Announcements

- **Project Milestone 2 due Wednesday, April 5 at 8pm**

- Homework 6 released Wednesday, April 5
  - Due Wednesday, April 19 at 8pm

Lecture 21: Reinforcement Learning

CIS 4190/5190

Spring 2023

Reinforcement Learning Problem

- At a high level, we need to specify the following:
  - **State space:** What are the observations the agent may encounter?
  - **Action space:** What are the actions the agent can take?
  - **Transitions/dynamics:** How the state is updated when taking an action
  - **Rewards:** What rewards the agent receives for taking an action in a state

- For most of today, assume state and action spaces are finite

Markov Decision Process (MDP)

- An MDP (S, A, P, R, γ) is defined by:
  - Set of states $s \in S$
  - Set of actions $a \in A$
  - Transition function $P(s' \mid s, a)$ (also called “dynamics” or the “model”)
  - Reward function $R(s, a, s')$
  - Discount factor $\gamma < 1$
- Also assume an initial state distribution $D(s)$
  - Often omitted since optimal policy does not depend on D

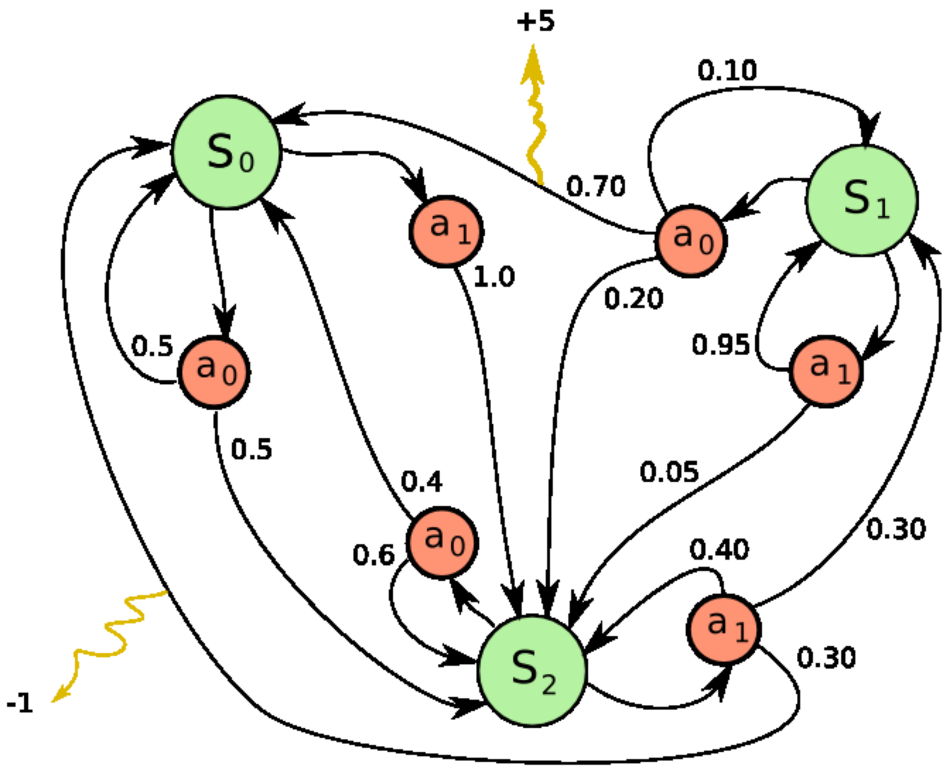


Image: https://towardsdatascience.com/reinforcement-learning-demystified-markov-decision-processes-part-1-bf00dda41690

Policy Gradient Algorithm

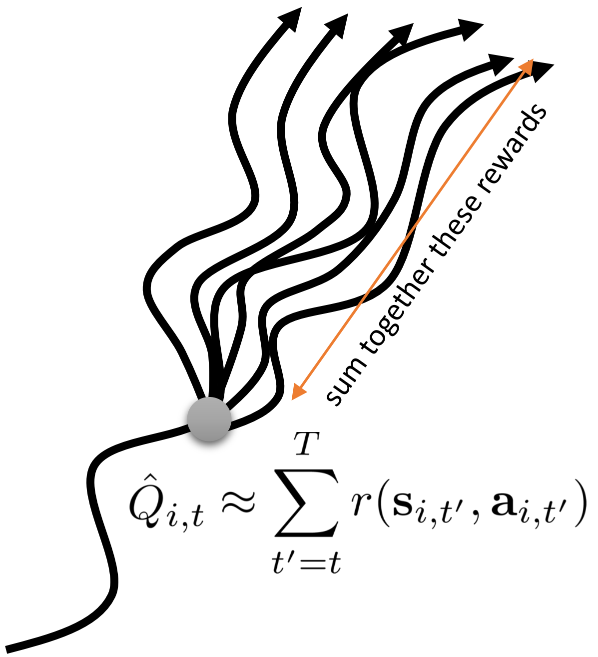
1. sample $\{\tau^i\}$ from $\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(\mathbf{a}_t^i|\mathbf{s}_t^i)\right)\left(\sum_t r(\mathbf{s}_t^i, \mathbf{a}_t^i)\right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

Reward-to-Go Function

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \underbrace{\left(\sum_{t'=t}^{T} r(\mathbf{s}_{i,t'}, \mathbf{a}_{i,t'}) \right)}_{\text{“reward to go” } \hat{Q}_{i,t}}$$

$\hat{Q}_{i,t}$: estimate of expected reward if we take action $\mathbf{a}_{i,t}$ in state $\mathbf{s}_{i,t}$

can we get a better estimate?



$$\hat{Q}_{i,t} \approx \sum_{t'=t}^{T} r(\mathbf{s}_{i,t'}, \mathbf{a}_{i,t'})$$

Reward-to-Go Function

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \underbrace{\left(\sum_{t'=t}^{T} r(\mathbf{s}_{i,t'}, \mathbf{a}_{i,t'}) \right)}_{\text{"reward to go" } \hat{Q}_{i,t}}$$

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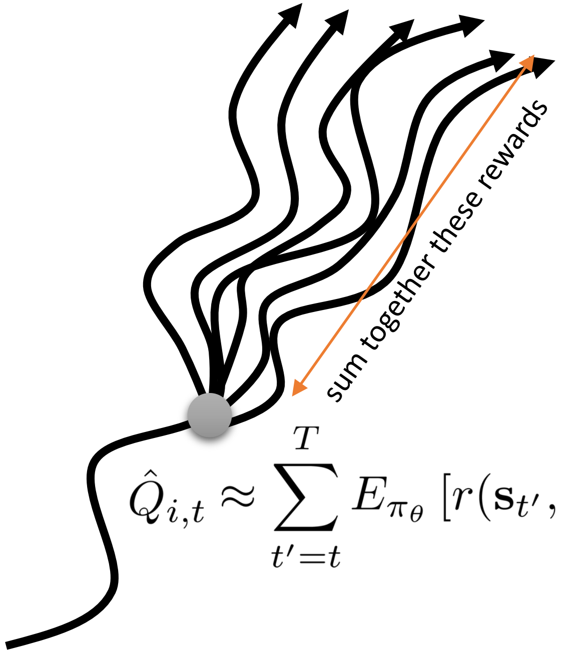
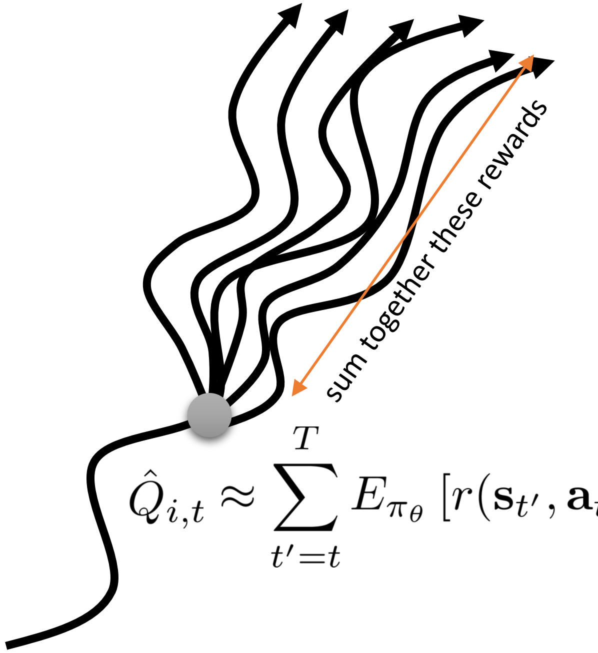


$$\hat{Q}_{i,t} \approx \sum_{t'=t}^{T} E_{\pi_\theta} \left[r(\mathbf{s}_{t'}, \mathbf{a}_{t'}) | \mathbf{s}_t, \mathbf{a}_t \right]$$

Reward-to-Go Function

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta \log \pi_\theta(\mathbf{a}_{i,t}|\mathbf{s}_{i,t}) \underbrace{\left(\sum_{t'=t}^{T} r(\mathbf{s}_{i,t'},\mathbf{a}_{i,t'})\right)}_{\text{“reward to go” } \hat{Q}_{i,t}}$$



$$\hat{Q}_{i,t} \approx \sum_{t'=t}^{T} E_{\pi_\theta}\left[r(\mathbf{s}_{t'},\mathbf{a}_{t'})|\mathbf{s}_t,\mathbf{a}_t\right]$$

$\hat{Q}_{i,t}$: estimate of expected reward if we take action $\mathbf{a}_{i,t}$ in state $\mathbf{s}_{i,t}$

can we get a better estimate?

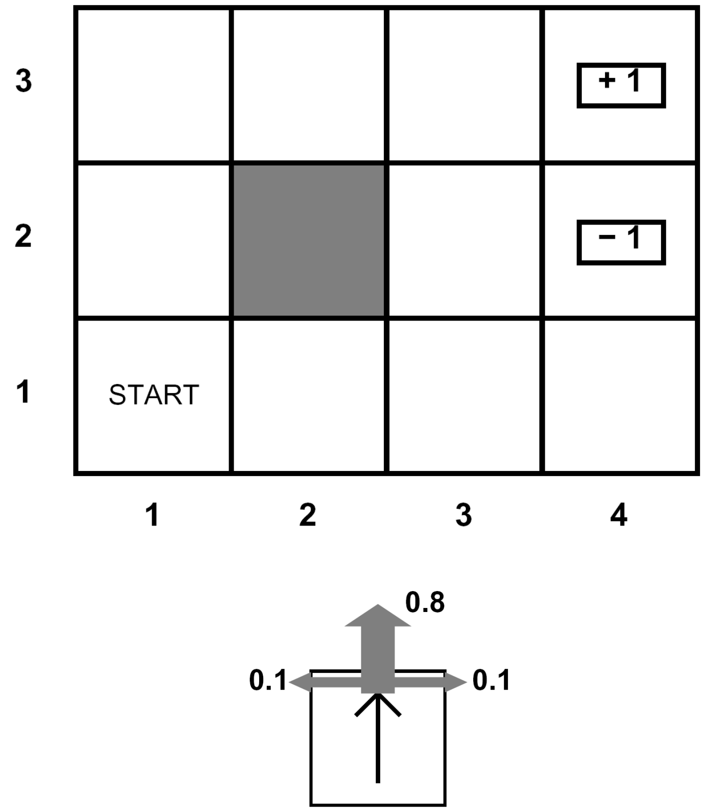
$Q(\mathbf{s}_t,\mathbf{a}_t) = \sum_{t'=t}^{T} E_{\pi_\theta}\left[r(\mathbf{s}_{t'},\mathbf{a}_{t'})|\mathbf{s}_t,\mathbf{a}_t\right]$: true *expected* reward-to-go

