CS446: Machine Learning	**Fall 2016**

October 25^{th}, 2016

- This is a closed book exam. Everything you need in order to solve the problems is supplied in the body of this exam.
- This exam booklet contains **four** problems. You need to solve all problems to get 100%.
- Please check that the exam booklet contains **14** pages, with the appendix at the end.
- The exam ends at 1:45 PM. You have 75 minutes to earn a total of 100 points.
- Answer each question in the space provided. If you need more room, write on the reverse side of the paper and indicate that you have done so.
- A list of potentially useful functions has been provided in the appendix at the end.
- **Besides having the correct answer, being concise and clear is very important. For full credit, you must show your work and explain your answers.**

Good Luck!

Name (NetID): (1 Point)

Decision Trees		/20
PAC Learning		/29
Neural Networks		/25
Short Questions		/25
Total		/100

Decision Trees [20 points]

You work in a weather forecasting company and your job as a machine learning expert is to design a decision tree which would predict whether it is going to rain today ('WillRain?' $= 1$) or not ('WillRain?' $= 0$). You are given a dataset D with the following attributes: *IsHumid* $\in \{0, 1\}$, *IsCloudy* $\in \{0, 1\}$, *RainedYesterday* $\in \{0, 1\}$ and *Temp>20* $\in \{0, 1\}$.

IsHumid	IsCloudy	RainedYesterday	Temp>20	WillRain?
1	1	1	0	1
0	1	0	0	0
1	0	0	0	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
0	1	0	0	0
1	0	1	1	0

To simplify your computations please use: $\log_2(3) \approx \frac{3}{2}$.

(a) **(4 points)** What is the entropy of the label 'WillRain?'?

(b) **(4 points)** What should the proportion of the examples labeled 'WillRain?'=1 be, in order to get the maximum entropy value for the label?

(c) **(4 points)** Compute the Gain(D, IsCloudy).

(d) **(4 points)** You are given that:

- Gain(D, IsHumid) = 0.25,
- Gain(D, RainedYesterday) = 0.11,
- Gain(D, Temp>20) = 0
- Gain(D, IsCloudy) is as computed in part c.

i. Which node should be the root node?

ii. Without any additional computation, draw a decision tree that is consistent with the given dataset and uses the root chosen in (i).

if(IsHumid):
else:
else:

(e) **(4 points)** Express the function 'WillRain?' as a simple Boolean function over the features defining the data set D. That is, define a Boolean function that returns true if an only if 'WillRain?'=1.

PAC Learning [29 points]

We define a set of functions

$$T = \{f(x) = \mathbb{1}[x > a] : a \in \mathbb{R}\},$$

where $\mathbb{1}[x > a]$ is the indicator function returning 1 if $x > a$ and returning 0 otherwise. For input domain $\mathcal{X} = \mathbb{R}$, and a *fixed* positive number k, consider a concept class DT_k consisting of all decision trees of depth at most k where the function at each non-leaf node is an element of T. Note that if the tree has only one decision node (the root) and two leaves, then $k = 1$.

(a) **(4 points)** We want to learn a function in DT_k. Define

i. The Instance Space X

ii. The Label Space Y

iii. Give an example of $f \in DT_2$.

iv. Give 3 examples that are consistent with your function f and one that is not consistent with it.

(b) **(7 points)** Determine the VC dimension of DT_k, and prove that your answer is correct.

(c) **(5 points)** Now consider a concept class DT_∞ consisting of all decision trees of unbounded depth where the function at each node is an element of T. Give the VC dimension of DT_∞, and prove that your answer is correct.

(d) **(7 points)** Assume that you are given a set S of m examples that are consistent with a concept in DT_k. Give an efficient learning algorithm that produces a hypothesis h that is consistent with S.

Note: The hypothesis you learn, h, **does not need to be** in DT_k. You can represent it any way you want.

(e) **(6 points)** Is the concept class DT_k PAC learnable? Explain your answer.

Neural Networks [25 points]

Consider the following set S of examples over the feature space $X = \{X_1, X_2\}$. These examples were labeled based on the XNOR (NOT XOR) function.

X_1	X_2	y^* (Label)
0	0	1
0	1	0
1	0	0
1	1	1

(a) **(4 points)** The set of 4 examples given above is not linearly separable in the $X = \{X_1, X_2\}$ space. Explain this statement in **one sentence**.

