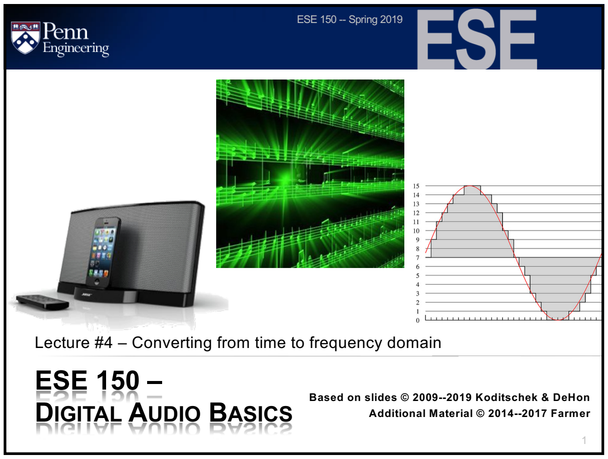
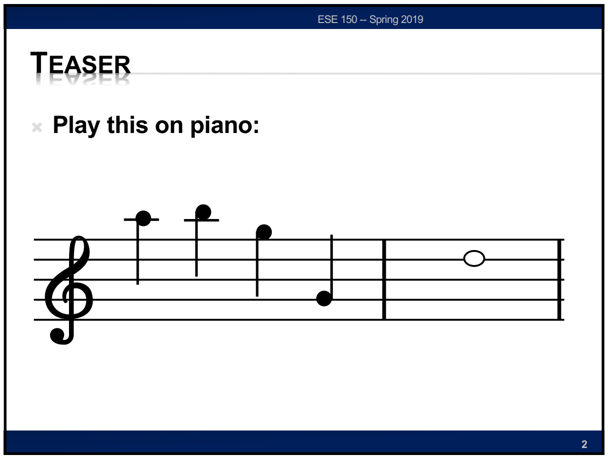


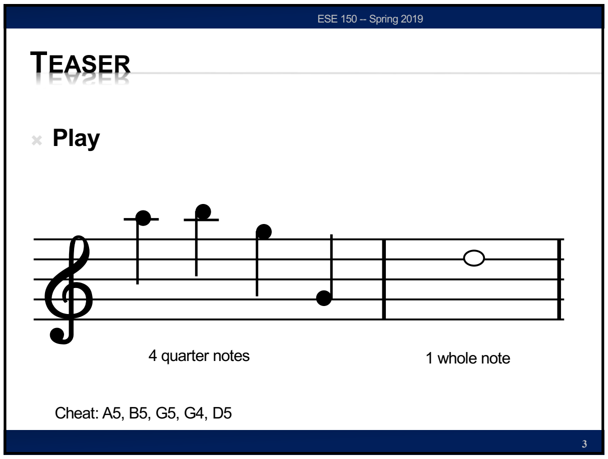
Lecture #4 – Converting from time to frequency domain

ESE 150 –
DIGITAL AUDIO BASICS



TEASER

- Play this on piano:



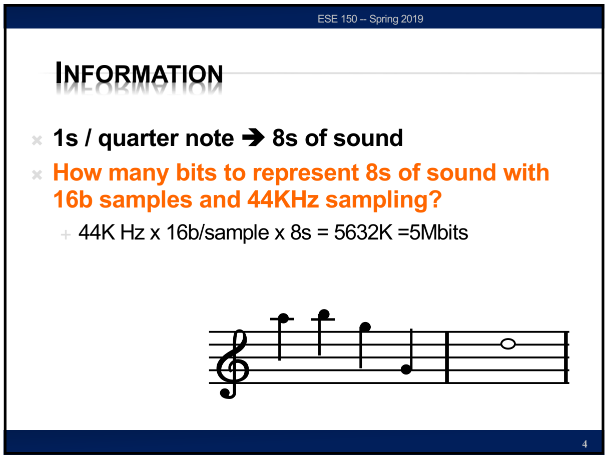
TEASER

- Play

4 quarter notes

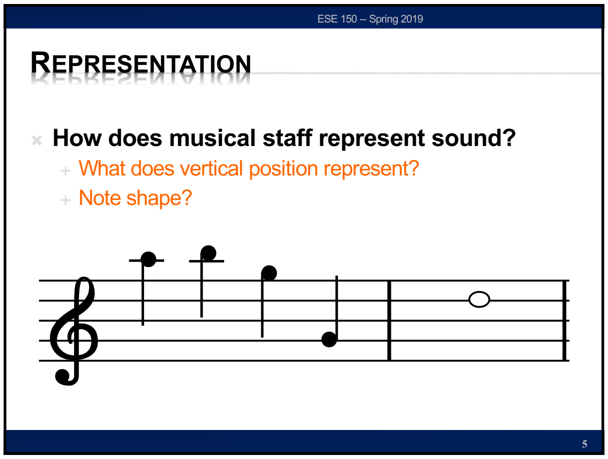
1 whole note

Cheat: A5, B5, G5, G4, D5



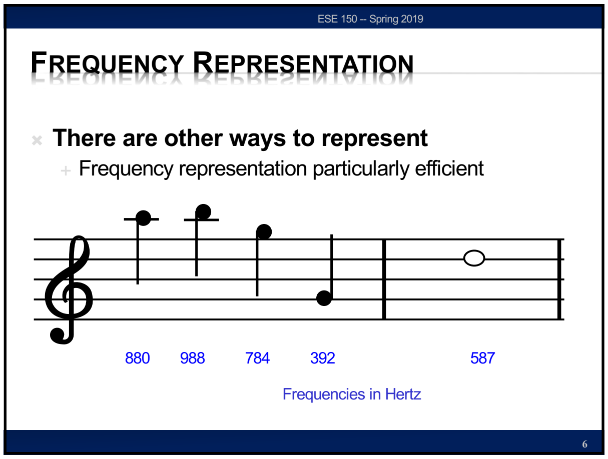
INFORMATION

- 1s / quarter note ➔ 8s of sound
- How many bits to represent 8s of sound with 16b samples and 44KHz sampling?
  - 44K Hz x 16b/sample x 8s = 5632K =5Mbits



REPRESENTATION

- How does musical staff represent sound?
  - What does vertical position represent?
  - Note shape?



