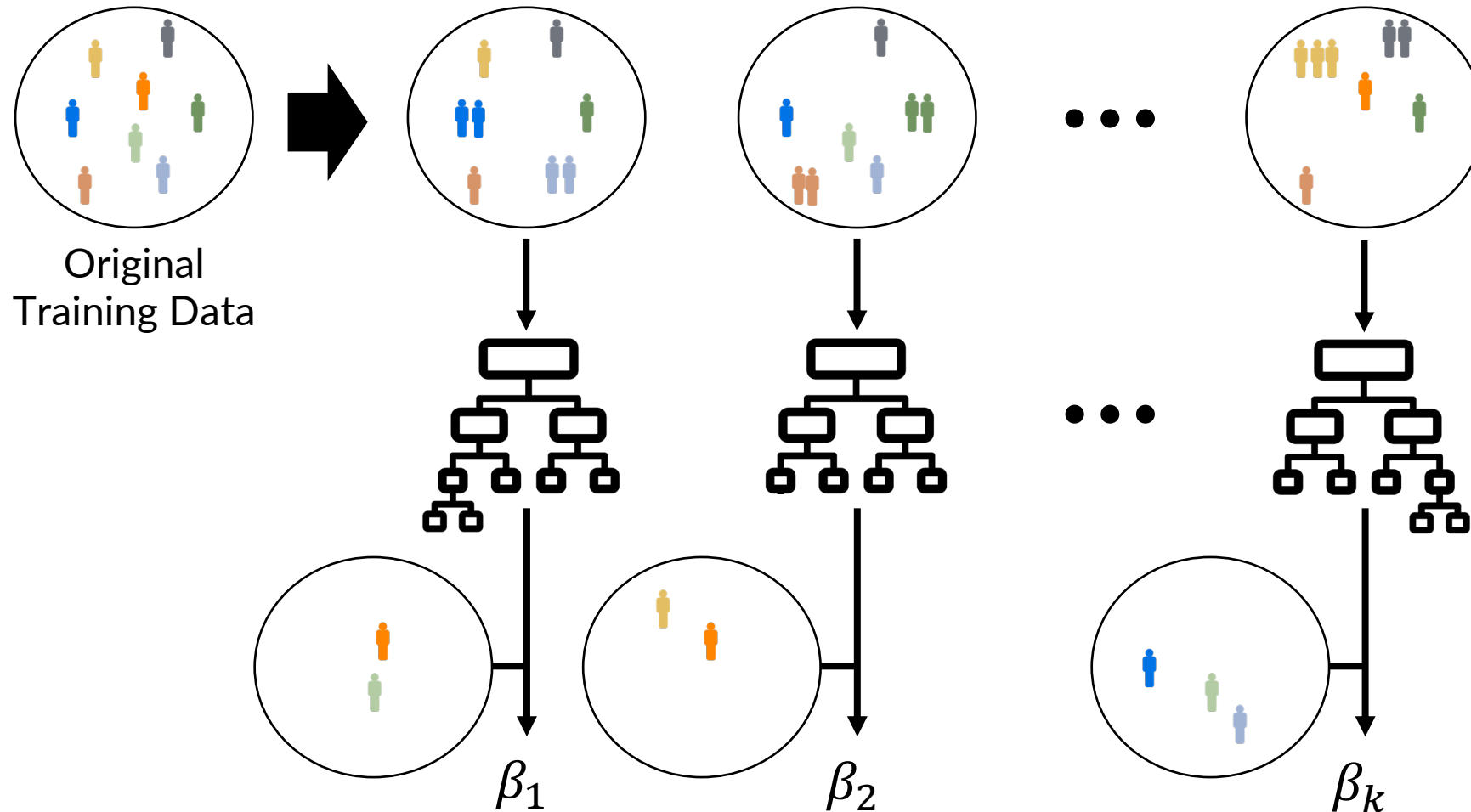
Randomizing Learning Algorithms



Ensemble Learning

- **Step 1:** Create bootstrap replicates of the original training dataset
- **Step 2:** Train a classifier for each replicate
- **Step 3 (Optional):** Use held-out validation set to weight models
 - Can just use average predictions

Ensemble Learning



Random Forests

- Ensemble of decision trees using bagging
 - Typically use simple average (over probabilities for classification)
- **Intuition:**
 - Large decision trees are good nonlinear models, but high variance
 - Random forests average over many decision trees to reduce variance without increasing bias

Random Forests

- **Tweak 1:** Randomize features in learning algorithm instead of bagging
 - At DT node splitting step, subsample $\approx \sqrt{d}$ features
 - Allows each tree to use all features, but not at every node
 - **Aside:** If a few features are highly predictive, then they will be selected in many trees, causing the base models to be highly correlated
- **Tweak 2:** Train **unpruned** decision trees
 - Ensures base models have higher capacity
 - **Intuition:** Skipping pruning increases variance

Bias Variance Tradeoff for Random Forests

- Naïvely, skipping pruning yields high variance
- Introduce randomness to average away “excess” variance
 - Without randomness, all models in the random forest would be the same (large) decision tree, so the random forest would still have very large variance
- Randomness should ideally make base models more independent

AdaBoost (Freund & Schapire 1997)

- Like bagging, meta-algorithm that turns base models into ensemble
 - **Provably learns** for base models achieving any error rate > 0.5
- Uses **different training example weights** (instead of different subsamples or different features) to introduce diversity
 - In particular, **upweights** currently incorrectly predicted examples
- Base models should satisfy the following:
 - High-bias/low-capacity (e.g., depth-limited decision trees, linear classifiers)
 - Able to incorporate sample weights during learning
 - **Specific to classification (discuss general losses later)**

AdaBoost (Freund & Schapire 1997)

- **Input**

- Training dataset Z
- Learning algorithm $\text{Train}(Z, w)$ that can handle weights w
- Hyperparameter T indicating number of models to train

- **Output**

- Ensemble of models $F(x) = \sum_{t=1}^T \beta_t \cdot f_t(x)$

Aside: Learning with Weighted Examples

- Many algorithms can directly incorporate weights into the loss
- For maximum likelihood estimation:

