Lecture #2 – Nyquist-Shannon Sampling Theorem

ESE 150 – Digital Audio Basics

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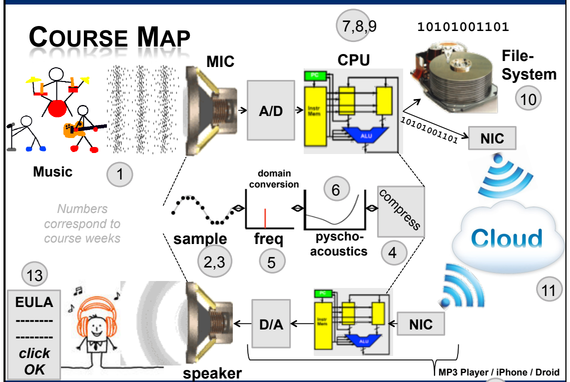
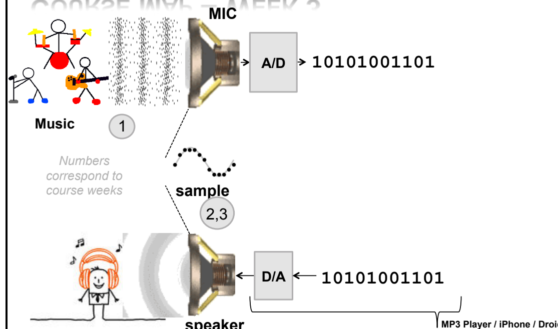
Lecture Topics

- Where are we on course map?
- What we did in lab last week
  - How it relates to this week
- Sampling/Quantization Review
- Nyquist Shannon Sampling Rate
- Next Lab
- References

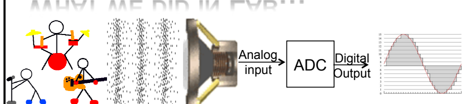
Course Map



Course Map – Week 3



What we did in Lab…



- Week 1: Converted Sound to analog voltage signal
  - a "pressure wave" that changes air molecules w/ respect to time
  - a "voltage wave" that changes amplitude w/ respect to time
  - Sample: Break up independent variable, take discrete 'samples'
  - Quantize: Break up dependent variable into n-levels (need 2^n bits to digitize)

Sampling vs Quantization Review

ADC – Sampling & Quantization

- **Analog-to-Digital (ADC) Conversion**
  - Converting analog (continuous) signal to digital signal
  - Digitization process has two important aspects:
    - *1) Sampling*
      - Converting ***independent*** variable of signal from continuous to discrete
      - e.g.: breaking continuous *time* down into intervals
    - *2) Quantization*
      - Converting ***dependent*** variable of signal from continuous to discrete
      - e.g.: breaking continuous *voltage* down into levels

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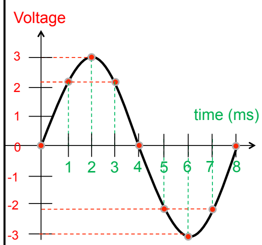
ADC – Sampling

- ***Sampling***: **breaking independent variable (time) into intervals**
- ***Quantization***: **breaking dependent variable (voltage) into levels**



Samples @ 1ms intervals:	Quantized into 7 levels	Levels digitized into 3-bits
{ 0 ms, 0 Volts }	{ 0 ms, 0 Volts } →	011
{ 1 ms, 2.2 Volts }	{ 1 ms, 2 Volts } →	101
{ 2 ms, 3 Volts }	{ 2 ms, 3 Volts } →	110
{ 3 ms, 2.2 Volts }	{ 3 ms, 2 Volts } →	101
{ 4 ms, 0 Volts }	{ 4 ms, 0 Volts } →	011
{ 5 ms, -2.2 Volts }	{ 5 ms, -2 Volts } →	001
{ 6 ms, -3 Volts }	{ 6 ms, -3 Volts } →	000
{ 7 ms, -2.2 Volts }	{ 7 ms, -2 Volts } →	001
{ 8 ms, 0 Volts }	{ 8 ms, 0 Volts } →	011

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Two Knobs

- **Quantization level (bits/sample)**
- **Sampling rate (samples/second)**

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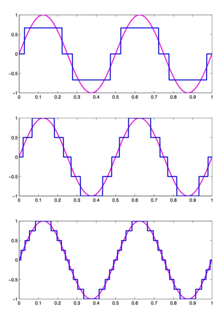
Effect of Increasing Quantization



- **Dividing dependent variable up into more levels**
  - Increasing resolution at each sample
  - Doesn't change the # of samples itself!