FREQUENCY REPRESENTATION

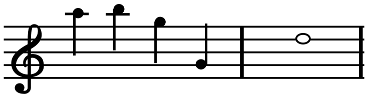
- There are other ways to represent
  - Frequency representation particularly efficient

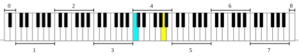
880 988 784 392 587

Frequencies in Hertz

FREQUENCY REPRESENTATION

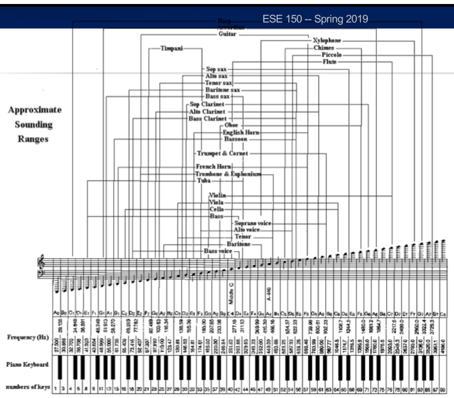
- How much information is this musical staff communicating?
- How many keys on piano? → bits/note





Hamburg Steinway D-274 Piano photo by Karl Kunde

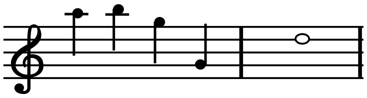
https://commons.wikimedia.org/wiki/File:D274.jpg



Larry Solomn: http://solomonsmusic.net/insrange.htm

FREQUENCY REPRESENTATION

- How much information is this musical staff communicating?
- How many keys on piano? → bits/note
- Let's say 8b duration
- How many bits for 5 notes?
  - (7b/note+8b/duration) x 5 note = 75 bits?

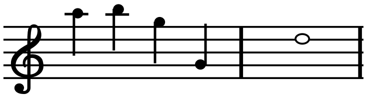


LAB 2 POSTLAB

- You reproduced 800 samples of a 300Hz sine wave at 1000Hz with 8b precision
  - 6400b
- What did you need to specify to do that?
- How may bits to represent that?

CONCLUDE

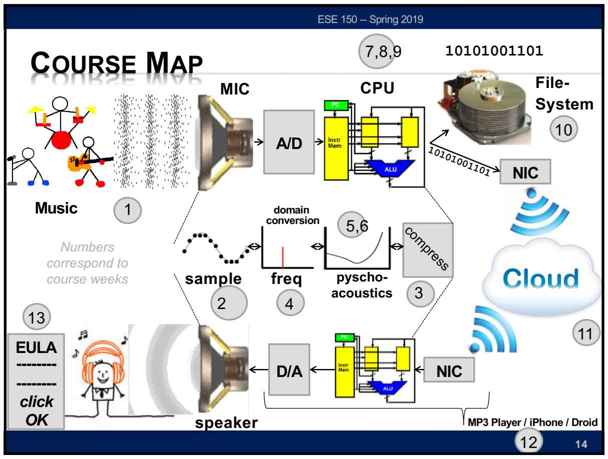
- Can represent common sounds much more compactly in frequency domain than in time-sample domain
  - Frequency domain ~ 75b
  - Time-sample domain ~ 5Mb



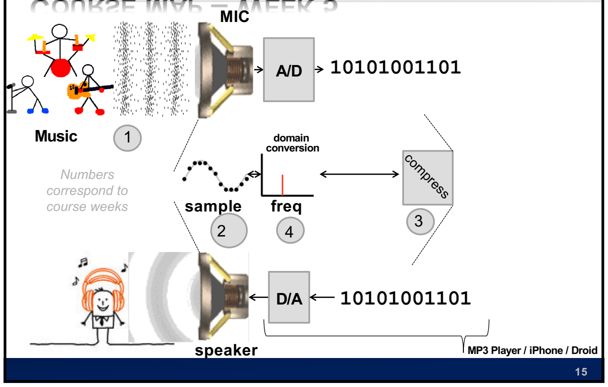
LECTURE TOPICS

- Teaser: frequency representation
- Where are we on course map?
- What we did in lab last week
  - How it relates to this week
- The Fourier Series – can represent any signal
- The Discrete Fourier Transform (DFT) – can translate
  - Change of basis
- Next Lab
- References

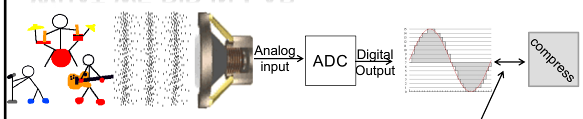
COURSE MAP



COURSE MAP – WEEK 5



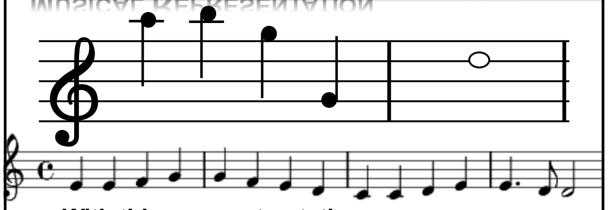
WHAT WE DID IN LAB…



- **Week 1: Converted Sound to analog voltage signal**
  - a "pressure wave" that changes air molecules w/ respect to time
  - a "voltage wave" that changes amplitude w/ respect to time
- **Week 2: Sampled voltage, then quantized it to digital sig.**
  - Sample: Break up independent variable, take discrete 'samples'
  - Quantize: Break up dependent variable into n-levels (need 2^n bits to digitize)
- **Week 3: Compress digital signal**
  - Use even less bits without using sound quality!
- **Week 4 (upcoming): Before we compress…**
  - Put our 'digital' data into another form…BEFORE we compress…less stuff to compress!