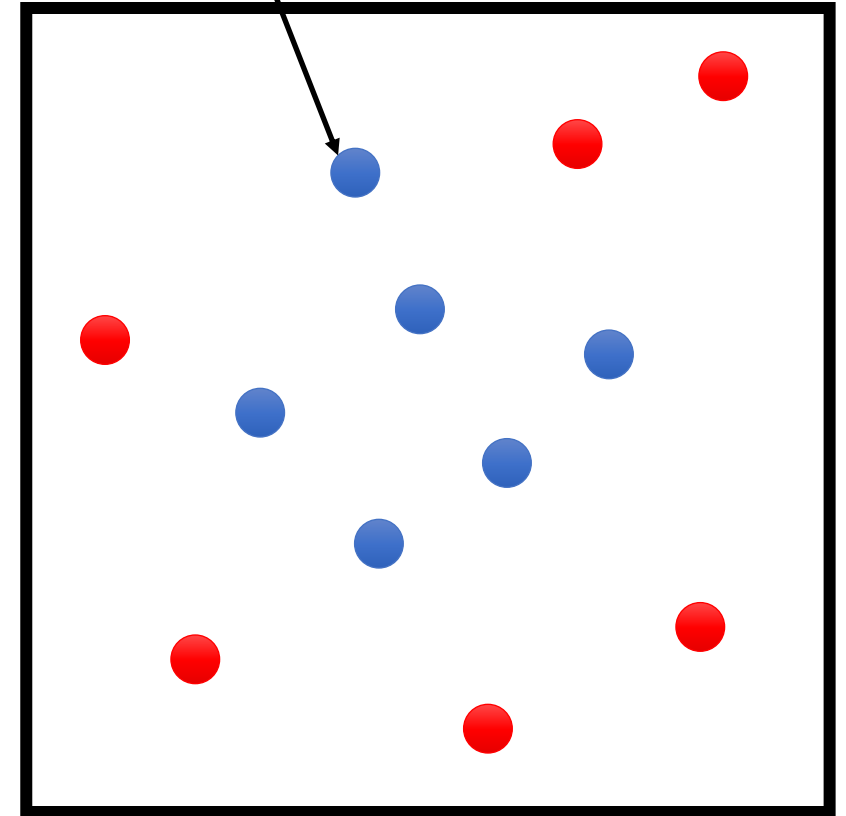
$$\ell(\beta; Z, w) = \sum_{i=1}^n w_i \cdot \log p_{\beta}(y_i | x_i)$$

- Alternatively, can subsample the data proportional to weights w_i

AdaBoost

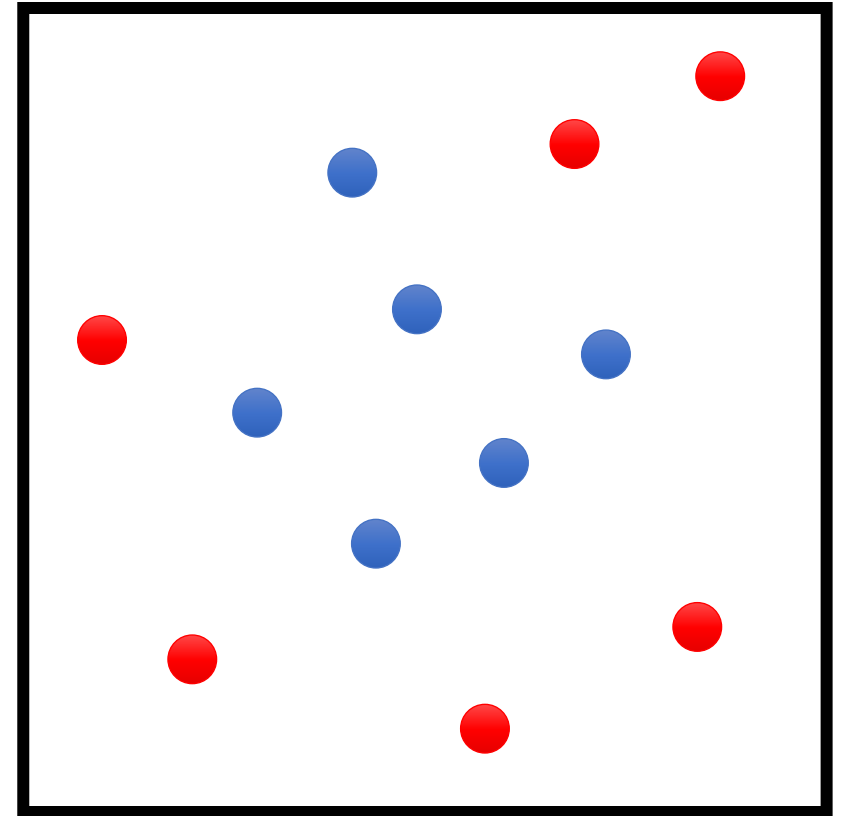
size represents weight w_i

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2. **for** $t \in \{1, \dots, T\}$
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6. $w_{t+1,i} \propto w_{t,i} \cdot e^{-\beta_t \cdot y_i \cdot f_t(x_i)}$ (for all i)
7. **return** $F(x) = \text{sign}(\sum_{t=1}^T \beta_t \cdot f_t(x))$