Space of RL Algorithms

Q-learning | actor-critic learning | REINFORCE

Toy Example

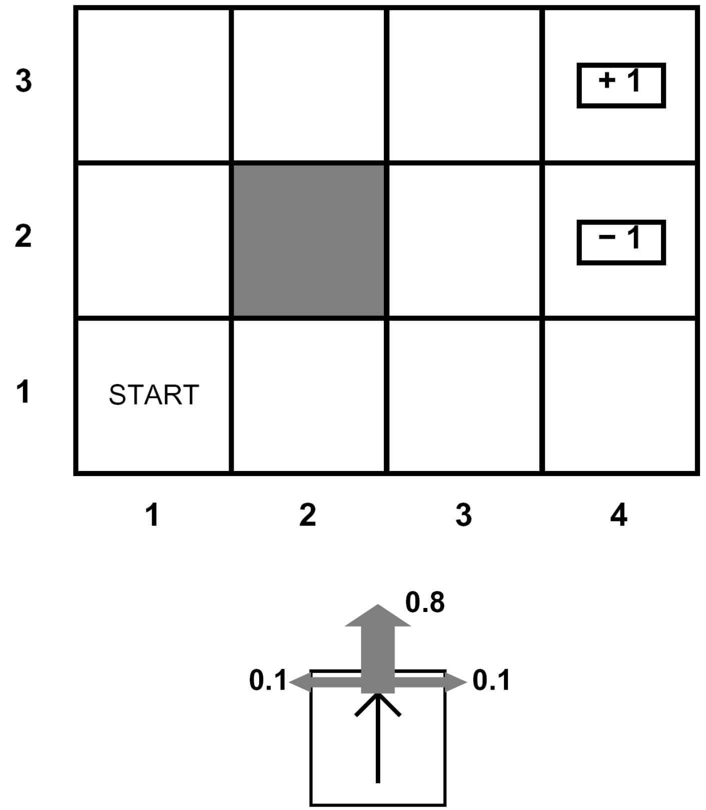
- Grid map with solid/open cells
- **State:** An open grid cell
- **Actions:** Move North, East, South, West



Based on slide by Dan Klein

Toy Example

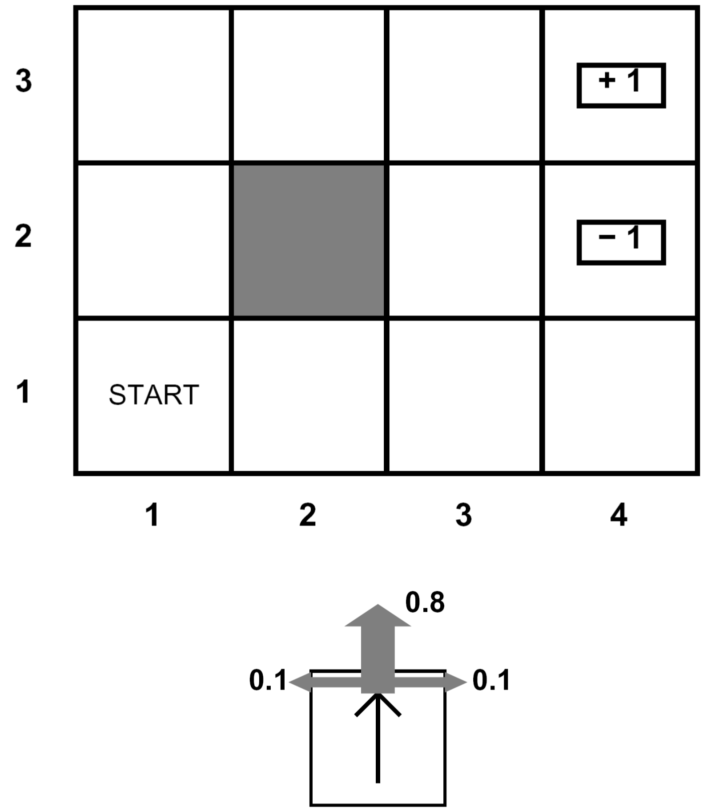
- **Dynamics**
  - Move in chosen direction, but not deterministically!
  - Succeeds 80% of the time
  - 10% of the time, end up $90°$ off
  - 10% of the time, end up $-90°$ off
  - The agent stays put if it tries to move into a solid cell or outside the world
  - At terminal states, any action ends **episode** (or **rollout**)



Based on slide by Dan Klein

Toy Example

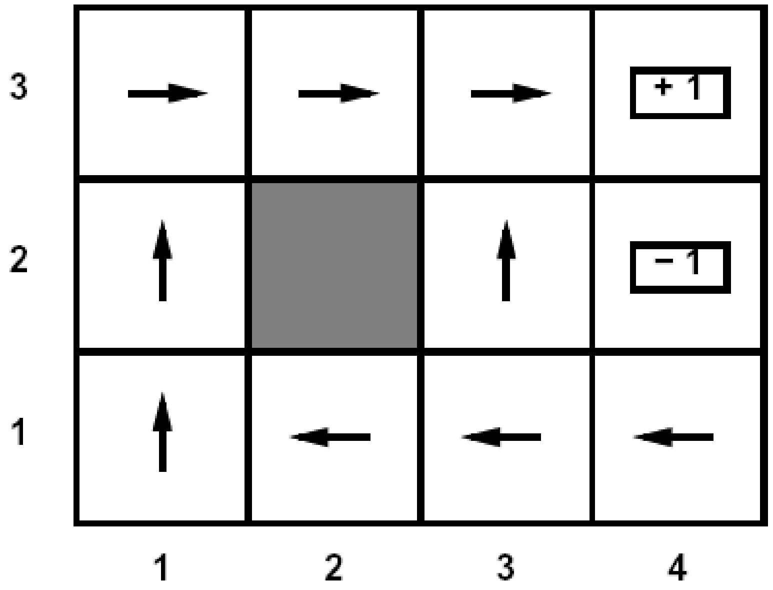
- **Rewards**
  - At terminal state, agent receives the specified reward
  - For each timestep outside terminal states , the agent pays a small cost, e.g., a “reward” of -0.03



Based on slide by Dan Klein

Optimal Policy

- **Optimal policy:** Following π^* maximizes total reward received
  - **Discounted:** Future rewards are downweighted
  - **In expectation:** On average across randomness of environment and actions



Based on slide by Dan Klein

Markov Decision Process (MDP)

- **Goal:** Maximize **cumulative expected discounted reward:**

$$\pi^* = \max_\pi J(\pi) \quad \text{where} \quad J(\pi) = \mathbb{E}_\zeta\left[\sum_{t=0}^{\infty} \gamma^t \cdot r_t\right]$$

- Expectation over **episodes** $\zeta = (s_0, a_0, r_0, s_1, \ldots)$, where
  - $s_0 \sim D$
  - $a_t = \pi(s_t)$
  - $s_{t+1} \sim P(\cdot \mid s_t, a_t)$
  - $r_t = R(s_t, a_t, s_{t+1})$

Policy Value Function

- **Policy Value Function:** Expected reward if we start in s and use π:

$$V^{\pi}(s) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^{t} \cdot r_t \mid s_0 = s\right)$$

- **Bellman equation:**

$$\underbrace{V^{\pi}(s)}_{\text{current value}} = \sum_{s' \in S} \underbrace{P(\, s' \mid s, \pi(s)\,)}_{\substack{\text{expectation} \\ \text{over next state}}} \cdot \underbrace{\big(R(s, \pi(s), s') + \gamma \cdot V^{\pi}(s')\big)}_{\substack{\text{current reward +} \\ \text{discounted future reward}}}$$

Optimal Value Function

- **Optimal value function:** Expected reward if we start in s and use π^*:

$$V^*(s) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t \cdot r_t \mid s_0 = s\right)$$

- **Bellman equation:**

Optimal policy selects action that maximizes future expected reward from state s

$$\underbrace{V^*(s)}_{\text{current value}} = \max_{a \in A} \sum_{s' \in S} \underbrace{P(s' \mid s, a)}_{\text{expectation over next state}} \cdot \underbrace{\left(R(s, a, s') + \gamma \cdot V^*(s')\right)}_{\text{current reward + discounted future reward}}$$

Optimal Value Function

- **Bellman equation:**

$$V^*(s) = \max_{a \in A} \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s, a, s') + \gamma \cdot V^*(s')\right)$$

- Do not need to know the optimal policy π^*!

- **Strategy:** Compute V^* and then use it to compute π^*
  - **Caveat:** Latter step requires knowing P

Policy Action-Value Function

- **Policy Action-Value Function (or Q function):** Expected reward if we start in s, take action a, and then use π thereafter:

$$Q^{\pi}(s,a) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^{t} \cdot r_t \mid s_0 = s, a_0 = a\right)$$

- **Bellman equation:**

$$Q^{\pi}(s,a) = \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s,a,s') + \gamma \cdot Q^{\pi}\big(s', \pi(s')\big)\right)$$

Optimal Action-Value Function

- **Optimal Action-Value Function (or Q function):** Expected reward if we start in s, take action a, and then act optimally thereafter:

$$Q^*(s,a) = \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t \cdot r_t \mid s_0 = s, a_0 = a\right)$$

- **Bellman equation:**

$$Q^*(s,a) = \sum_{s' \in S} P(s' \mid s,a) \cdot \left(R(s,a,s') + \gamma \cdot \max_{a' \in A} Q^*(s',a')\right)$$

Relationship

- We have

$$V^{\pi}(s) = Q^{\pi}\big(s, \pi(s)\big)$$

- Similarly, we have

$$V^{*}(s) = \max_{a} Q^{*}(s, a)$$

Q Iteration

- We have

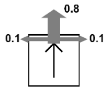
$$\pi^*(s) = \max_{a \in A} Q^*(s, a)$$

- **Strategy:** Compute Q^* and then use it to compute π^*

Q Iteration

- Initialize $Q_1(s, a) \leftarrow 0$ for all s, a
- For $i \in \{1,2, \ldots\}$ until convergence:

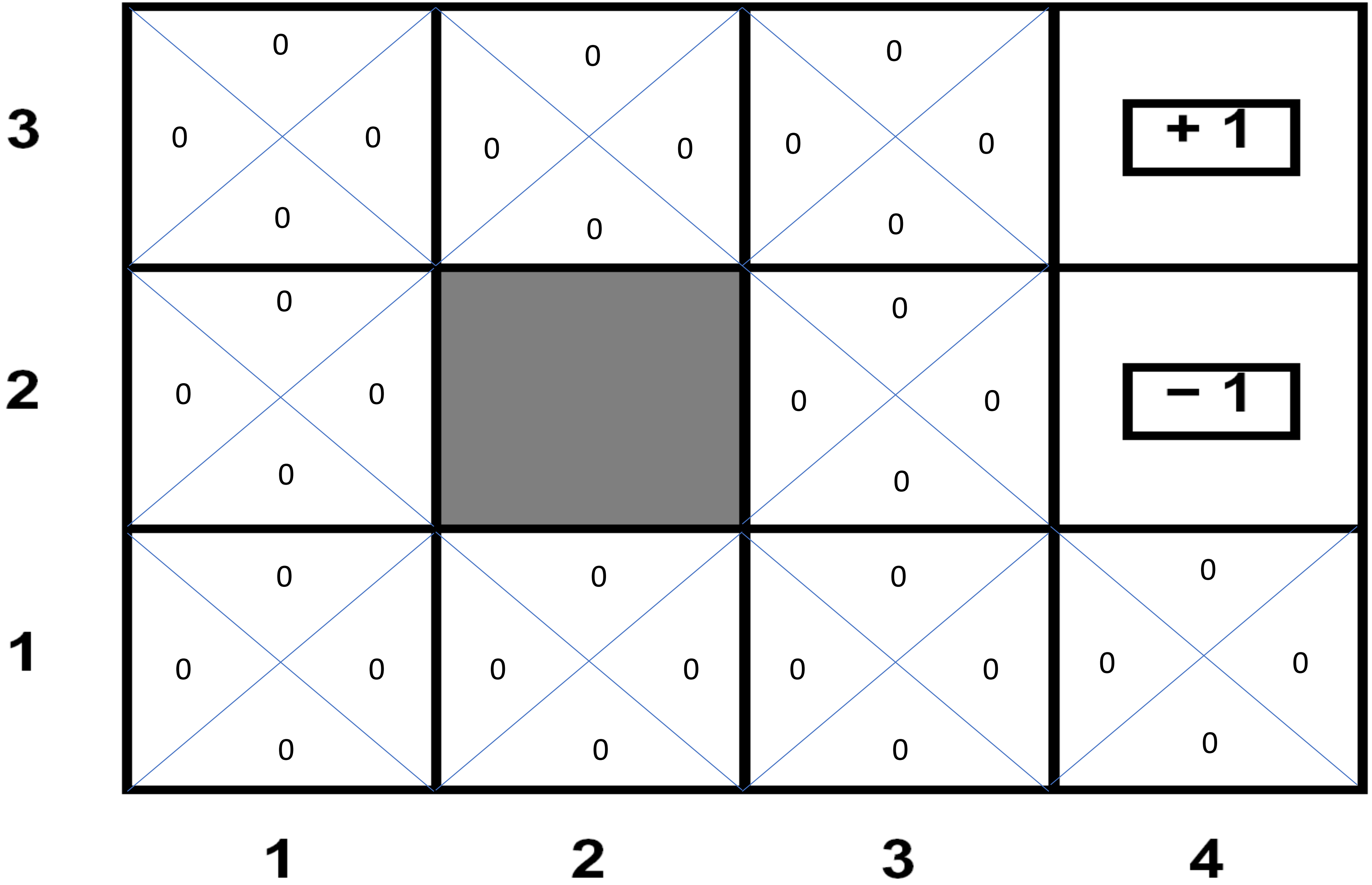
$$Q_{i+1}(s,a) \leftarrow \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s', a')\right)$$

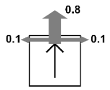


Living cost 0

0.9

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$





0 0.9

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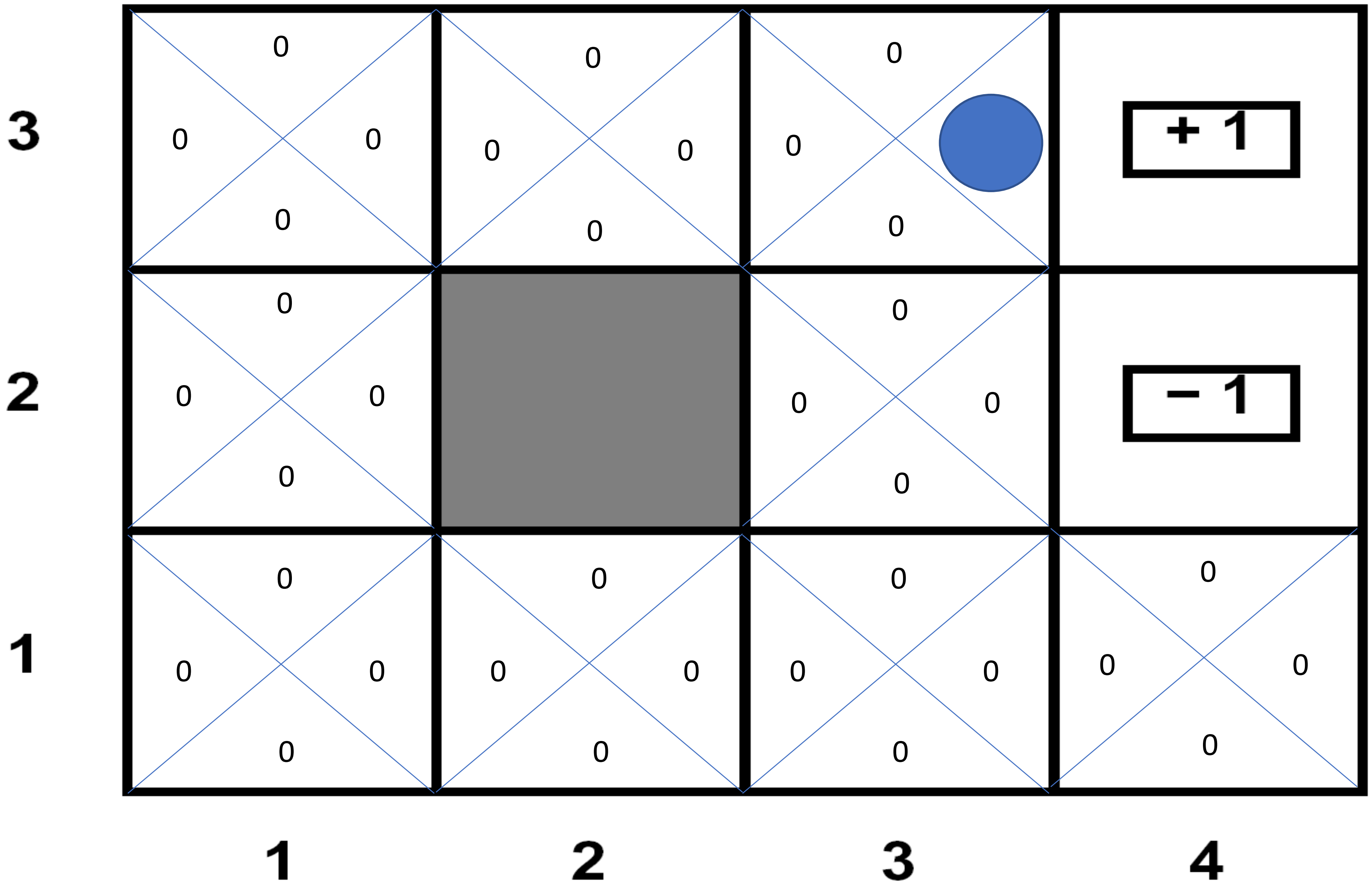
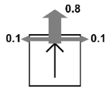


0.8x[0+0.9x1]
+ 0.1x[0 + 0]
+0.1x[0+0]
=0.72



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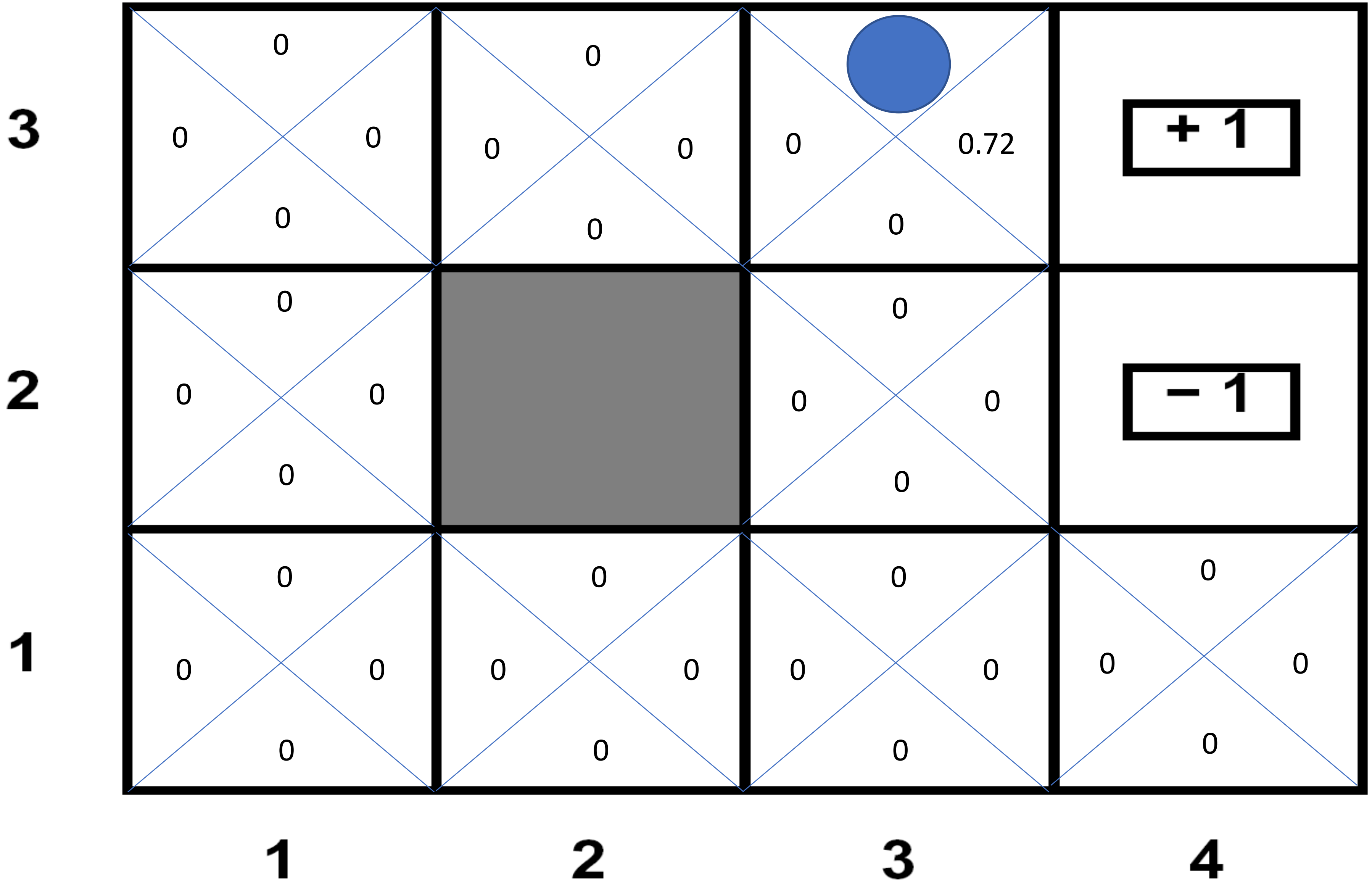
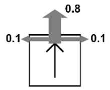
0.8x[0+0]
+ 0.1x[0+0.9x1]
+0.1x[0+0]
=0.09