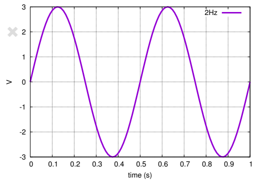
Background

WHAT IS THE FREQUENCY DOMAIN?

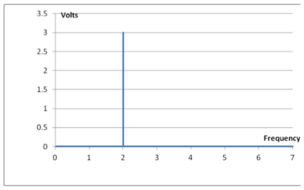
MUSICAL REPRESENTATION

- With this compact notation
  - Could communicate a sound to pianist
  - Much more compact than 44KHz time-sample amplitudes (fewer bits to represent)
  - Represent frequencies

TIME-DOMAIN & FREQUENCY-DOMAIN

- **As an example…let's say we have a pure tone**
  - If period: T = 1/2 and Amplitude = 3 Volts
  - $s(t) = A\,sin(2\pi f t) = A\,sin(2\pi 2 t)$



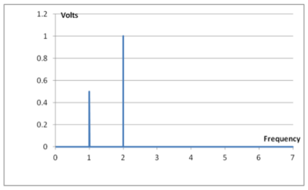
Time domain representation



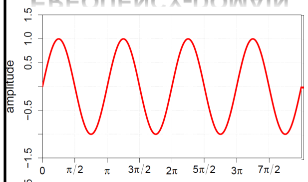
Frequency domain representation

FREQUENCY-DOMAIN

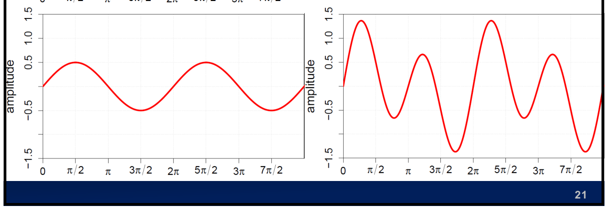
- **Of course, not all signals are this simple**
- **For example:** $\mathbf{s(t)} = \sin(2{\times}2\pi{\times}t) + \frac{1}{2}\sin(2\pi{\times}t)$
- **Question: What will the frequency representation look like?**



FREQUENCY-DOMAIN

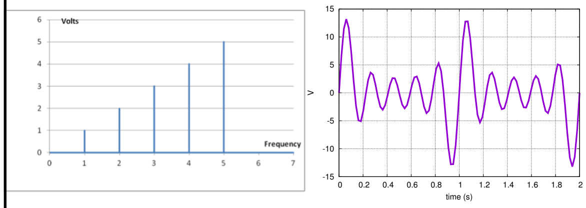


- **How about the time-domain?**
  - Plot $\sin(2{\times}2\pi{\times}t)$
  - Plot $\frac{1}{2}\sin(2\pi{\times}t)$
  - Sum: $\sin(2{\times}2\pi{\times}t) + \frac{1}{2}\sin(2\pi{\times}t)$
  - ***Notice how it was easier to plot the frequency domain representation***



FREQUENCY-DOMAIN

- **Another example**



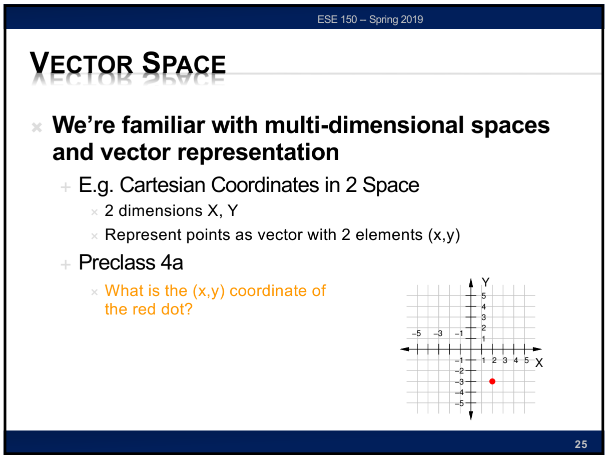
The time domain plot on the right is really the sum of 5 sinusoids, where 5 Hz is the strongest component of the signal

FREQUENCY-DOMAIN

- **So far…**
  - we have seen how a signal written as:
    - a sum of sines of different frequencies
  - can have a ***frequency domain representation***
- **Each sine component…**
  - is more or less important depending on its **coefficient**
  - Example: $\boldsymbol{s(t)} = 1\sin(2{\times}2\pi{\times}t) + \frac{1}{2}\sin(2\pi{\times}t)$
- **Can any arbitrary signal be represented as a sum of sines?**
  - No. But the idea has potential, let's explore it!