AdaBoost

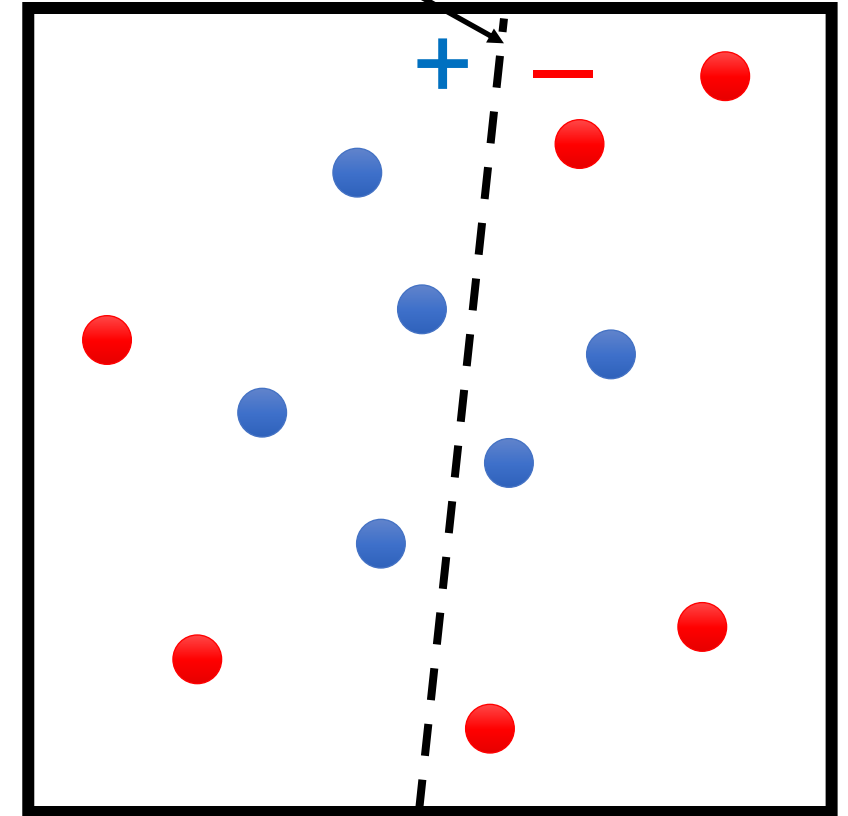
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AdaBoost

focus on linear classifiers f_t

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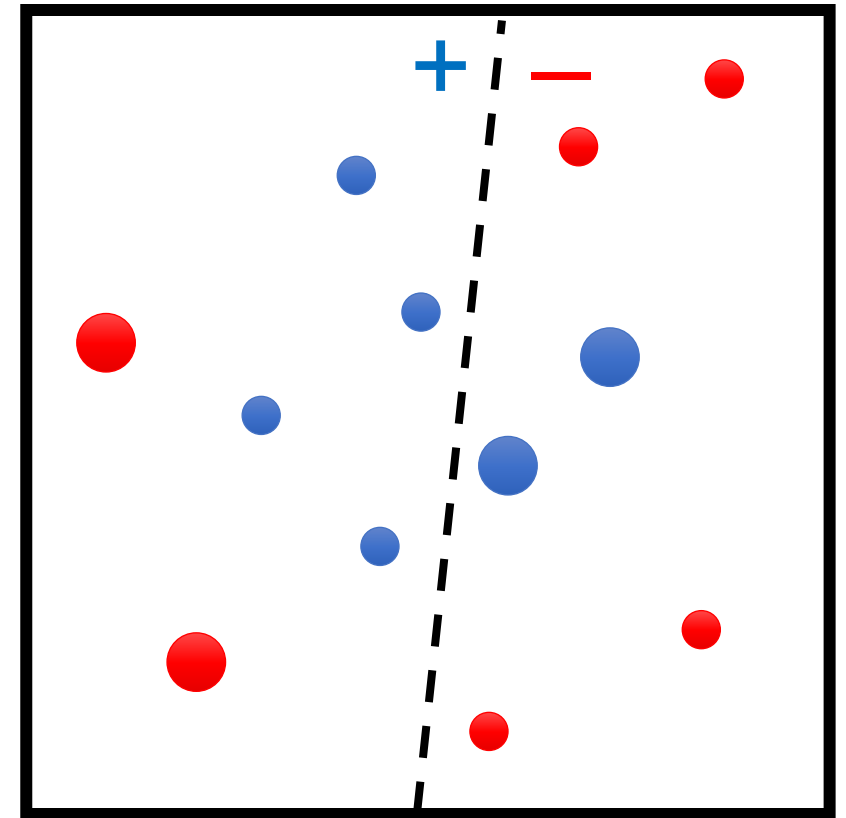
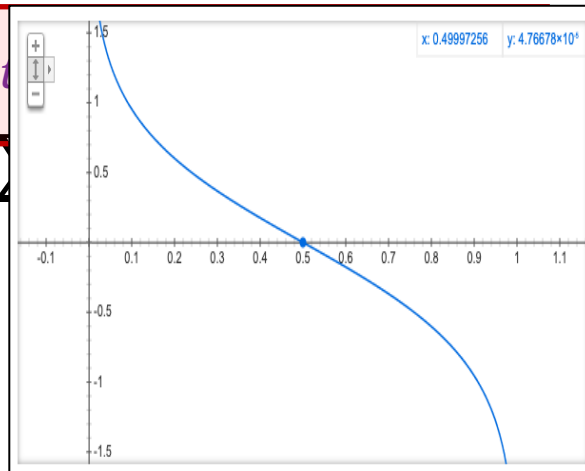


$t = 1$

AdaBoost

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β_t becomes larger as
 ϵ_t becomes smaller



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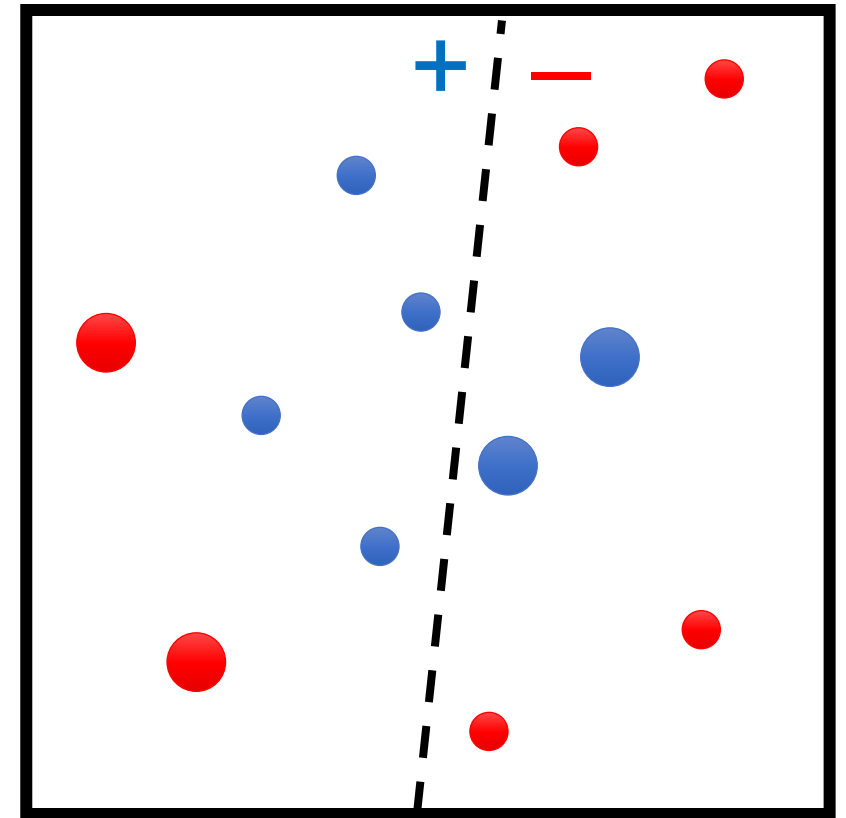
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Use convention $y_i \in \{-1, +1\}$