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+ 0.1x[0+0.9x1]
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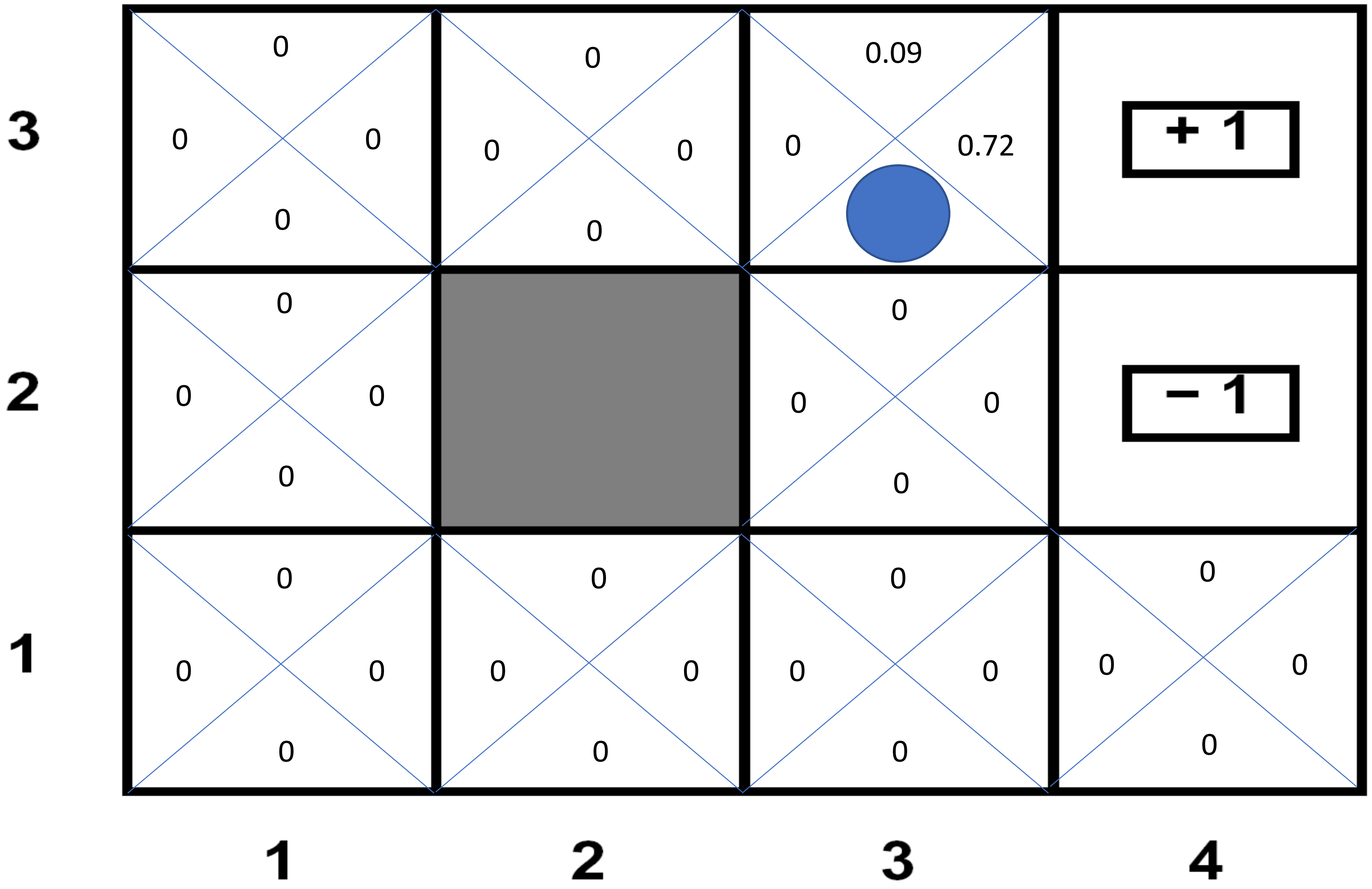
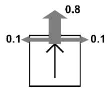
=0.09





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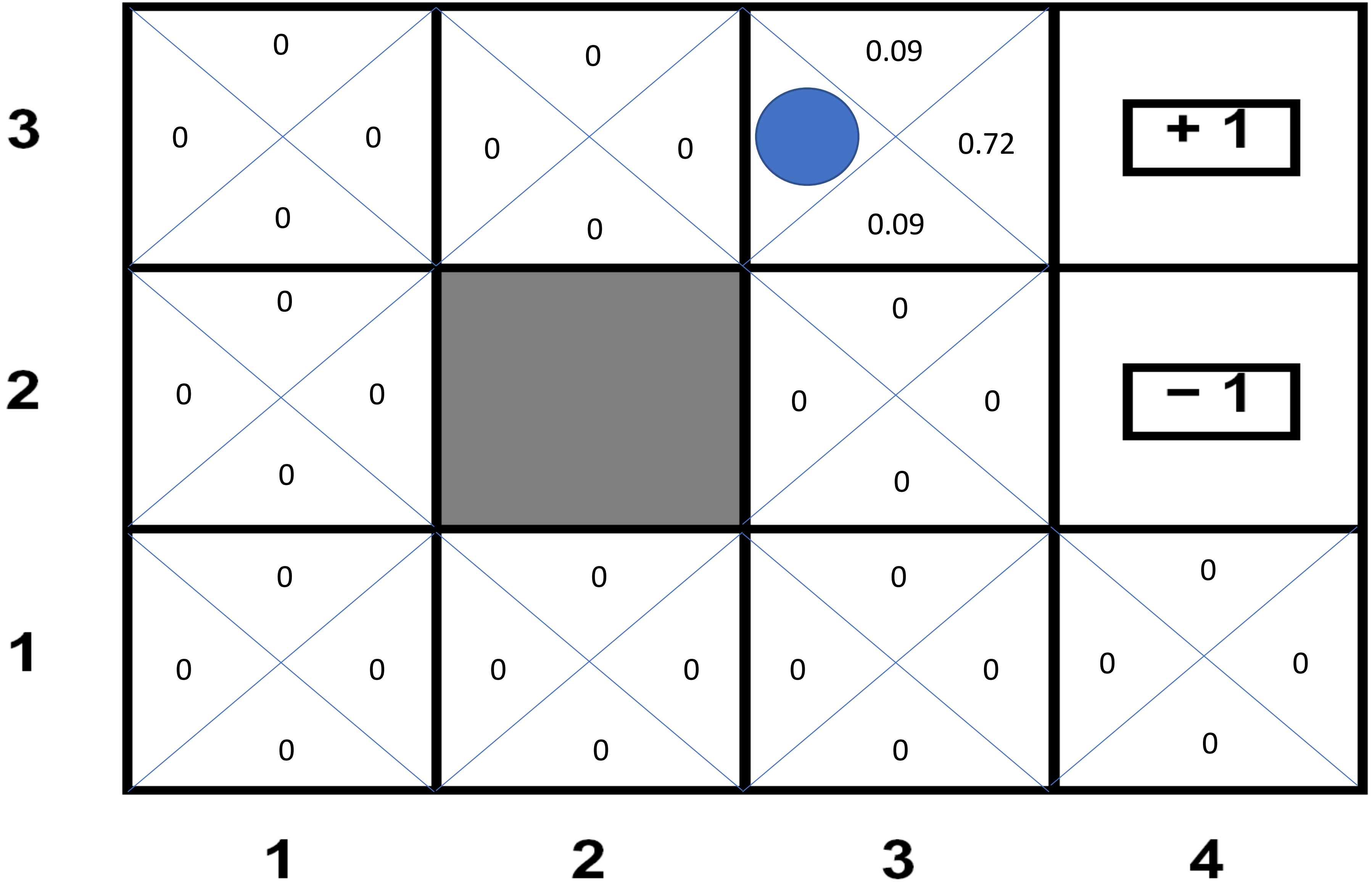
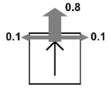


0.8x[0+0]
+ 0.1x[0+0]
+0.1x[0+0]
=0



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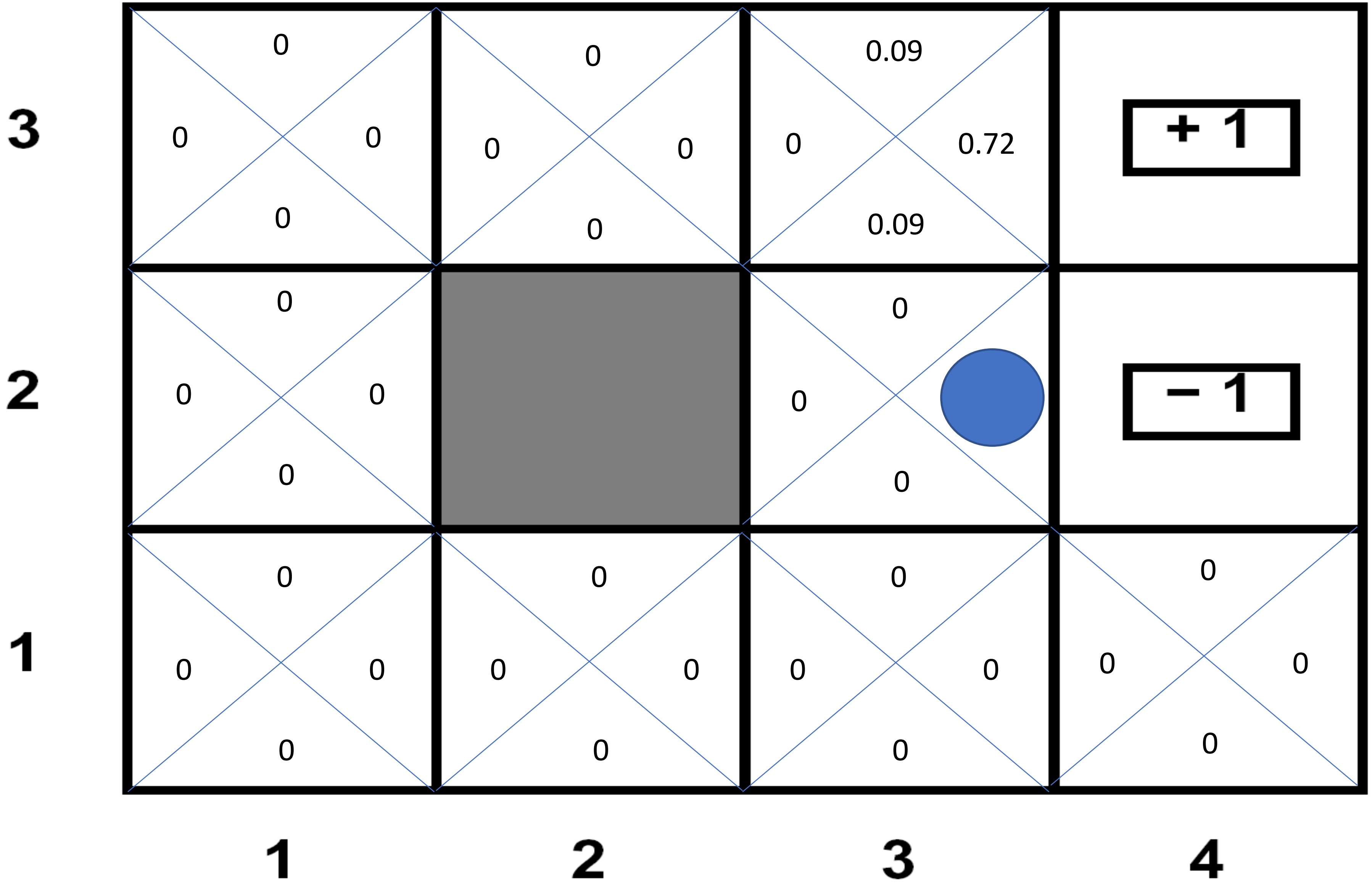
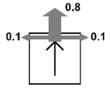