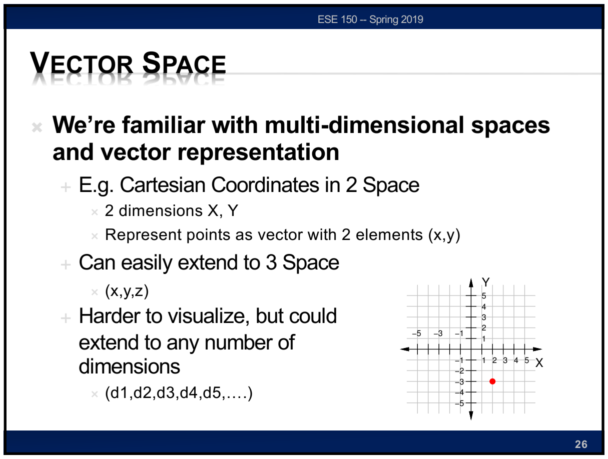
(new background setup)

VECTOR BACKGROUND



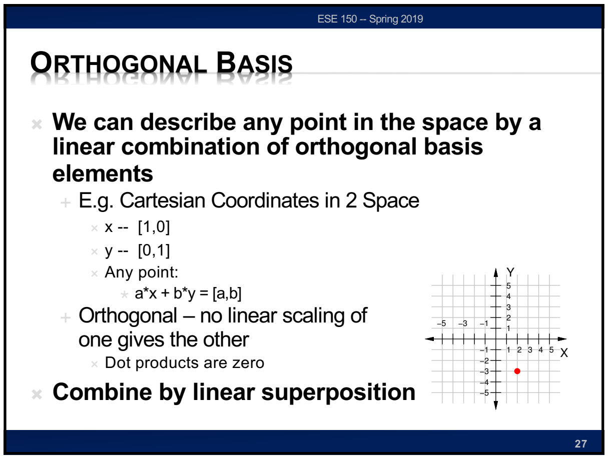
VECTOR SPACE

- **We're familiar with multi-dimensional spaces and vector representation**
  - E.g. Cartesian Coordinates in 2 Space
    - 2 dimensions X, Y
    - Represent points as vector with 2 elements (x,y)
  - Preclass 4a
    - What is the (x,y) coordinate of the red dot?



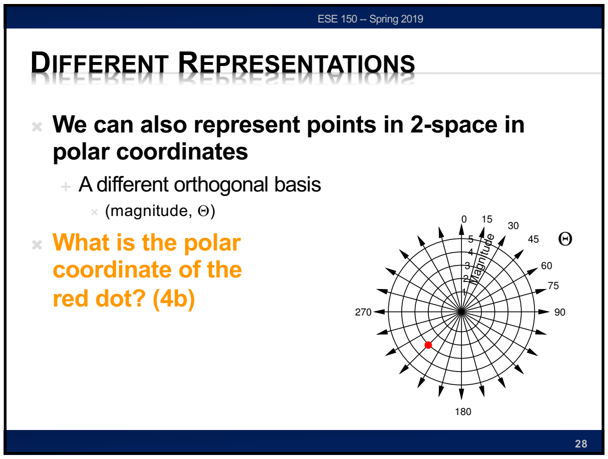
VECTOR SPACE

- **We're familiar with multi-dimensional spaces and vector representation**
  - E.g. Cartesian Coordinates in 2 Space
    - 2 dimensions X, Y
    - Represent points as vector with 2 elements (x,y)
  - Can easily extend to 3 Space
    - (x,y,z)
  - Harder to visualize, but could extend to any number of dimensions
    - (d1,d2,d3,d4,d5,….)



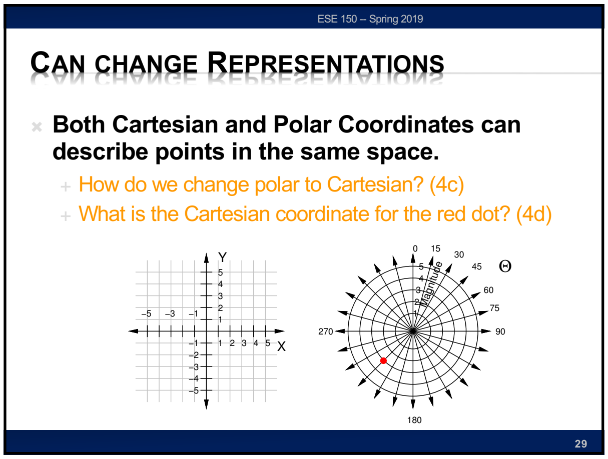
ORTHOGONAL BASIS

- **We can describe any point in the space by a linear combination of orthogonal basis elements**
  - E.g. Cartesian Coordinates in 2 Space
    - x -- [1,0]
    - y -- [0,1]
    - Any point:
      - a*x + b*y = [a,b]
  - Orthogonal – no linear scaling of one gives the other
    - Dot products are zero
- **Combine by linear superposition**



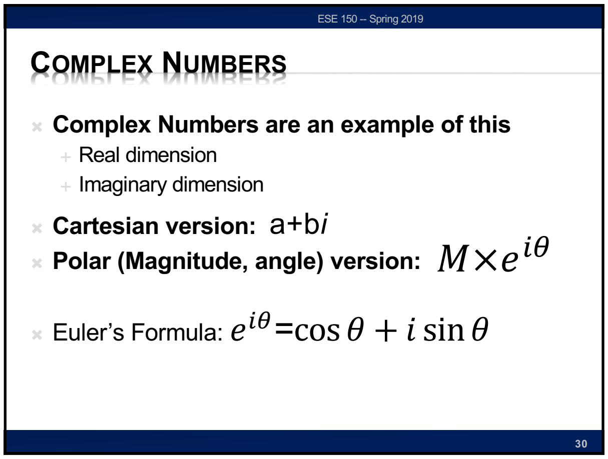
DIFFERENT REPRESENTATIONS

- **We can also represent points in 2-space in polar coordinates**
  - A different orthogonal basis
    - (magnitude, Θ)
- **What is the polar coordinate of the red dot? (4b)**