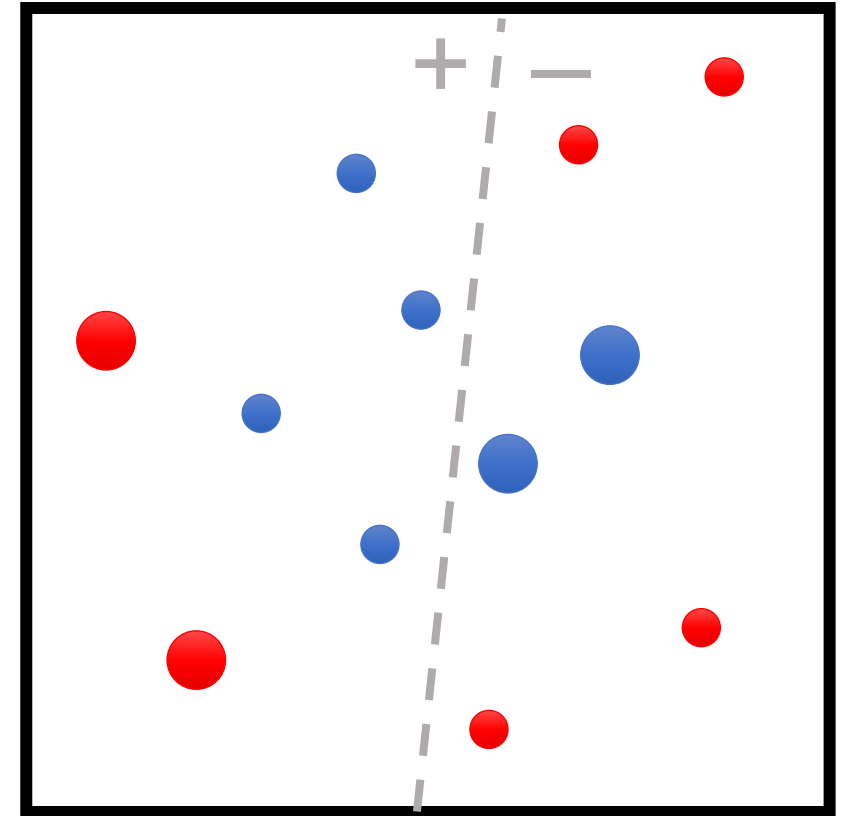
If correct ($y_i = f_t(x_i)$) then multiply by $e^{-\beta_t}$