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Effect of Increasing Sampling Rate

- **Increasing how often we take samples also helps**
  - Much like quantization…
    - 1 bit was too few, 16 bits was more than enough
    - Is there a sweet spot for the sampling rate?

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Both (Quantization, Sampling) impact storage

- **How many bytes for 3 minute song sampled at 8b precision and 1000 samples/s?**
- **at 2000 samples/s?**
- **16b precision at 2000 samples/s?**

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Key Question

- **What sampling rate should we use?**

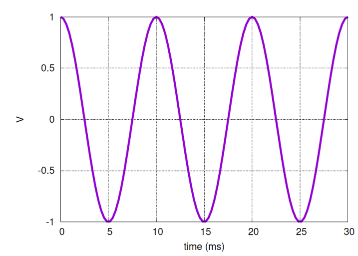
Definition of Good Sampling

- **Definition of proper sampling:**
  - Let's say you've sampled an analog signal…
  - If you can exactly reconstruct the analog signal from the samples
    - You have done the sampling properly!
  - Essentially: if you can reverse the process…
    - You've capture enough information about the signal
- **Can we formalize this a bit more?**
  - Yes, next few slides will try….

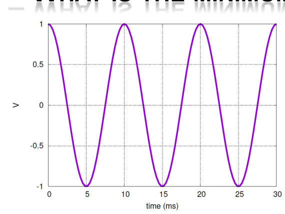
Preclass

- **Identify frequencies**
- **Samples**
- **What's indistinguishable at various sample rates?**

Sampling – What is the minimum?

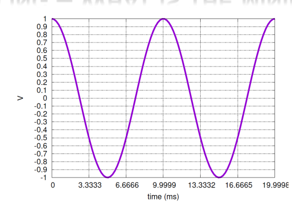


Sampling – What is the minimum?



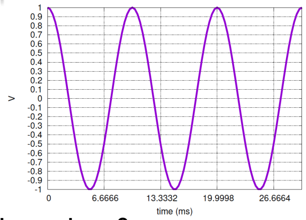
- **How much do we need to capture to reconstruct it?**
  - If we sample at 200 Hz, capture peaks & troughs of signal
  - Sample rate: 2 x frequency = 200 Hz

Sampling – What is the minimum?



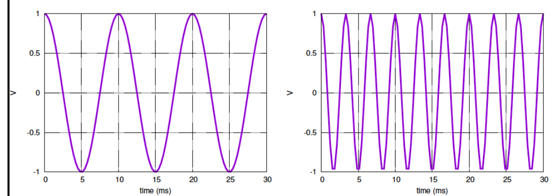
- **How much do we need to capture to reconstruct it?**
  - If we sample at 3 x frequency or 300 Hz
    - more than enough samples to capture it!
  - We are actually wasting space!
  - …more samples…more bits per sample…more storage required

Sampling – What is the minimum?



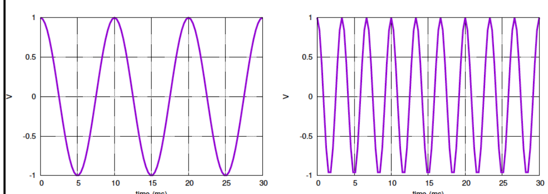
- Could we go lower?
  - If we sample at rate 1.5 x frequency or 150 Hz
    - We aren't capturing all peaks/troughs of signal
  - Yes, we lose information
    - but it gets worse!

200 Hz Sample



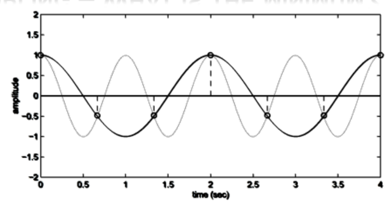
- What happened here?

200 Hz Sample



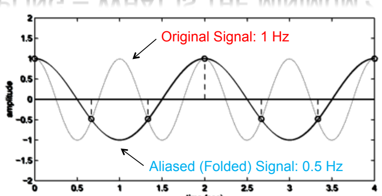
- Cannot let signal "wiggle" around between samples
- Sample too infrequently, can miss signal behavior

Sampling – What is the minimum?