0.8x[0+0.9x-1]

+ 0.1x[0+0]

+0.1x[0+0]

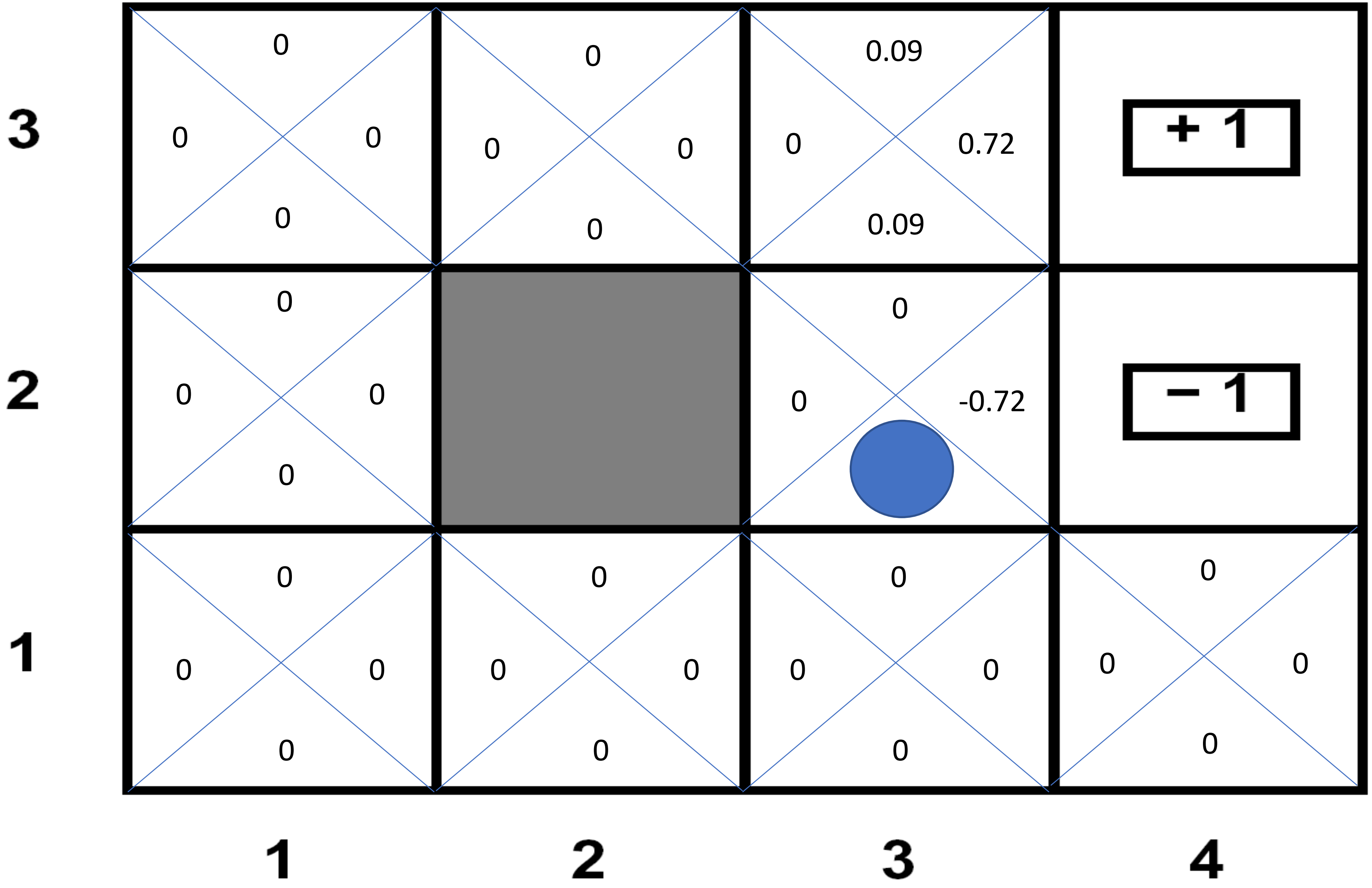
=-0.72

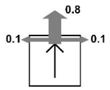


0 0.9

$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$

0.8x[0+0]
+ 0.1x[0+0]
+0.1x[0+0.9x-1]
=-0.09





$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma}\,\boxed{\max_{a'} Q_i(s',a')}\right]$$

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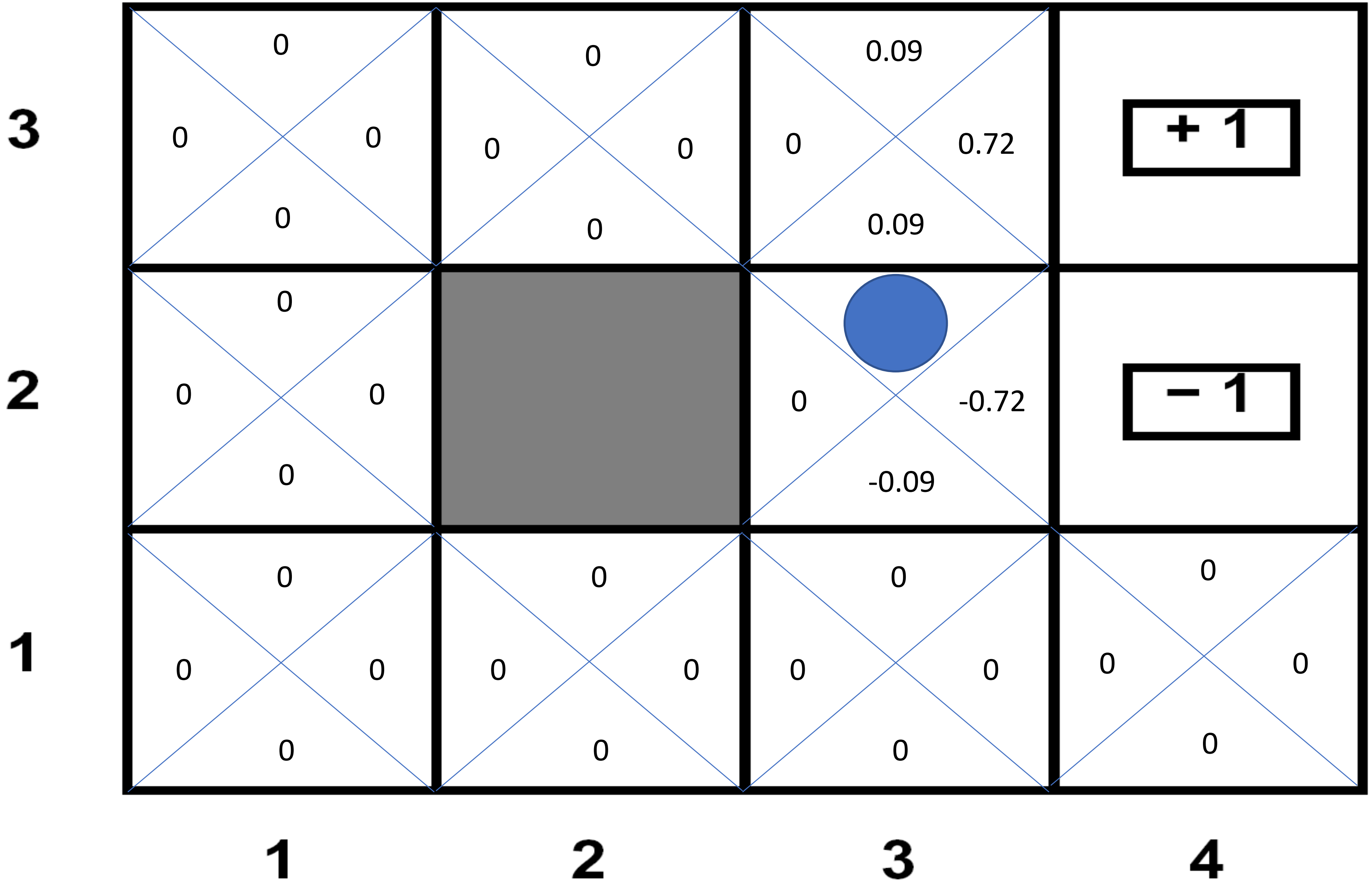

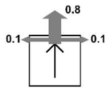

0 0.9

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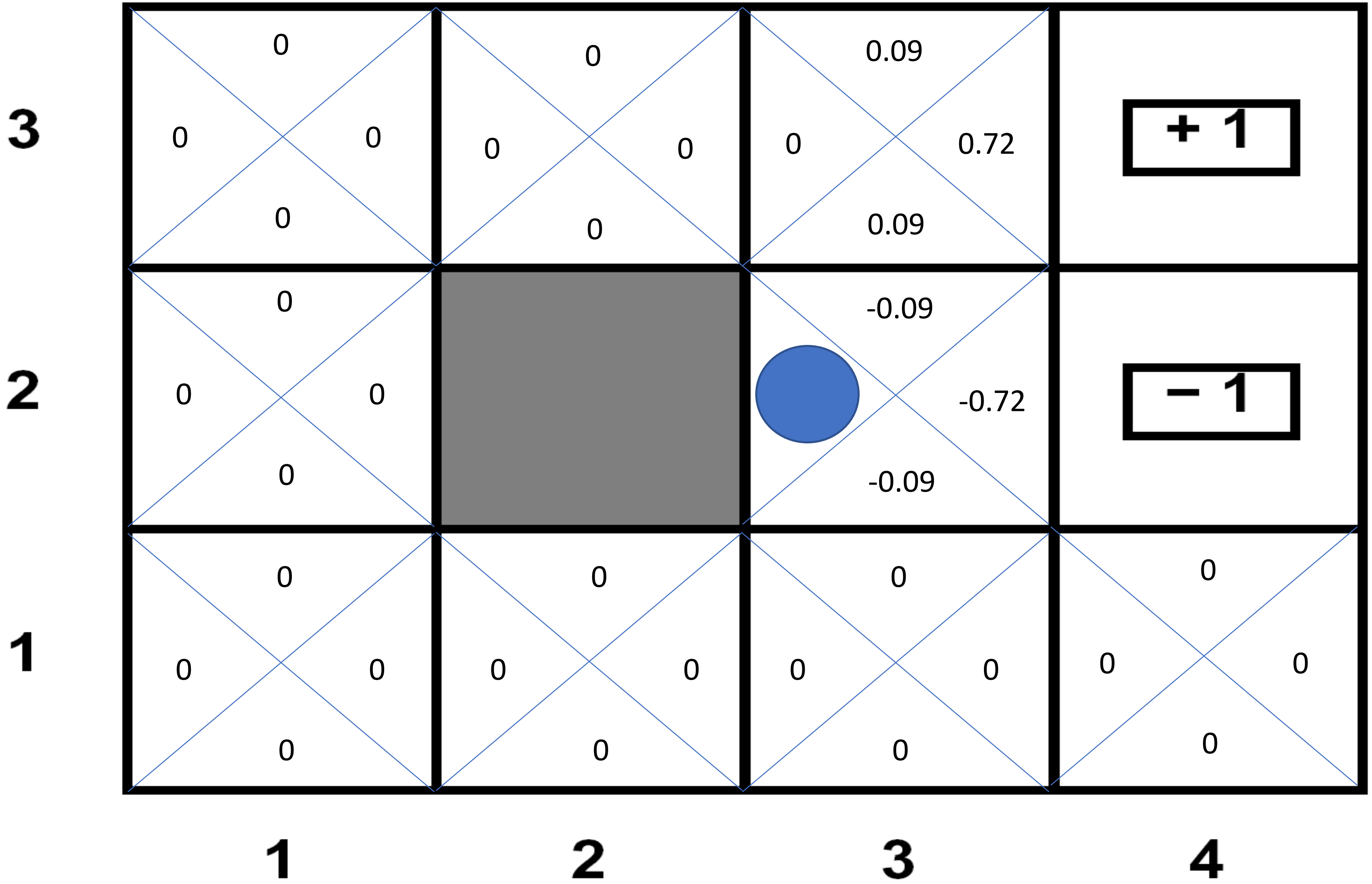
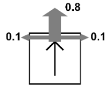
+ 0.1x[0+0]
+0.1x[0+0]
=0