CAN CHANGE REPRESENTATIONS

- **Both Cartesian and Polar Coordinates can describe points in the same space.**
  - How do we change polar to Cartesian? (4c)
  - What is the Cartesian coordinate for the red dot? (4d)



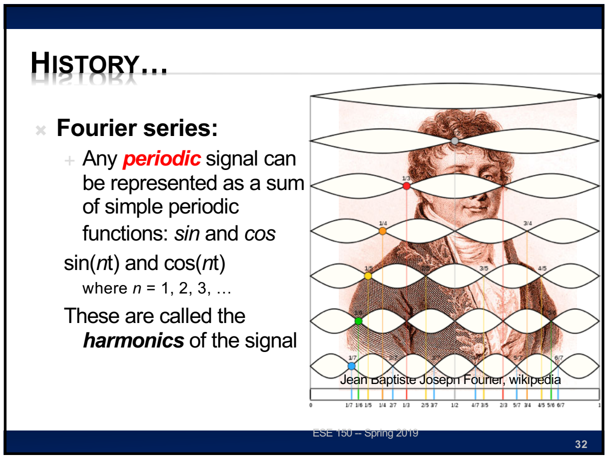
COMPLEX NUMBERS

- **Complex Numbers are an example of this**
  - Real dimension
  - Imaginary dimension
- **Cartesian version:** $a+bi$
- **Polar (Magnitude, angle) version:** $M \times e^{i\theta}$
- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

(revised)

The frequency domain &

THE FOURIER SERIES



HISTORY…

- Fourier series:
  - Any ***periodic*** signal can be represented as a sum of simple periodic functions: *sin* and *cos*

sin(*n*t) and cos(*n*t)

where *n* = 1, 2, 3, …

These are called the ***harmonics*** of the signal

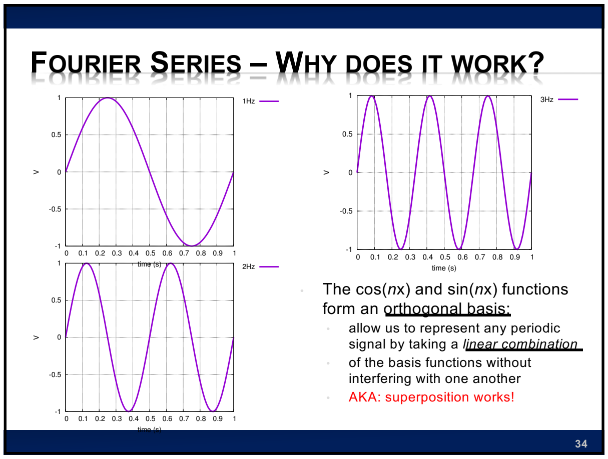
Jean Baptiste Joseph Fourier, wikipedia

FOURIER SERIES – MORE FORMALLY

The Fourier Theorem states that any *periodic* function f(t) of period L can be cast in the form:

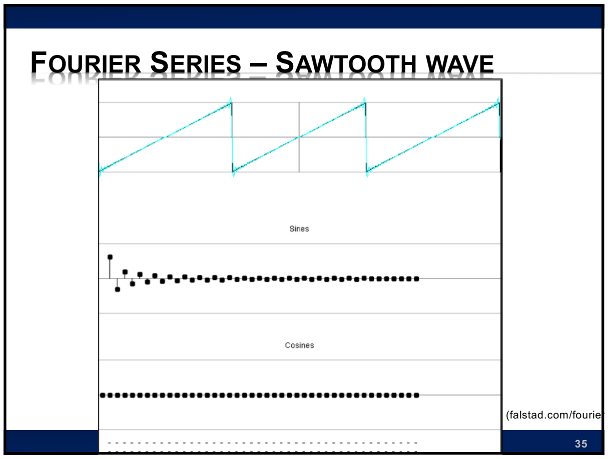
$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

The constants: a_0, a_n , and b_n are called the Fourier coefficients of *f(t)*



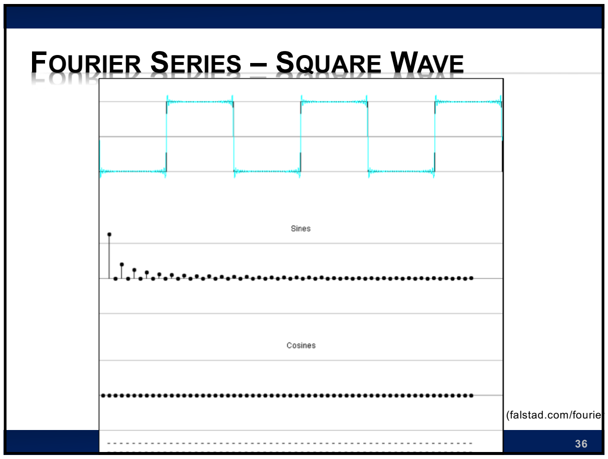
FOURIER SERIES – WHY DOES IT WORK?

- The cos(*n*x) and sin(*n*x) functions form an orthogonal basis:
  - allow us to represent any periodic signal by taking a *linear combination*
  - of the basis functions without interfering with one another
  - AKA: superposition works!