If incorrect ($y_i \neq f_t(x_i)$) then multiply by e^{β_t}



AdaBoost

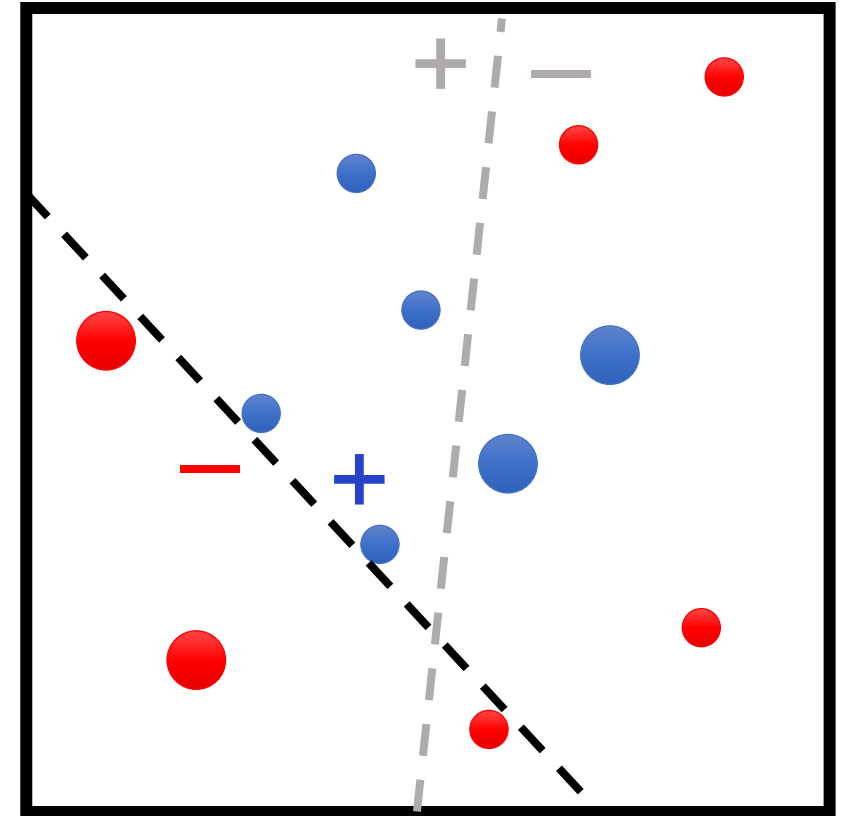
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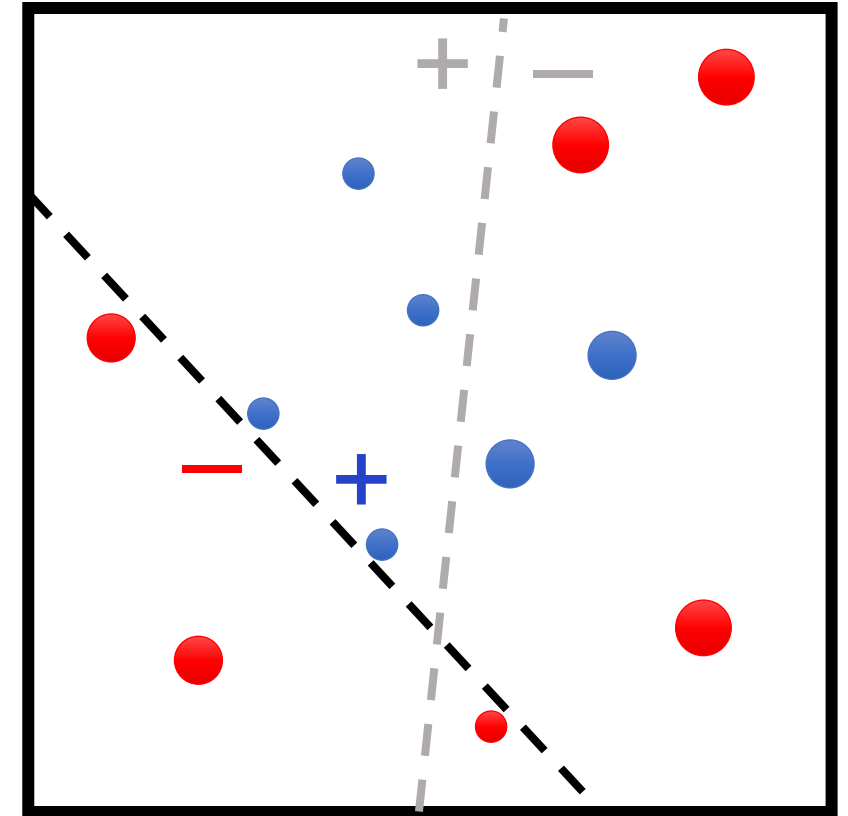
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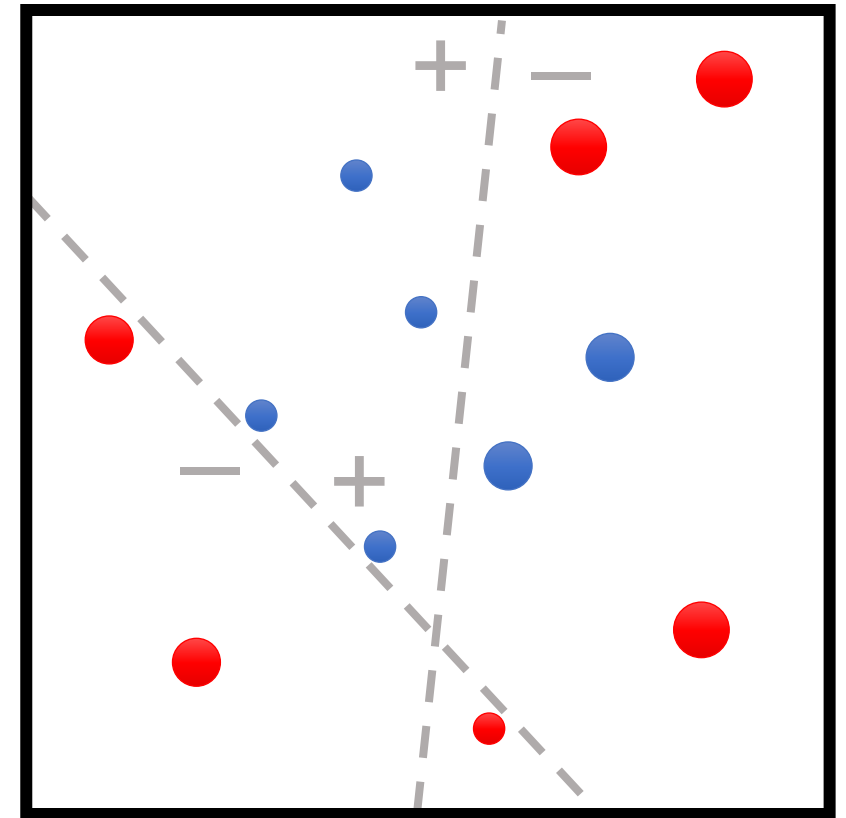
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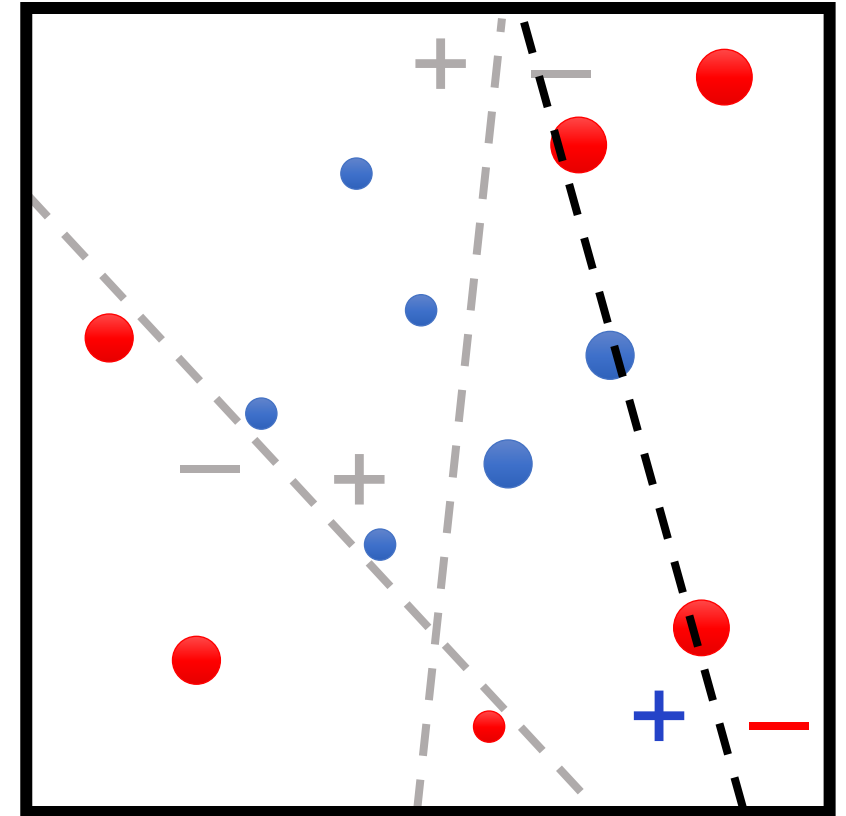
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$t = 2$