(b) **(6 points)** Propose a new set of features $Z = \{Z_1, \ldots Z_k\}$ such that in the Z space, this set of examples is linearly separable.

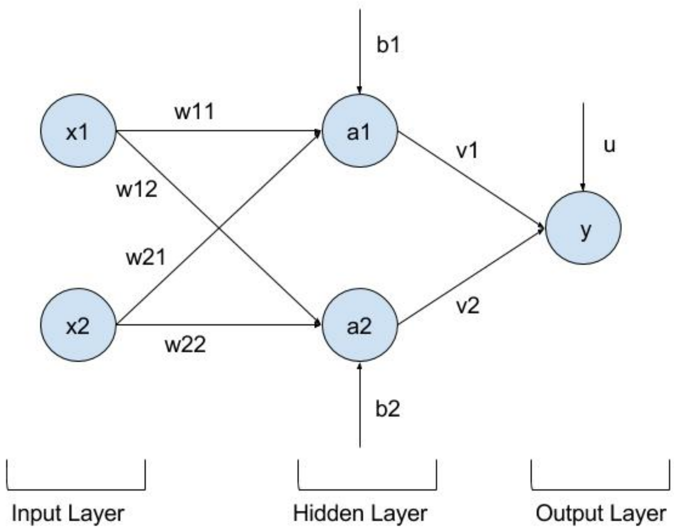
i. Define each Z_i as a function of the X_is.

ii. Write down the set of 4 examples given above in the new Z space.

iii. Show that the data set is linearly separable. (**Show**, don't just say that it is separable.)

(c) **(5 points)** Now consider running the set S of examples presented above in the space X through a neural network with a single hidden layer, as shown in the figure below.

Note that numbers on the edges correspond to weights and the arrows into the units indicate the bias term. Recall that the output of a node (denoted by the terms inside the nodes in the graph e.g. a_1, a_2, y) in the neural network is given by $f(w^T x + b)$, where x is the input to the unit, w are the weights on the input, b is the bias in the unit, and f is the activation function.



For the sake of simplicity, assume that the function $sgn(x)$ ($sgn(x) = 1$ if $x \geq 0$, 0 otherwise) is used as the activation function at all the nodes of the network.

Which of the following sets of weights guarantees that the neural network above is *consistent* with all the examples in S? (That is, the 0-1 loss is 0).

The correct set of weights is ______________________________
{option (1) | option (2) | option (3)}

Options:

Options	w_{11}	w_{21}	b_1	w_{12}	w_{22}	b_2	v_1	v_2	u
1	1	0	- 0.5	0	1	- 0.5	- 1	- 1	0.9
2	1	1	0.5	- 1	- 1	2.5	0	- 1	0.5
3	1	1	- 0.5	- 1	- 1	1.5	- 1	- 1	1.5

(d) **(10 points)** We now want to use the data set S to *learn* the neural network depicted earlier.

We will use the **sigmoid** function, $sigmoid(x) = (1 + exp^{-x})^{-1}$, as the activation function in the hidden layer, and **no activation function in the output layer** (i.e. it's just a linear unit). As the loss function we will use the Hinge Loss:

$$\text{Hinge loss(w, x, b, y*)} = \begin{cases} 1 - y^*(w^T x + b), & \text{if } y^*(w^T x + b) > 1 \\ 0, & \text{otherwise} \end{cases}$$

Write down the BackPropagation update rules for the weights in the output layer ($\boldsymbol{\Delta v_i}$), and the hidden layer ($\boldsymbol{\Delta w_{ij}}$).

Short Questions [25 points]

(a) **(10 points)** In this part of the problem we consider Adaboost. Let D_t be the probability distribution in the tth round of Adaboost, h_t be the weak learning hypothesis learned in the tth round, and ϵ_t its error.

i. Denote by $D_t(i)$ the weight of the ith example under the distribution D_t. Use it to write an expression for the error ϵ_t of the AdaBoost weak learner in the tth round.

ii. Consider the following four statements **with respect to the hypothesis at time t,** h_t. Circle the one that is true, and provide a short explanation.