FOURIER SERIES – SAWTOOTH WAVE

(falstad.com/fourie



FOURIER SERIES – SQUARE WAVE

(falstad.com/fourie

FOURIER SERIES (REVIEW OF KEY POINTS)

- **The idea of the series:**
  - Any ***PERIODIC*** wave can be represented as simple sum of sine waves
- **2 Caveats:**
  - Linearity:
    - The series only holds while the system it is describing is *linear* because it relies on the superposition principle
    - -aka – adding up all the sine waves is superposition in action
  - Periodicity:
    - The series only holds if the waves it is describing are periodic
    - Non-periodic waves are dealt with by the Fourier Transform
      - We will examine that in the 2nd half of lecture

NYQUIST

- **Remember we said we needed to sample at twice the maximum frequency**
  - Now see all signals can be represented as a linear sum of frequencies
  - …and the frequency components are orthogonal
    - Can be extracted and treated independently

INTERLUDE

- **Close Encounters Mothership**
- **https://www.youtube.com/watch?v=S4PYI6TzqYk**

WHAT NOW?

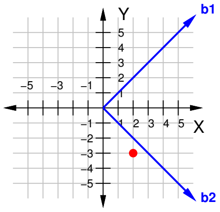
- **In the first half of the lecture we introduced:**
  - The idea of frequency domain
  - The Fourier Series
- **In the second half of the lecture:**
  - Fourier Transform
  - See how to perform this time-frequency translation
  - Examples

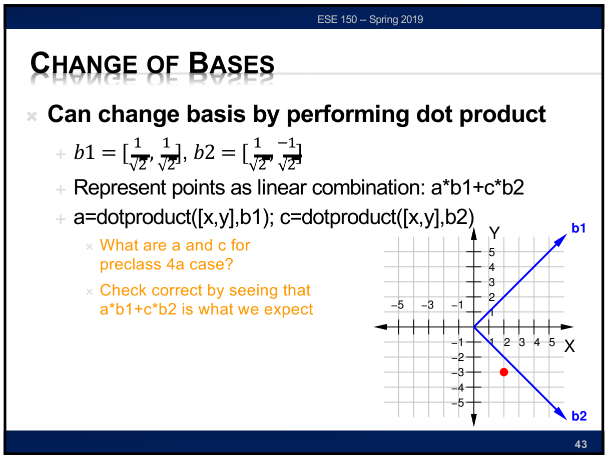
(new background setup)

VECTOR BACKGROUND

CHANGE OF BASES

- **There are more than one set of basis vectors that span a space**
  - For example, might rotate 90 degrees in Cartesian coordinates
    - $b1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}], b2 = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$
    - Note dotproduct(b1,b2)=0
    - Represent points as linear combination: a*b1+c*b2





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CHANGE OF BASES

- **Can change basis by performing dot product**
  - $b1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}], b2 = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$
  - Represent points as linear combination: a*b1+c*b2
  - a=dotproduct([x,y],b1); c=dotproduct([x,y],b2)
    - What are a and c for preclass 4a case?
    - Check correct by seeing that a*b1+c*b2 is what we expect

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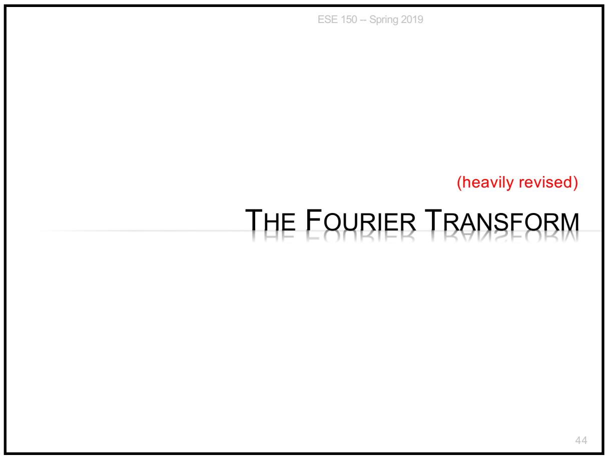

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(heavily revised)

THE FOURIER TRANSFORM

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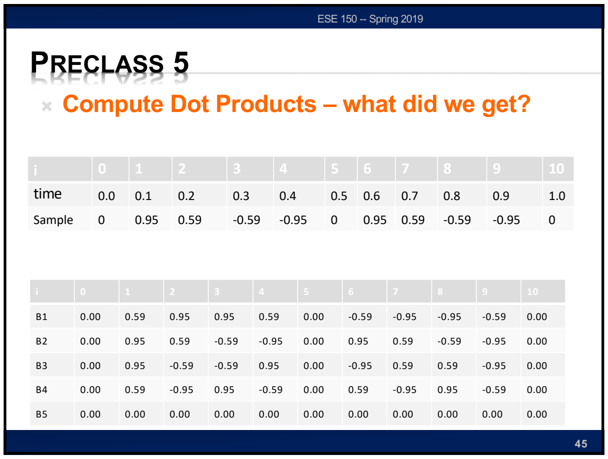

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PRECLASS 5

- Compute Dot Products – what did we get?

i	0	1	2	3	4	5	6	7	8	9	10
time	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Sample	0	0.95	0.59	-0.59	-0.95	0	0.95	0.59	-0.59	-0.95	0

i	0	1	2	3	4	5	6	7	8	9	10
B1	0.00	0.59	0.95	0.95	0.59	0.00	-0.59	-0.95	-0.95	-0.59	0.00
B2	0.00	0.95	0.59	-0.59	-0.95	0.00	0.95	0.59	-0.59	-0.95	0.00
B3	0.00	0.95	-0.59	-0.59	0.95	0.00	-0.95	0.59	0.59	-0.95	0.00
B4	0.00	0.59	-0.95	0.95	-0.59	0.00	0.59	-0.95	0.95	-0.59	0.00
B5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

