Generalized Linear Models (GLM) and Radial Basis Functions (RBF)

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Learning Objectives
Extend linear regression with link functions, basis functions
Know RBF algorithm and its uses
Generalized linear models (GLM)

- Linear Model:  \( \hat{y}(x) = \sum_{j=1}^{p} w_j x_j \)

- GLM with link fn \( f() \):  \( \hat{y}(x) = f(\sum_{j=1}^{p} w_j x_j) \)

- Basis transformation:  \( \hat{y}(x) = \sum_{j=1}^{d} w_j \phi_j(x) \)

Based on slide by Geoff Hinton
Link functions

- Link function $\hat{y} = f(w^T x)$
  - $f(x) = e^x$
  - $f(x) = \log(x)$
- Equivalent to $f^{-1}(\hat{y}(x)) = w^T x$
Linear Basis Function Models

- Generally, 
  \[ \hat{y}(x) = \sum_{j=1}^{d} w_j \phi_j(x) \]

- Typically, \( \phi_0(x) = 1 \) so that \( \theta_0 w_0 \) acts as a bias
- In the simplest case, we use linear basis functions
  \( \phi_j(x) = x_j \)
- Could use polynomials or Gaussians

Based on slide by Christopher Bishop (PRML)
Linear Basis Function Models

- **Polynomial basis functions**
  \[ \phi_j(x) = x^j \]
  — Global — mostly crappy

- **Gaussian basis functions:**
  \[ \phi_j(x) = \exp \left\{ -\frac{(x - \mu_j)^2}{2s^2} \right\} \]
  — Local — good!

Based on slide by Christopher Bishop (PRML)
Fitting a Polynomial Curve with a Linear Model

\[
y = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_p x^p = \sum_{j=0}^{p} \theta_j x^j
\]
Radial Basis Functions

Originally by Andrew Moore; now heavily edited by Lyle Ungar

http://www.it.uu.se/research/project/rbf/rbf.png
Radial Basis Functions (RBFs)

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

$x = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ \vdots & \vdots \end{bmatrix}$

$y = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$z = \text{(list of radial basis function evaluations)}$

$w = (Z^T Z)^{-1} (Z^T y)$

$\hat{y} = w_0 + w_1 x_1 + \ldots$
1-D RBFs

\[ \hat{y} = w_1 \phi_1(x) + w_2 \phi_2(x) + w_3 \phi_3(x) \]

where

\[ \phi_i(x) = k(\|x - \mu_j\| / C) \]

For RBF:

\[ k(\|x - \mu_j\| / C) = \exp\{-\|x - \mu_j\|_2^2 / C\} \]

C = “Kernel Width”

k = kernel function
Example

\[ \hat{y} = 2\phi_1(x) + 0.05\phi_2(x) + 0.5\phi_3(x) \]

where

\[ \phi_j(x) = k(\|x - \mu_j\| / C) \]
Radial Basis Functions in 2-d

Two inputs.
Outputs (heights sticking out of page) not shown.
Too small! Even before seeing the data, you should understand that this is a disaster!
Too big!!!

Center

Sphere of significant influence of center

x_1

x_2
So what do we do? 

Search to find the optimal size “width” for the Gaussians (on a test set, of course!)
RBFs can do ...

- Use $d < p$ basis vectors
  - Dimensionality reduction
  - Good for high dimensional feature spaces

- Use $d > p$ basis vectors
  - Increases the dimensionality
  - Can make a formerly nonlinear problem linear

- Use $d = n$ basis vectors
  - We can use this to switch to a *dual* representation
How to find the kernel centers?

- Pick random points
  - Generally a bad idea
- **Standard RBF: do k-means clustering and use the centers of the clusters**
  - Works great!
- Use all n of the training data points as kernel centers
  - Requires regularization
- **Estimate them: nonlinear regression**
  - A good initialization helps
What you should know

- Link functions give a nonlinear regression
- Basis functions allow one to fit a nonlinear function using linear regression
- RBF
  - Cluster points
  - Put a Gaussian basic function at each cluster center
    - Pick the Gaussian width
  - Fit a linear regression
How is my speed?

Slow

Good

Fast