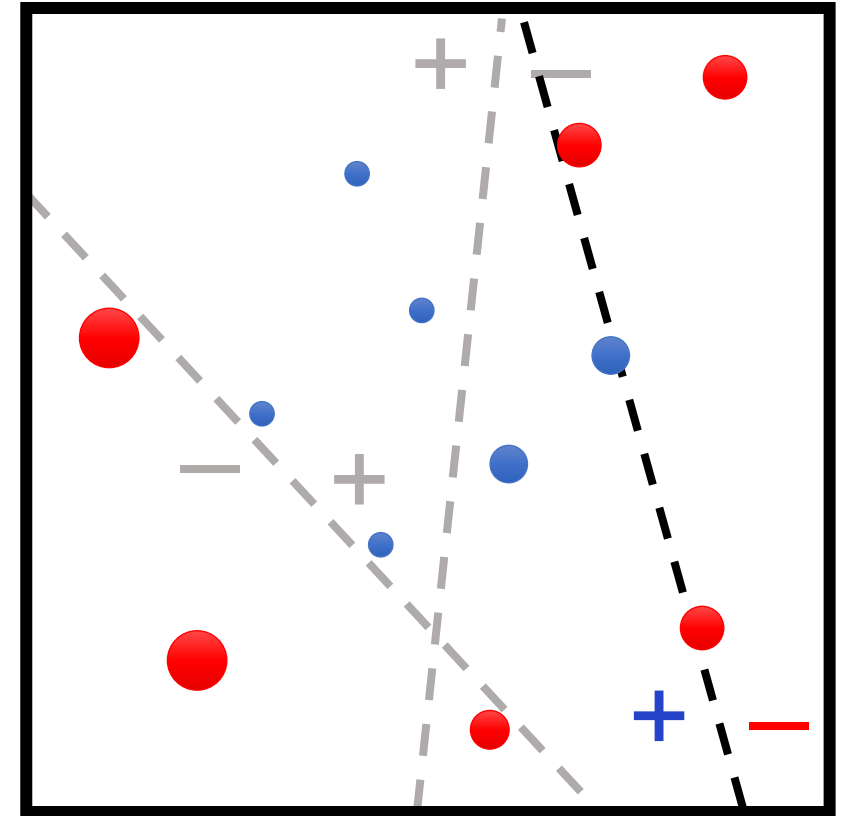
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AdaBoost

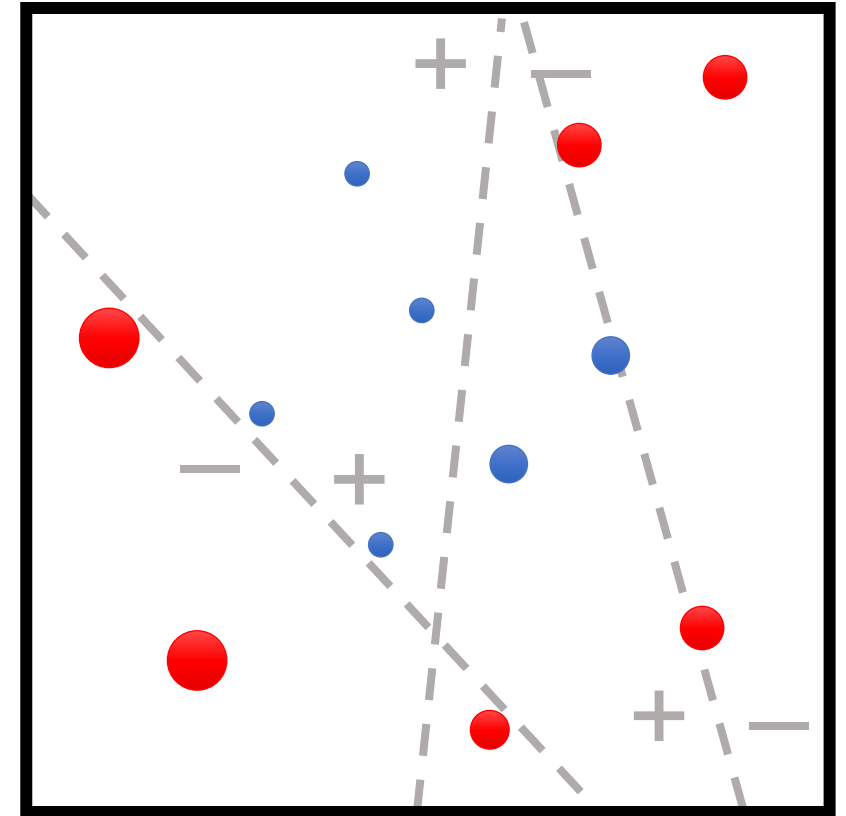
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$t = 3$