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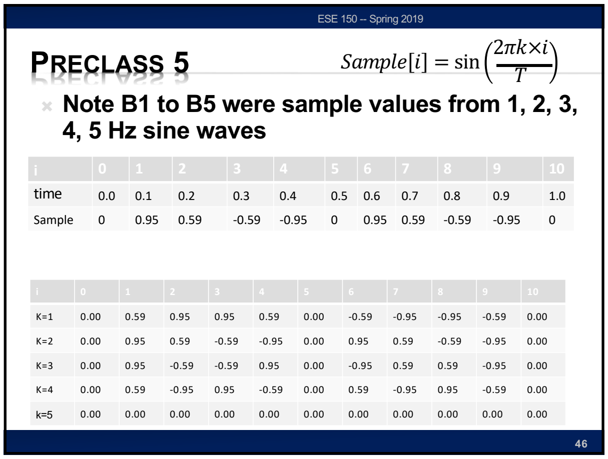

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PRECLASS 5

$$Sample[i] = \sin\left(\frac{2\pi k \times i}{T}\right)$$

- **Note B1 to B5 were sample values from 1, 2, 3, 4, 5 Hz sine waves**

i	0	1	2	3	4	5	6	7	8	9	10
time	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Sample	0	0.95	0.59	-0.59	-0.95	0	0.95	0.59	-0.59	-0.95	0

i	0	1	2	3	4	5	6	7	8	9	10
K=1	0.00	0.59	0.95	0.95	0.59	0.00	-0.59	-0.95	-0.95	-0.59	0.00
K=2	0.00	0.95	0.59	-0.59	-0.95	0.00	0.95	0.59	-0.59	-0.95	0.00
K=3	0.00	0.95	-0.59	-0.59	0.95	0.00	-0.95	0.59	0.59	-0.95	0.00
K=4	0.00	0.59	-0.95	0.95	-0.59	0.00	0.59	-0.95	0.95	-0.59	0.00
k=5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

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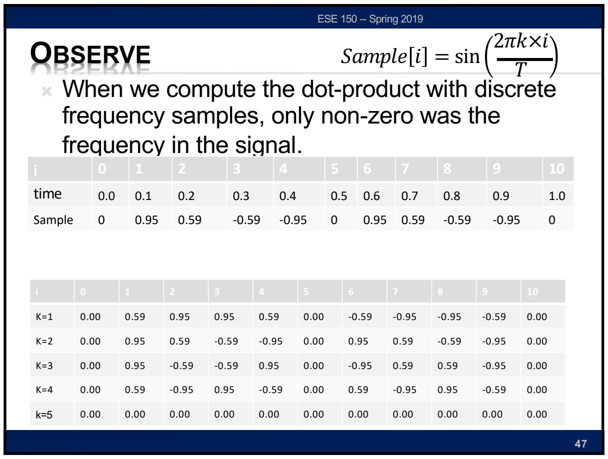

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OBSERVE

$$Sample[i] = \sin\left(\frac{2\pi k \times i}{T}\right)$$

- When we compute the dot-product with discrete frequency samples, only non-zero was the frequency in the signal.

i	0	1	2	3	4	5	6	7	8	9	10
time	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Sample	0	0.95	0.59	-0.59	-0.95	0	0.95	0.59	-0.59	-0.95	0

i	0	1	2	3	4	5	6	7	8	9	10
K=1	0.00	0.59	0.95	0.95	0.59	0.00	-0.59	-0.95	-0.95	-0.59	0.00
K=2	0.00	0.95	0.59	-0.59	-0.95	0.00	0.95	0.59	-0.59	-0.95	0.00
K=3	0.00	0.95	-0.59	-0.59	0.95	0.00	-0.95	0.59	0.59	-0.95	0.00
K=4	0.00	0.59	-0.95	0.95	-0.59	0.00	0.59	-0.95	0.95	-0.59	0.00
k=5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

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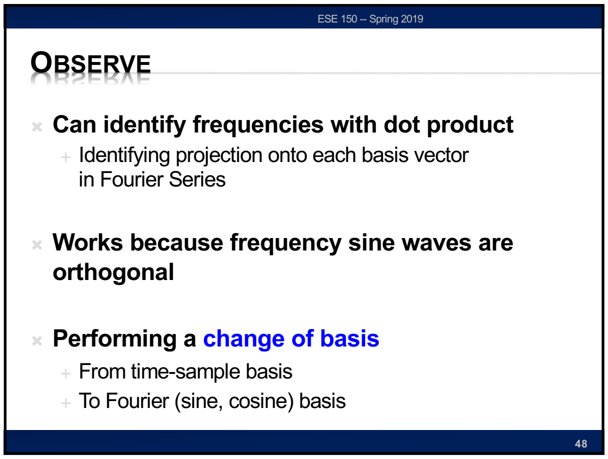

ESE 150 -- Spring 2019

OBSERVE

- **Can identify frequencies with dot product**
  - Identifying projection onto each basis vector in Fourier Series
- **Works because frequency sine waves are orthogonal**
- **Performing a change of basis**
  - From time-sample basis
  - To Fourier (sine, cosine) basis

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TIME AND FREQUENCY BASES

- **Time Sample basis**
  - Also a multi-dimensional space
  - Dimension = # time samples
  - Vector [t_0,t_1,t_2,t_3, ….]
- **Frequency basis**
  - Multi-dimensional
  - Dimensions = Coefficients of sine and cosine components
  - $f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$
  - Vector [a_0,a_1,b_1,a_2,b_2,…]

FREQUENCY-DOMAIN

- **How to make a song appear "periodic"**
  - Treat the entire song as 1 period of a very complicated sinusoid!
  - *This is the assumption of the Fourier Transform*