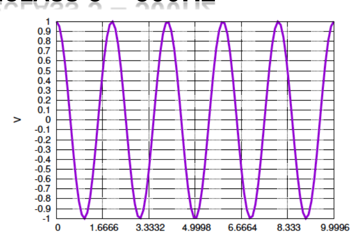
- Cannot sample lower without reconstruction error
  - We not only lose information…
    - …but when we 'reconstruct' the signal from the samples alone…
    - We will reconstruct at a lower frequency!
    - This phenomenon is called: aliasing

Sampling – What is the minimum?



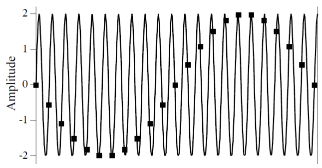
- What frequency does aliasing occur?
  - Original Signal's Frequency: 1 Hz
    - Sampling Rate: 1.5 Hz
  - Aliasing occurs at: 1.5 Hz – 1 Hz = 0.5 Hz
    - Also referred to as "Folding" – signal has "folds over" as if it were lower frequency

Preclass 3 – 500Hz



- Properly sampled?
- What did we get?

Sampling – What is the minimum?

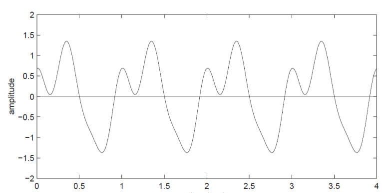


- **Another example/effect of aliasing**
  - The dots represent the samples, we can see an inverse sine-wave
  - Not only has the frequency of the original signal changed…
    - But phase of the signal has changed too!
    - Original signal: sine wave + 0° phase
    - Aliased signal: sine wave w/different frequency + 180° phase shift!

Sampling – What is the minimum?

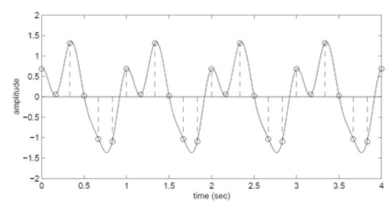
- **Harry Nyquist**
  - Electronic Engineer for AT&T from 1917 to 1954
  - Published paper in 1928 defining the: Sampling Theorem
    - **Nyquist Sampling Rate** = 2 x frequency of signal
      - Anything less: *under-sampling* – leads to aliasing
      - Anything more: *over-sampling* – waste of space?

What about Multirate Signals?



- **Fourier's Theorem (week 4 preview!):**
  - We can decompose continuous signal in terms of a sum of sines and cosines at different frequencies
  - This waveform: sum of sine waves at 1 Hz, 2 Hz, 3 Hz
    - What's the Nyquist Sampling Rate then?

What about Multirate Signals?



- **Fourier's Theorem & Nyquist Rate:**
  - Highest component's frequency: 3 Hz
  - What is Nyquist Sampling Rate?
    - 2 x highest frequency contained in the signal = 6 Hz
    - Sampling at this rate: avoids aliasing problem

Nyquist Rate vs Frequency

- **Nyquist Sampling Rate:**
  - f_s = 2 x highest frequency component of signal
    - Minimum sampling rate that satisfies: Nyquist Sampling Criterion for a given signal or family of signals
    - Minimum sampling rate that avoids aliasing
    - Property of a continuous-time signal
- **Nyquist Frequency:**
  - $\frac{1}{2} f_s$ = ½ sampling rate
    - Highest frequency that can be recovered from samples
    - Property of a discrete-time signal

Interlude

VIDEO

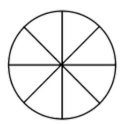
- http://www.youtube.com/watch?v=jHS9JGkEOmA

ALIASING IN MOVIES

- **Called visual aliasing**
  - See it all the time on TV/Film
    - Wheels tend to move backwards on moving cars…why?
  - What is it?
    - Primer: Movies are just pictures (frames) flying by quickly
    - Movies “sample” real life at roughly 24 frames per second
  - What do we know from Nyquist Sampling Theorem?
    - Aliasing will occur if changes occur faster than ½ f_s
    - Film Example:
      - If **light to dark transitions** occur faster than ½ f_s aka: 12 frame/sec
      - Aliasing will occur…