AdaBoost

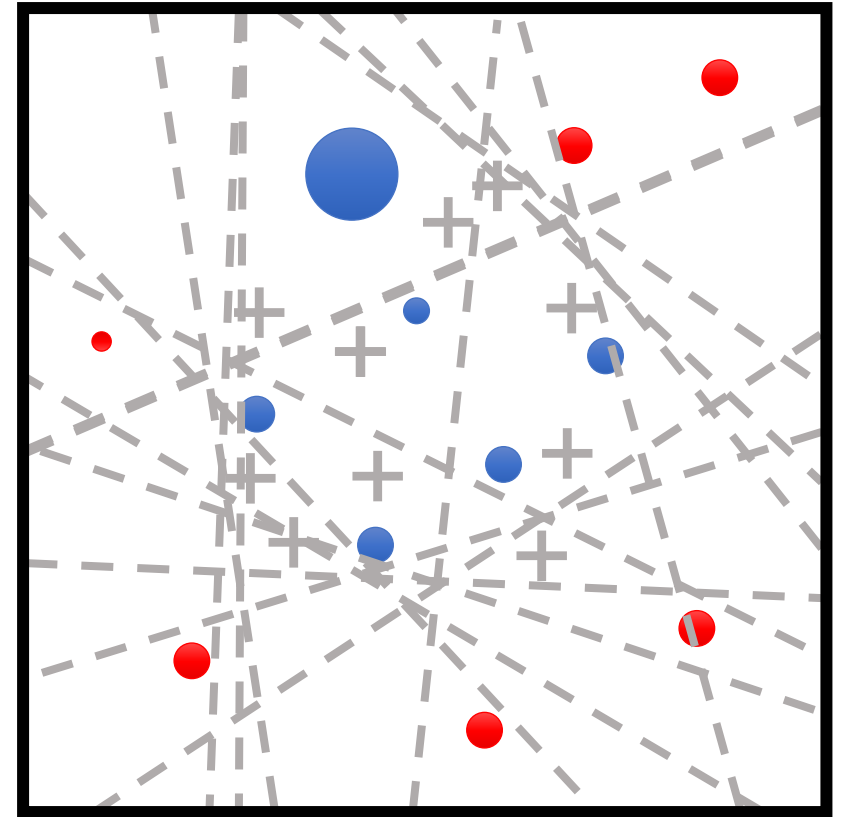
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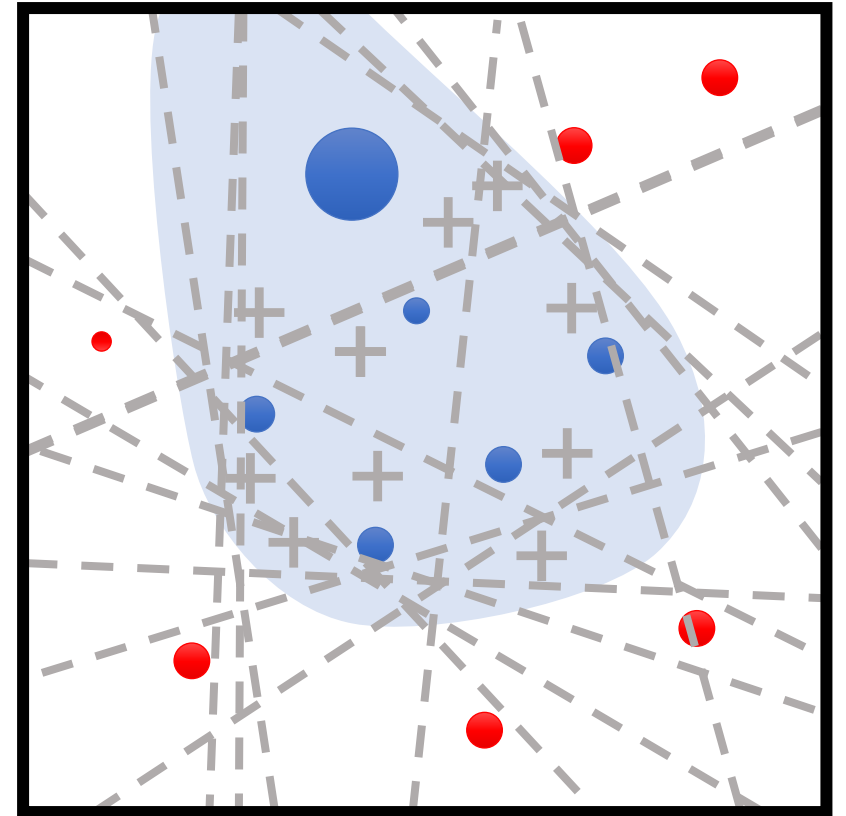


Under certain assumptions, training error goes to zero in $O(\log n)$ iterations $\rightarrow t = T$

AdaBoost

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final model is average of base models
weighted by their performance

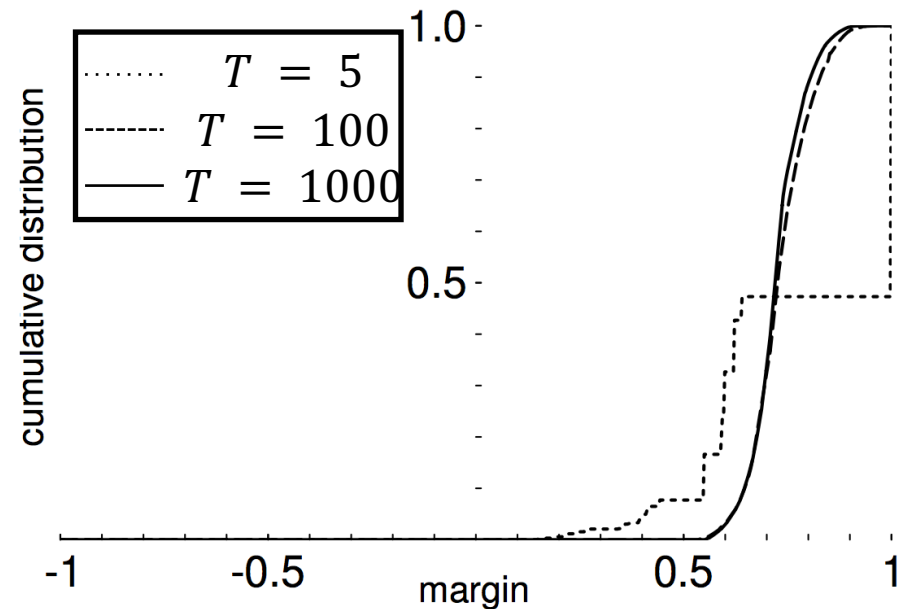
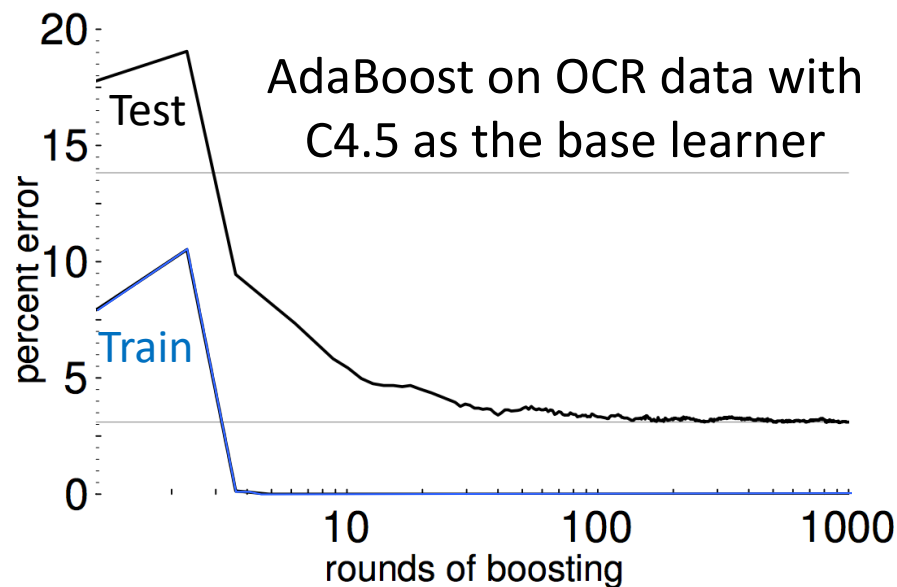