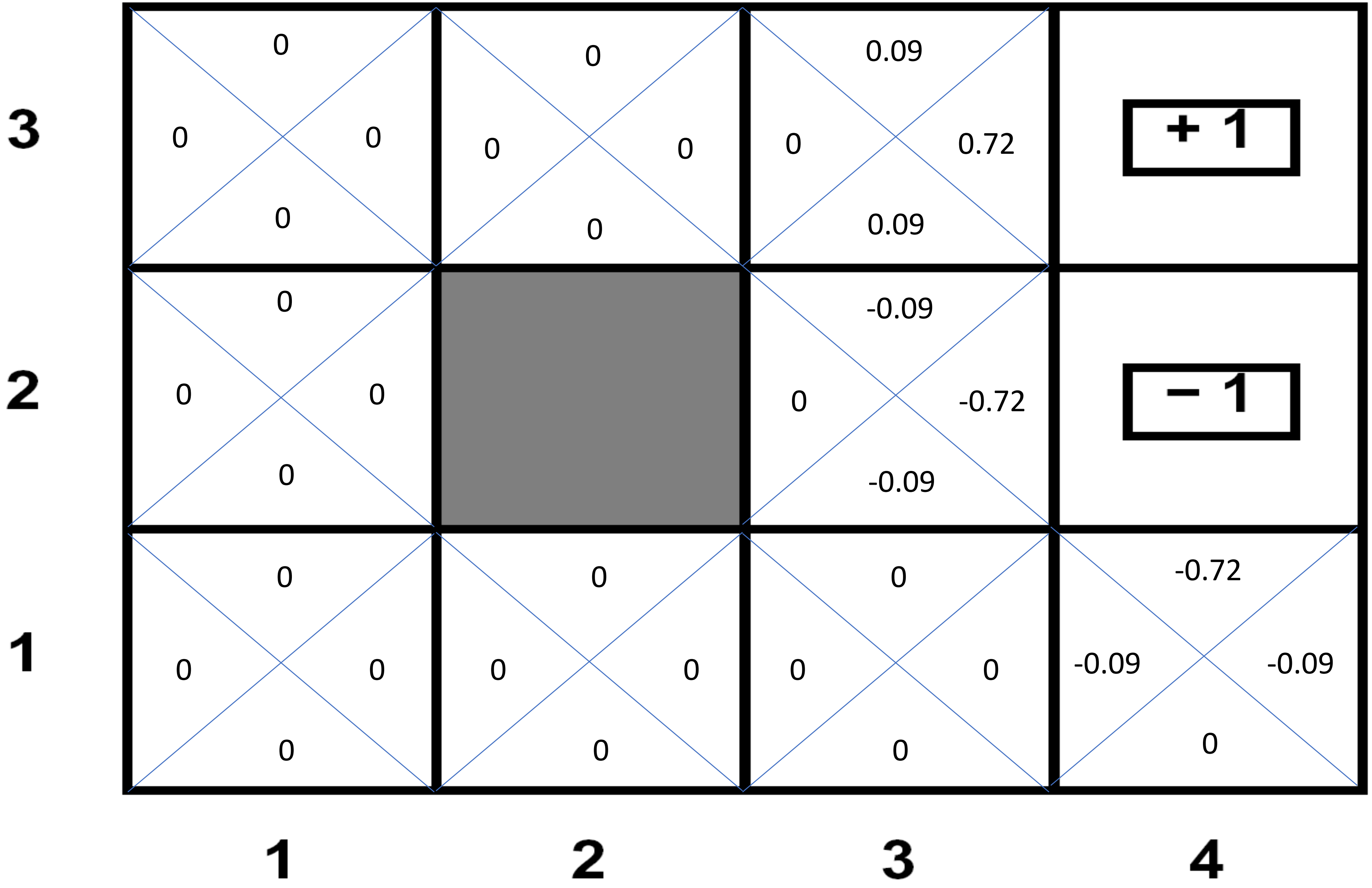


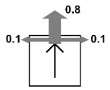


0 0.9

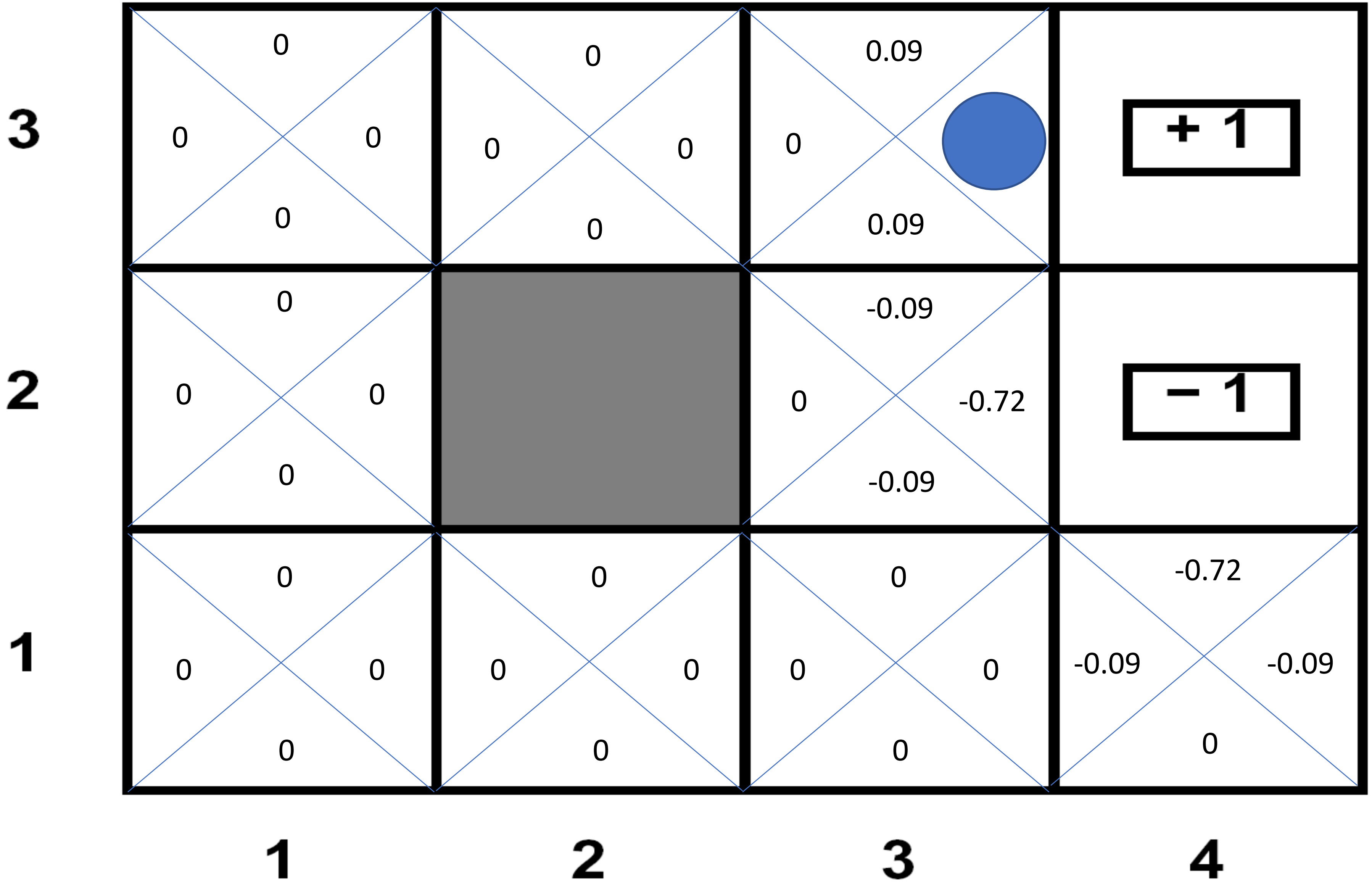
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \boxed{\max_{a'} Q_i(s',a')}\right]$$

Now we have
$Q_1(s,a)$ for all (s,a)

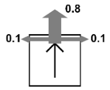




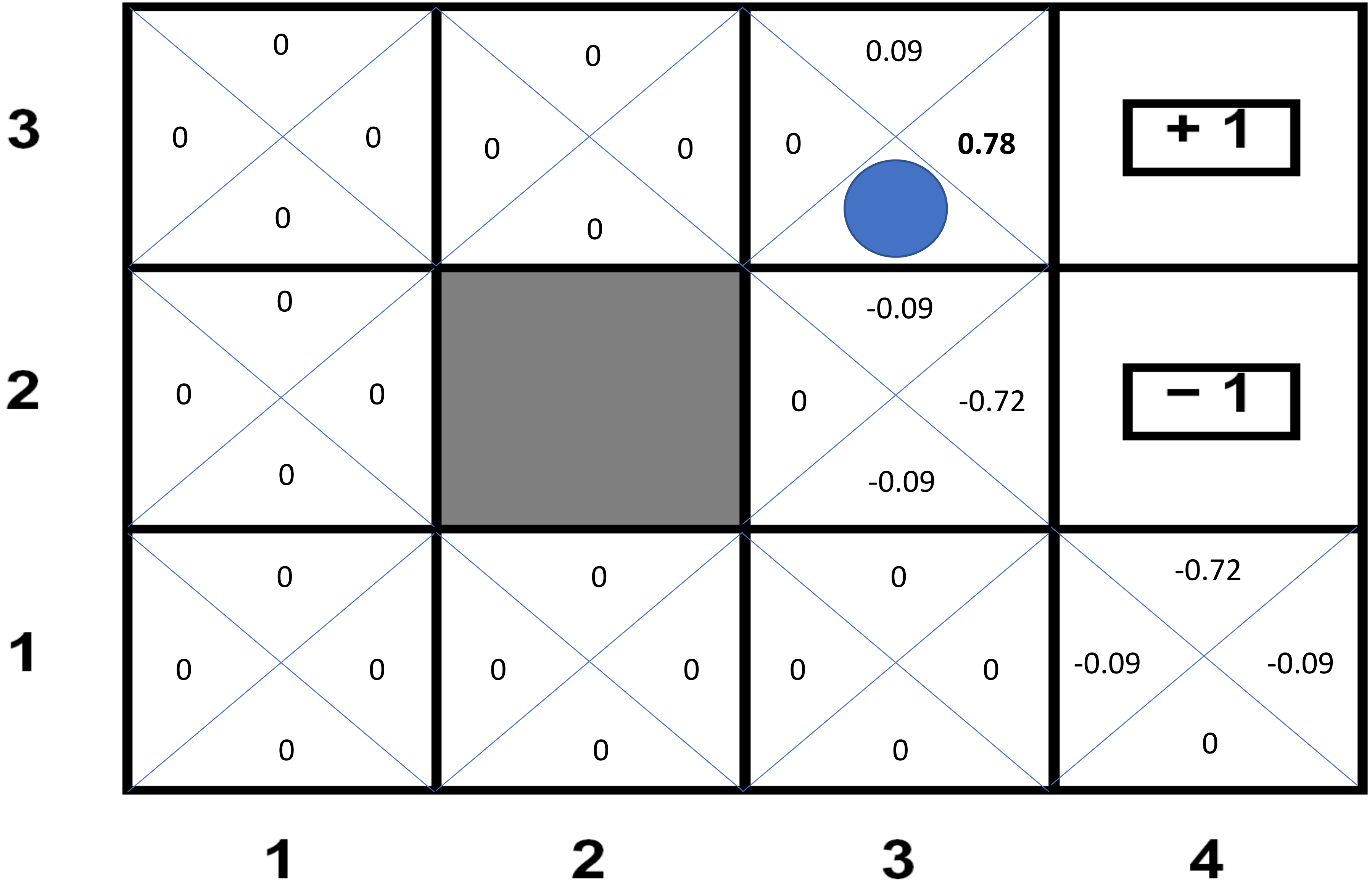
$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma}\boxed{\max_{a'} Q_i(s',a')}\right]$$