THE “WAGON WHEEL” EFFECT

- **Consider a wagon with 8 spokes:**

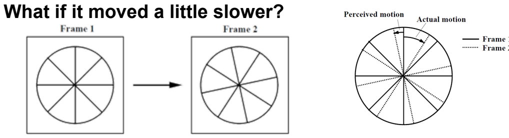


  - Let’s say it turns at a rate of 3 revolutions per second clockwise
    - That’s 180 rpm
  - On film this wheel will appear to stand still! Why?

$$\frac{\left(3\,\frac{revolutions}{sec}\right)\times\left(8\,\frac{spokes}{revolution}\right)}{\left(24\,\frac{frames}{sec}\right)} = 1\,\frac{spoke}{frame}$$

THE “WAGON WHEEL” EFFECT

- **What if it moved a little slower?**



  - Let’s say it turns at a rate of 2.5 revolutions per second clockwise

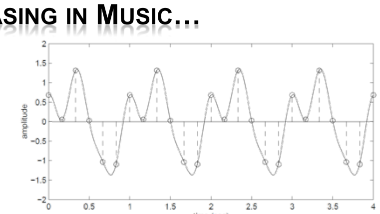
$$\frac{\left(2.5\,\frac{revolutions}{sec}\right)\times\left(8\,\frac{spokes}{revolution}\right)}{\left(24\,\frac{frames}{sec}\right)} = .83\,\frac{spoke}{frame}$$

  - Our brain could interpret this in two possible ways:
    - Wheel has moved clockwise by 83% of spoke interval in clockwise direction
    - OR: wheel has moved counter-clockwise by 17%
      - Our brains prefer this view! So we see the wheel moving backwards! *(thanks aliasing!)*

Fool your brain: http://www.youtube.com/watch?v=jHS9JGkEOmA

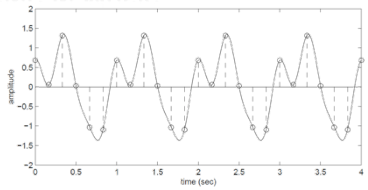
EFFECTS OF ALIASING

ALIASING IN MUSIC…



- **Multirate Signals and Aliasing**
  - Imagine the above is a music signal (1 Hz, 2 Hz and 3 Hz chord)
    - What happens if we undersample? Should sample at 6 Hz, but instead 4 Hz
    - The 1Hz & 2Hz signals will be sampled just fine (as 4 Hz is *2 x* 2Hz)
    - But what happens to 3 Hz Signal?
      - Fold baby fold!

ALIASING IN MUSIC…



- **Multirate Signals and Aliasing**
  - Imagine the above is a music signal (1 Hz, 2 Hz and 3 Hz chord)
    - Where will folding occur?
      - folding occurs at: 4 Hz – 3 Hz = 1 Hz
      - Sample rate – frequency = aliasing/folding frequency
    - But what happens to 3 Hz signal? It will "fold over" and sound like a 1 Hz signal!
    - To us that can sound like a low frequency noise in our music

ANTI-ALIASING

HOW DO WE FIX THIS?

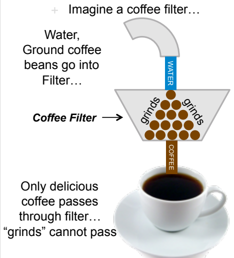
- **It's simple…sample at the Nyquist Rate**
  - But…what if your rate is fixed? Like 24 frames/sec?
  - Or our eye's sampling rate: 60 cycles/degree
    - Spatial variations finer than this are undetectable!

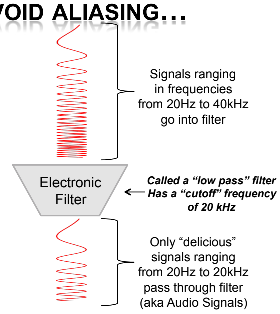
HOW TO AVOID ALIASING WITH DIGITAL MUSIC?