AdaBoost Weighting Strategy

- On each iteration:
 - Misclassified examples are upweighted
 - Correctly classified are downweighted
- If an example is repeatedly misclassified, it will eventually be upweighted so much that it is correctly classified
- Emphasizes “hardest” parts of the input space
 - Instances with highest weight are often outliers

AdaBoost and Overfitting

- Basic ML theory predicts AdaBoost always overfits as $T \rightarrow \infty$
 - Hypothesis keeps growing more complex!
 - In practice, AdaBoost often does not overfit



AdaBoost Summary

- **Strengths:**

- Fast and simple to implement
- No hyperparameters (except for T)
- Very few assumptions on base models

- **Weaknesses:**

- Can be susceptible to noise/outliers when there is insufficient data
- No way to parallelize
- Small gains over complex base models
- **Specific to classification!**

Boosting as Gradient Descent

- Both algorithms: **new model** = **old model** + **update**
- **Gradient Descent:**

$$\theta_{t+1} = \theta_t - \alpha \cdot \nabla_{\theta} L(\theta_t; Z)$$

- **Boosting:**

$$F_{t+1}(x) = F_t(x) + \beta_{t+1} \cdot f_{t+1}(x)$$

- Here, $F_t(x) = \sum_{i=1}^t \beta_i \cdot f_i(x)$

Boosting as Gradient Descent

- Assuming $\beta_t = 1$ for all t , then:

$$F_t(x_i) + f_{t+1}(x_i) = F_{t+1}(x_i)$$

Boosting as Gradient Descent

- Assuming $\beta_t = 1$ for all t , then:

$$F_t(x_i) + f_{t+1}(x_i) = F_{t+1}(x_i) \approx y_i$$

- Rewriting this equation, we have

$$f_{t+1}(x_i) = F_{t+1}(x_i) - F_t(x_i) \approx \underbrace{y_i - F_t(x_i)}$$

“residuals”, i.e., error of the current model

Boosting as Gradient Descent

- In other words, at each step, boosting is training the next model f_{t+1} to approximate the residual:

$$f_{t+1}(x_i) \approx \underbrace{y_i - F_t(x_i)}$$

“residuals”, i.e., error of the current model

- **Idea:** Train f_{t+1} directly to predict residuals $y_i - F_t(x_i)$
- **This strategy works for regression as well!**

Boosting as Gradient Descent

- **Algorithm:** For each $t \in \{1, \dots, T\}$:
 - **Step 1:** Train f_{t+1} using dataset

$$Z_{t+1} = \left\{ (x_i, y_i - F_t(x_i)) \right\}_{i=1}^n$$

- **Step 2:** Take

$$F_{t+1}(x) = F_t(x) + f_{t+1}(x)$$

- Return the final model F_T

Boosting as Gradient Descent

- Consider losses of the form

$$L(F; Z) = \frac{1}{n} \sum_{i=1}^n \tilde{L}(F(x_i); y_i)$$

- In other words, sum of individual label-level losses $\tilde{L}(\hat{y}; y)$ of a prediction $\hat{y} = F(x)$ if the ground truth label is y
- For example, $\tilde{L}(\hat{y}; y) = \frac{1}{2} (\hat{y} - y)^2$ yields the MSE loss

Boosting as Gradient Descent

- Residuals are the gradient of the squared error $\tilde{L}(y, \hat{y}) = \frac{1}{2} (y - \hat{y})^2$:

$$-\frac{\partial \tilde{L}}{\partial \hat{y}}(F_t(x_i); y_i) = y_i - F_t(x_i) = \text{residual}_i$$

- For general \tilde{L} , instead of $\{(x_i, y_i - F_t(x_i))\}_{i=1}^n$ we can train f_{t+1} on

$$Z_{t+1} = \left\{ \left(x_i, -\frac{\partial \tilde{L}}{\partial \hat{y}}(F_t(x_i); y_i) \right) \right\}_{i=1}^n$$

Boosting as Gradient Descent

- **Algorithm:** For each $t \in \{1, \dots, T\}$:
 - **Step 1:** Train f_{t+1} using dataset

$$Z_{t+1} = \{(x_i, y_i - F_t(x_i))\}_{i=1}^n$$

- **Step 2:** Take

$$F_{t+1}(x) = F_t(x) + f_{t+1}(x)$$

- Return the final model F_T

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Boosting as Gradient Descent

- Casts ensemble learning in the **loss minimization framework**
 - **Model family:** Sum of base models $F_T(x) = \sum_{t=1}^T f_t(x)$
 - **Loss:** Any differentiable loss expressed as

$$L(\textcolor{teal}{F}; \textcolor{teal}{Z}) = \sum_{i=1}^n \textcolor{red}{\tilde{L}}(\textcolor{teal}{F}(\textcolor{teal}{x}_i), \textcolor{teal}{y}_i)$$

- Gradient boosting is a general paradigm for training ensembles with specialized losses (e.g., most NLL losses)

Gradient Boosting in Practice

- Gradient boosting with depth-limited decision trees (e.g., depth 3) is one of the most powerful off-the-shelf classifiers available
 - **Caveat:** Inherits decision tree hyperparameters
- XGBoost is a very efficient implementation suitable for production use
 - A popular library for gradient boosted decision trees
 - Optimized for computational efficiency of training and testing
 - Used in many competition winning entries, across many domains
 - <https://xgboost.readthedocs.io>