A. $\forall t, Error_{D_t}(\boldsymbol{h_t}) = Error_{D_{t+1}}(\boldsymbol{h_t})$

B. $\forall t, Error_{D_t}(\boldsymbol{h_t}) > Error_{D_{t+1}}(\boldsymbol{h_t})$

C. $\forall t, Error_{D_t}(\boldsymbol{h_t}) < Error_{D_{t+1}}(\boldsymbol{h_t})$

D. The relation between $Error_{D_t}(\boldsymbol{h_t})$ and $Error_{D_{t+1}}(\boldsymbol{h_t})$ cannot be determined in general.

Explanation:

(b) **(10 points)** We consider Boolean functions in the class $L_{10,20,100}$. This is the class of 10 *out of* 20 *out of* 100*, defined over* $\{x_1, x_2, \ldots x_{100}\}$.

Recall that a function in the class $L_{10,20,100}$ is defined by a set of 20 *relevant variables*. An example $x \in \{0,1\}^{100}$ is positive if and only if *at least* 10 out these 20 are **on**.

In the following discussion, for the sake of simplicity, whenever we consider a member in $L_{10,20,100}$, we will consider the function f in which the *first* 20 coordinates are the relevant coordinates.

i. Show that the perceptron algorithm can be used to learn functions in the class $L_{10,20,100}$. In order to do so,

A. Show a linear threshold function h that behaves just like $f \in L_{10,20,100}$ on $\{0,1\}^{100}$.

B. Write h as a weight vector that goes through the origin and has size (as measured by the L_2 norm) equal to 1.

ii. Let R be the set of 20 variables defining the target function. We consider the following two data sets, both of which have examples with 50 **on** bits.

  $\mathbf{D_1}$: In all the negative examples exactly 9 of the variables in R are **on**; in all the positive examples exactly 11 of the variables in R are **on**.

  $\mathbf{D_2}$: In all the negative examples exactly 5 of the variables in R are **on**; in all the positive examples exactly 15 of the variables in R are **on**.

Consider running perceptron on D_1 and on D_2. On which of these data sets do you expect Perceptron to make less mistakes?

Perceptron will make less mistakes on the data set ______________________
$\{D_1 \mid D_2\}$

iii. Define the *margin* of a data set D with respect to weight vector w. Explain your answer to (ii) using the notion of the *margin*.

(c) **(5 points)** Let f be a concept that is defined on examples drawn from a distribution D. The "true" error of the hypothesis h is defined as

$$Error_D(h) = Pr_{x \in D}\left(h(x) \neq f(x)\right).$$

In the class, we saw that the true error of a classifier is bounded above by two terms that relate to the training data and the hypothesis space. That is

$$Error_D(h) < \quad A \quad + \quad B$$

What are A and B? (If you do not remember the exact functional forms of these terms, it is sufficient to **briefly** describe what they mean.)

Appendix

(a) $Entropy(S) = -p_+log_2(p_+) - p_-log_2(p_-)$

(b) $Gain(S,a) = Entropy(S) - \sum_{v \in values(a)} \frac{|S_v|}{|S|} Entropy(S_v)$

(c) $\text{sgn(x)} = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$

(d) $\text{sigmoid(x)} = \frac{1}{1 + exp^{-x}}$

(e) $\frac{\partial}{\partial x} sigmoid(x) = sigmoid(x)\Big(1 - sigmoid(x)\Big)$

(f) $\text{ReLU(x)} = max(0, x)$

(g) $\frac{\partial}{\partial x} ReLU(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$

(h) $\text{tanh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(i) $\frac{\partial}{\partial x} tanh(x) = 1 - tanh^2(x)$

(j) $\text{Zero-One loss}(y,\, y^*) = \begin{cases} 1, & \text{if } y \neq y^* \\ 0, & \text{if } y = y^* \end{cases}$

(k) $\text{Hinge loss(w, x, b, y*)} = \begin{cases} 1 - y^*(w^Tx + b), & \text{if } y^*(w^Tx + b) > 1 \\ 0, & \text{otherwise} \end{cases}$

(l) $\frac{\partial}{\partial w} \text{ Hinge loss(w, x, b, y*)} = \begin{cases} -y^*(x), & \text{if } y^*(w^Tx + b) > 1 \\ 0, & \text{otherwise} \end{cases}$

(m) $\text{Squared loss}(w,\, x,\, y^*) = \frac{1}{2}(w^Tx - y^*)^2$

(n) $\frac{\partial}{\partial w} \text{ Squared loss}(w, x, y^*) = x(w^Tx - y^*)$