- **If we simply sample at *2 x highest frequency of signal*…**
  - (AKA: Nyquist Rate)
  - …we won't encounter aliasing!
- **But how do we guarantee highest frequency of our signal?**
  - Audio: this is easy!
    - We know the range of human ear: 20 Hz to 20 kHz…
    - The highest frequency component in music is then: 20 kHz
    - …so, before sound goes into ADC, we apply a filter!
      - Blocks any frequency above 20 kHz from going into ADC
    - Essentially, we are fixing our sampling rate & 'bluring' or filtering our incoming signal

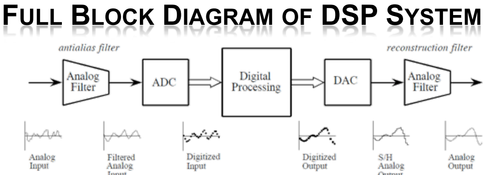
WE KNOW HOW TO AVOID ALIASING…

- **What is a filter you ask?**
  - Imagine a coffee filter…





FULL BLOCK DIAGRAM OF DSP SYSTEM



- **Before ADC, we put music signal through *antialias* filter**
  - Filter blocks any signals higher than 20 kHz (prevents aliasing!)
  - Then our ADC can safely sample at 2 x 20 kHz without aliasing
    - What is our Nyquist Rate?
      - f_s=2 x 20 kHz = 40 kHz, or 40 thousand samples per second!
    - What is our Nyquist Frequency?
      - ½ f_s = 20 kHz
  - Cutoff frequency of our filter? Has to be the Nyquist Frequency

Why do we need the antialias filter?

- **If we can't hear anything above 20kHz…**
  - Why do we need to filter it out?
    - Dog's can hear from 40 Hz to 60 kHz
      - so clearly there are sounds above 20 kHz
  - Let's imagine a high frequency noise in music studio
    - Let's say it's a vibration occurring at 25 kHz
      - No human can hear it, why filter it out?
    - Because of aliasing:
      - Frequency aliasing/folding will occur:
      - Sample rate – frequency = aliasing/folding frequency
      - 40 kHz - 25 kHz = 15 kHz
      - The 25 kHz vibration will fold-over to a 15 kHz "hum" or audible noise
        - It will ruin our recording and source of noise wouldn't be obvious!

Compact Disc (CD)

- **CD (late 20th century)**
  - First form of digitized music
    - ADC → DSP → DAC
  - Up until this time, music was…
    - …exact reproduction (record, tape)

  - Nyquist Sampling Rate: 44.1 kHz
  - Nyquist Frequency: ½ (44.1 kHz) = 22.05 kHz
    - AKA – upper range of audio
    - 22.05 = cutoff frequency for low-pass antialias filter

Compact Disc (CD)

- **CD (late 20th century)**
  - Quick Math:
  - Sampling Rate: 44.1 kHz
  - Sampling Rate: 44,100 Hz
  - That means we collect 44,100 Samples in 1 second!
    - In 60 seconds, we collect: 2,646,000 samples
      - *(44,100 samples/sec * 60 sec) =2,646,000 samples/minute*
    - For a 3 minute song: 7,938,000 samples / song!
      - *(2,646,000 samples/minute) * (3 minutes) = 7,938,000 samples/song*
    - If each sample requires 16-bits to store:
      - *(7,938,000 samples/song) * (16 bits/sample) = 127,008,000 bits/song*
      - That's 15,876,000 bytes per song
      - *15,504 kB = **15.14 MB** per song!*
    - What about stereo recordings? Double that!
      - **30.28 MB** per 3 minute stereo song!!
    - This is why a CD can only hold about 80 minutes of digital audio!

Big Ideas

- **Sample at twice the maximum frequency**
  - Can reconstruct perfectly

- **If have frequencies > sample_freq/2**
  - Will get aliasing … as high frequencies fold

- **Avoid aliasing with analog Anti-Alias prefilter before sampling**

This Week In Lab

- **Lab 2: D2A** – play back the samples you recorded last week

Piazza

- **Signup piazza (half not)**
  - Reminders and administrivia
  - Answer questions from lecture

Learn More

- ESE224 – Signal Processing

References

- S. Smith, "The Scientists and Engineer's Guide to Digital Signal Processing," 1997.
- http://en.wikipedia.org/wiki/Nyquist_frequency
- http://en.wikipedia.org/wiki/Nyquist_rate
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- http://en.wikipedia.org/wiki/Sampling_rate
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- B. Olshausen, "Aliasing", PSC 129 – Sensory Processes Course Notes, UC Davis