0.8x[0+0.9x1]
+ 0.1x[0+0.9x0.72]
+0.1x[0+0]
=0.7848

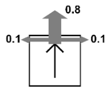


$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma}\, \boxed{\max_{a'} Q_i(s',a')}\right]$$

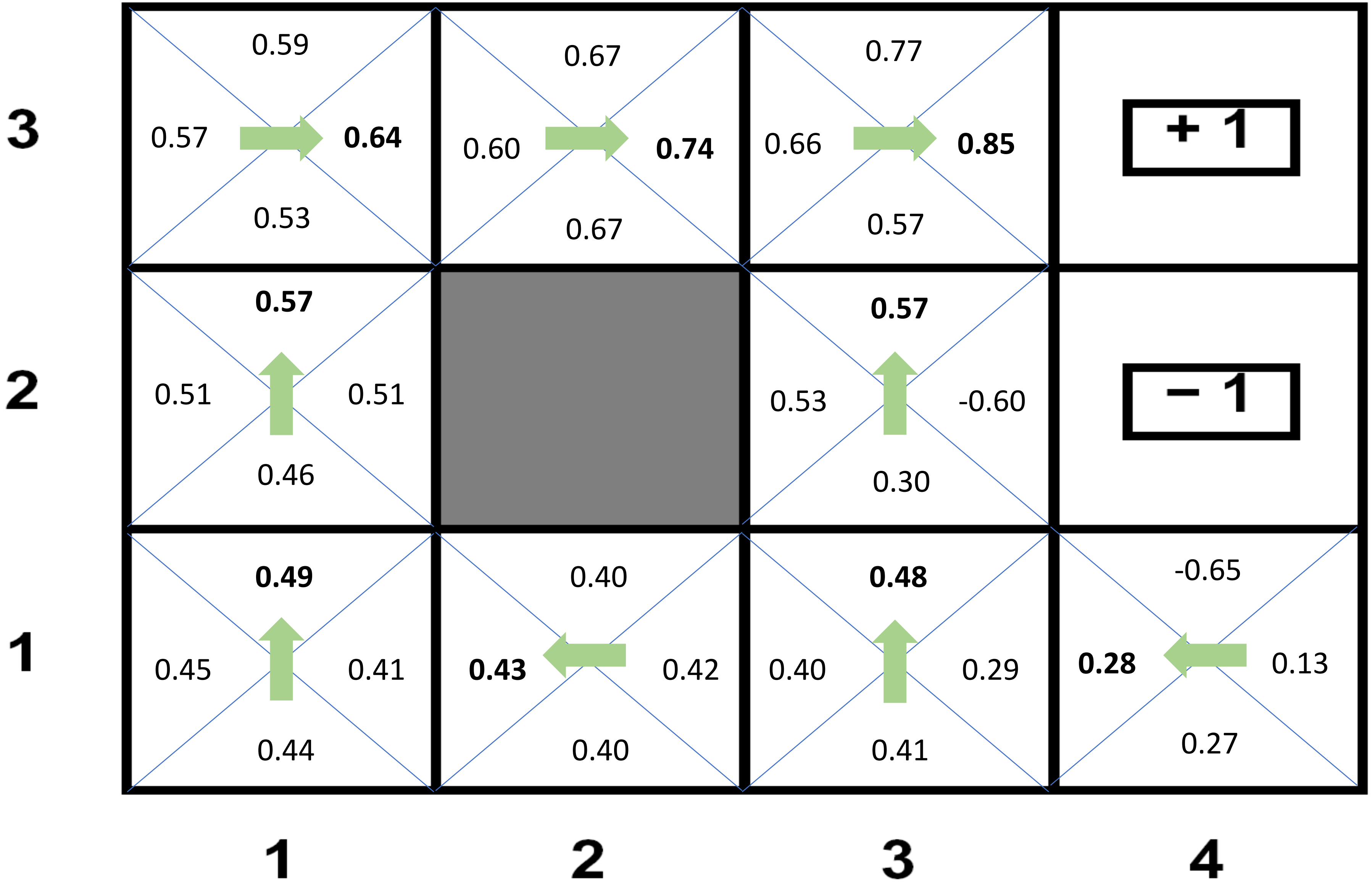


0.8x[0+0]
+ 0.1x[0+0.9x1]
+0.1x[0+0]
=0.09

After 1000 iterations:



$$Q_{i+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left[\overbrace{R(s,a,s')}^{0} + \gamma \overbrace{\boxed{\max_{a'} Q_i(s',a')}}^{0.9}\right]$$



Q Iteration

- Information propagates outward from terminal states
- Eventually all state-action pairs converge to correct Q-value estimates

Aside: Value Iteration

- Analogous to Q-Policy iteration but for computing the value function

- Initialize $V_1(s) \leftarrow 0$ for all s
- For $i \in \{1,2,\dots\}$ until convergence:

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s' \mid s, a) \cdot \left(R(s, a, s') + \gamma \cdot V_i(s')\right)$$

$$V_{i+1}(s) \leftarrow \max_{a\in A}\sum_{s'\in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP

	1	2	3	4
3				+1
2				−1
1				

V_0

	1	2	3	4
3	0	0	0	0
2	0		0	0
1	0	0	0	0

V_1

	1	2	3	4
3	0	0	0	+1
2	0		0	-1
1	0	0	0	0

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP

3				+1
2				− 1
1				
	1	**2**	**3**	**4**

V_1

3	0	0	0	+1
2	0		0	-1
1	0	0	0	0
	1	**2**	**3**	**4**

V_2

3	0	0	0.72	+1
2	0		0	-1
1	0	0	0	0
	1	**2**	**3**	**4**

$V_2(\langle 4,3 \rangle) \leftarrow 1$

$V_2(\langle 4,2 \rangle) \leftarrow -1$

$$V_{i+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} P(s'|s,a)[\overset{0}{R(s,a,s')} + \overset{0.9}{\gamma} V_i(s')]$$

Example MDP

3				+ 1
2				− 1
1				
	1	2	3	4

V_2

3	0	0	0.72	+1
2	0		0	-1
1	0	0	0	0
	1	2	3	4

V_3

3	0	0.52	0.78	+1
2	0		0.43	-1
1	0	0	0	0
	1	2	3	4

Reinforcement Learning

- Q iteration can be used to compute the optimal Q function when P and R are **known**

- How can we adapt it to the setting where these are unknown?
  - **Observation:** Every time you take action a from state s, you obtain one sample $s' \sim P(\cdot \mid s, a)$ and observe $R(s, a, s')$
  - Use single sample instead of full P

Q Learning

- Can we learn π^* without explicitly learning P and R?

$$Q_{i+1}(s, a) \leftarrow \sum_{s' \in S} P(\, s' \mid s, a \,) \cdot \left(R(s, a, s') + \gamma \cdot \max_{a' \in A} Q_i(s', a') \right)$$

Q Learning

- Can we learn π^* without explicitly learning P and R?

$$Q_{i+1}(s,a) \leftarrow \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a')\right]$$

Q Learning

- **Q Learning update:**

$$Q_{i+1}(s,a) \leftarrow R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a')$$

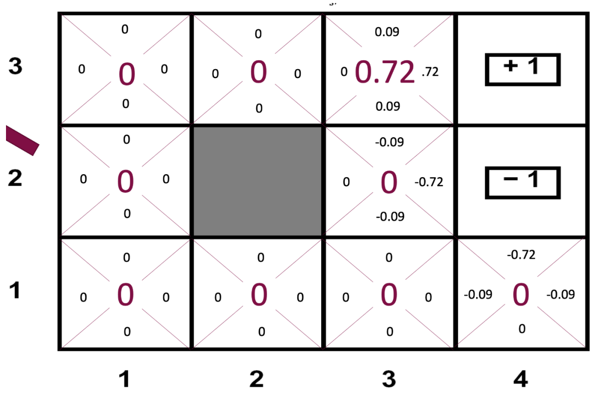
- **Q Iteration:** Update for all (s,a,s') at each step
- **Q Learning:** Update just for current (s,a), and approximate with the state s' we actually reached (i.e., a single sample $s' \sim P(\cdot \mid s,a)$)

Q Learning

- **Problem:** Forget everything we learned before (i.e., $Q_i(s,a)$)
- **Solution:** Incremental update:

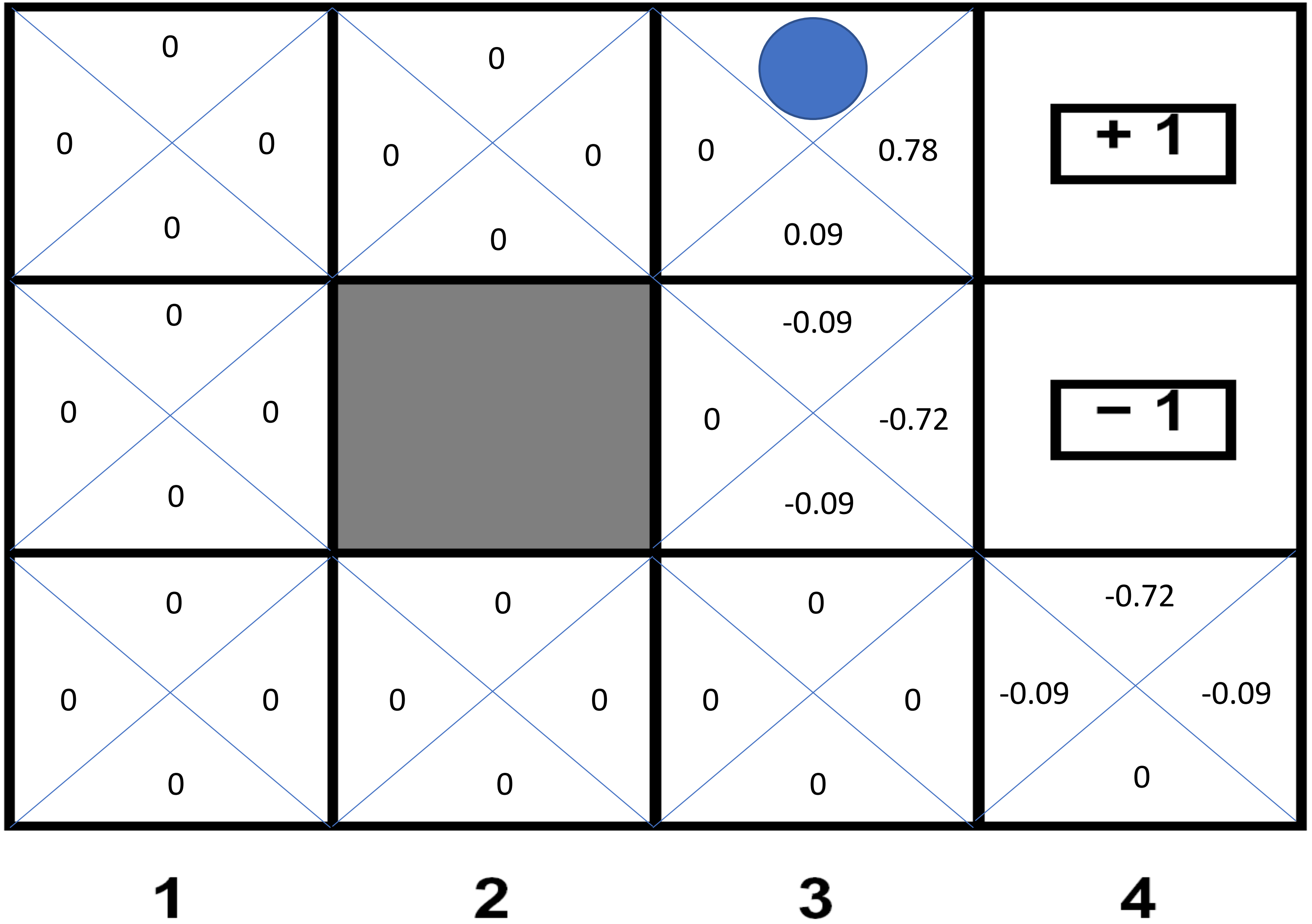
$$Q_{i+1}(s,a) \leftarrow (1-\alpha) \cdot Q_i(s,a) + \alpha \cdot \left(R(s,a,s') + \gamma \cdot \max_{a' \in A} Q_i(s',a')\right)$$

$$Q(s,a) \leftarrow Q(s,a) + \overset{0.1}{\alpha}\left(R(s,a,s') + \overset{0.9}{\gamma} \max_{a'} Q(s',a') - Q(s,a)\right)$$



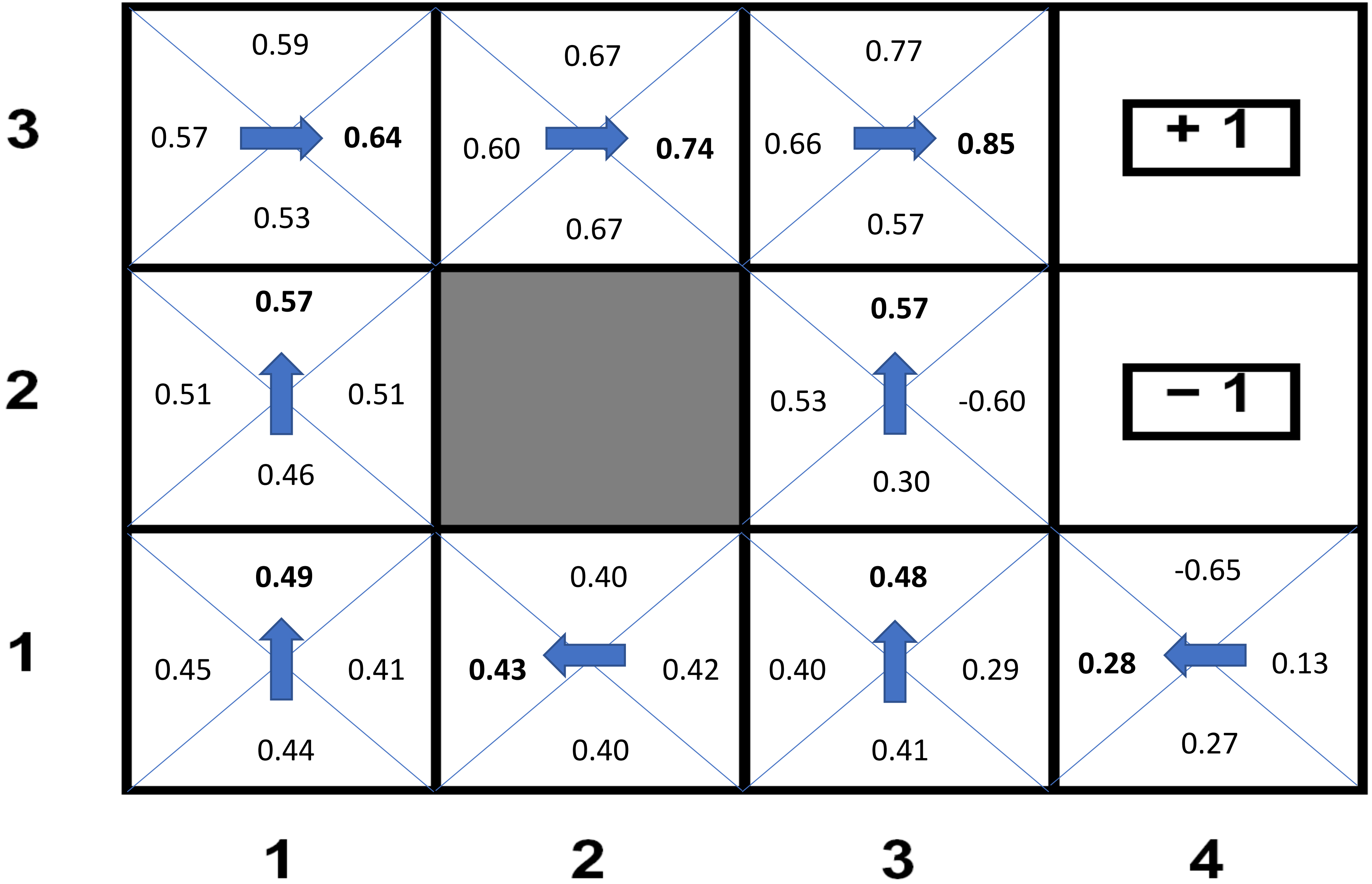
Sample $R + \gamma \max Q$ =
0+0.9x0.78 = 0.702

New Q =
0.09+0.1X(0.702-0.09)
= 0.1512



After 100,000 actions:

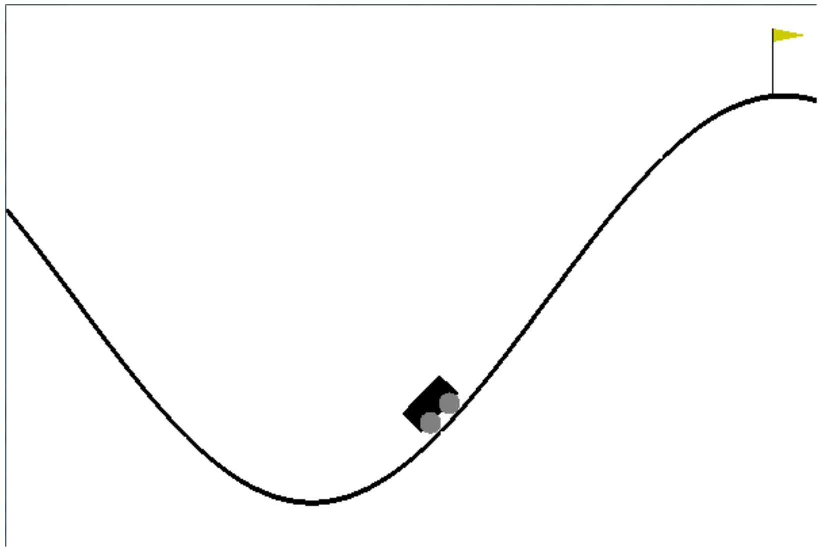
$$Q(s,a) \leftarrow Q(s,a) + \alpha\left(R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)\right)$$



Policy for Gathering Data

- **Strategy 1:** Randomly explore all (s, a) pairs
  - Not obvious how to do so!
  - E.g., if we act randomly, it may take a very long time to explore states that are difficult to reach

- **Strategy 2:** Use current best policy
  - Can get stuck in local minima
  - E.g., we may never discover a shortcut if it sticks to a known route to the goal



Policy for Gathering Data

- **ϵ-greedy:**
  - Play current best with probability $1 - \epsilon$ and randomly with probability ϵ
  - Can reduce ϵ over time
  - Works okay, but exploration is undirected
- **Visitation counts:**
  - Maintain a count $N(s, a)$ of number of times we tried action a in state s
  - Choose $a^* = \arg\max_{a \in A} \left\{ Q(s, a) + \frac{1}{N(s,a)} \right\}$, i.e., inflate less visited states

Summary

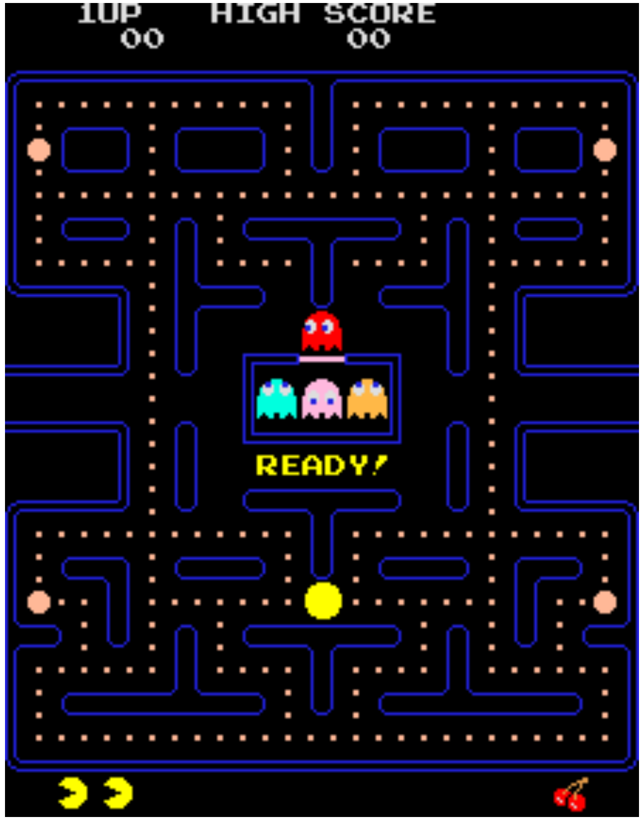
- **Q iteration:** Compute optimal Q function when the transitions and rewards are known
- **Q learning:** Compute optimal Q function when the transitions and rewards are unknown
- **Extensions**
  - Various strategies for exploring the state space during learning
  - Handling large or continuous state spaces

Curse of Dimensionality

- How large is the state space?
  - **Gridworld:** One for each of the n cells
  - **Pacman:** State is $(\text{player}, \text{ghost}_1, \dots, \text{ghost}_k)$, so there are n^k states!

- **Problem:** Learning in one state does not tell us anything about the other states!

- Many states → learn very slowly



State-Action Features

- Can we learn **across** state-action pairs?

- Yes, use features!
  - $\phi(s, a) \in \mathbb{R}^d$
  - Then, learn to predict $Q^*(s, a) \approx Q_\theta(s, a) = f_\theta\big(\phi(s, a)\big)$
  - Enables generalization to similar states