DISCRETE FOURIER TRANSFORMS

- **Fourier Transforms are nice,**
  - but we want to store and process our signals with computers
- **We extend Fourier Transforms into Discrete Fourier Transforms, or DFT**
  - We know our music signal is now discrete: $x(t) \rightarrow x_n$
  - The signal contains **N** samples: $0 \leq n \leq \boldsymbol{N} - 1$

WARNING

- **Don't get lost in mathematical notation**

DFT – DISCRETE FOURIER TRANSFORM

- **Represent any sequence of time samples as**

$$f(k) = a_0 + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi k}{N} + b_n \sin\frac{n\pi k}{N}\right)$$

- **Compute a_n, b_n by dot product**
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k] \times \cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k] \times \sin\left(\frac{n2\pi k}{N}\right)\right)$

DFT – DISCRETE FOURIER TRANSFORM (COMPLEX REPRESENTATION)

- **Represent any sequence of time samples as**

$$f(k) = a_0 + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi k}{N} + b_n \sin\frac{n\pi k}{N}\right)$$

- **From Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$**, can also express as exponential**

$$f(k) = \frac{1}{N}\sum_{n=0}^{N-1} \boldsymbol{X_K}\, e^{-i\left(\frac{2\pi nk}{N}\right)}$$

- Representation vector is [X_0,X_1,…X_{N-1}]; X_K complex

DFT – DISCRETE FOURIER TRANSFORM (COMPLEX REPRESENTATION)

- Represent any sequence of time samples as

$$f(k) = \frac{1}{N}\sum_{n=0}^{N-1} \boldsymbol{X_K}\, e^{-i\left(\frac{2\pi nk}{N}\right)}$$

- Compute X_K by dot product

$$X_k = \sum_{n=0}^{N-1} x_n \times e^{-i\frac{2\pi\ k\ n}{N}}$$

DFT – DISCRETE FOURIER TRANSFORM (COMPLEX REPRESENTATION)

- Compute X_K by dot product

$$X_k = \sum_{n=0}^{N-1} x_n \times e^{-i\frac{2\pi\ k\ n}{N}}$$

- Same as … compute a_n, b_n by dot product
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$

DON'T LET NOTATION CONFUSE YOU EXPANDING….

$$b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$$

- E.g. for n=2, this says

$$b_2 = \left(\frac{2}{N}\right) dotproduct\left(Sample, \left[\sin(0)\,, \sin\left(\frac{2*2\pi*1}{N}\right), \sin\left(\frac{2*2\pi*1}{N}\right), \ldots\right]\right)$$

$$b_2 = \left(\frac{2}{N}\right) dotproduct(Sample, [0,0.95,0.59,-0.59,-0.95,0,0.95,0.59,-0.59,-0.95,0])$$

- …which is dot product we performed in preclass 5

Don't let notation confuse you…

DISCRETE FOURIER TRANSFORMS

- A DFT transforms N samples of a signal in time domain
  - into a (periodic) frequency representation with N samples
  - So we don't have to deal with real signals anymore
- We work with sampled signals (quantized in time),
  - and the frequency representation we get is also quantized in time!

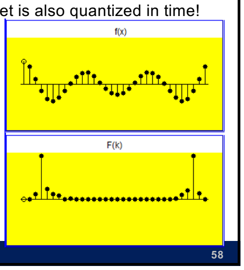
Music Signal in Discrete Time →

↓

$$DFT(x_n) = X_k = \sum_{n=0}^{N-1} x_n \times e^{-i\frac{2\pi\ k\ n}{N}}$$

↓

Music Signal "Transformed" To Frequency Domain →



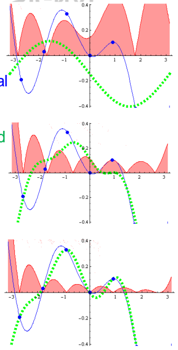
(sepwww.stanford.edu/oldsep/hale/FftLab.html)

DISCRETE FOURIER TRANSFORMS

- A smaller sampling period means:
  - → more points to represent the signal
    - larger N
  - → more harmonics used in DFT
    - N harmonics
  - → Smaller error compared to actual analog signal we capture/produce
- DFTs are extensively used in practice, since computers can handle them

APPROXIMATING THE SAMPLED SIGNAL

- A signal sampled in time can be approximated *arbitrarily* closely from the time-sampled values
- With a DFT, each sample gives us knowledge of one harmonic
- Each harmonic is a component used in the reconstruction of the signal
- The more harmonics we use, the better the reconstruction



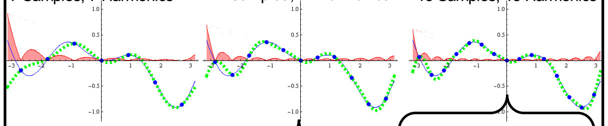
Original signal (samples)

Approximated

Error

{ Cos[0*t*], Sin[1*t*], Cos[1*t*] , Sin[2*t*], Cos[2*t*] , Sin[3*t*], Cos[3*t*] }

MORE SAMPLES → MORE HARMONICS→LESS ERROR



7 Samples; 7 Harmonics

11 Samples; 11 Harmonics

15 Samples; 15 Harmonics

t	v
-2.7	-0.2
-1.8	0
-0.9	0.3
0	0
0.9	0.1
1.8	-0.4
2.7	-0.9

DFT

f	A
0	-1.1
1s	-0.8
1c	1.3
2s	0.2
2c	0.7
3s	0.5
3c	-0.3

t	v
-2.9	0
-2.3	-0.3
-1.7	0.1
-1.1	0.4
-0.6	0.2
0	0
0.6	0.1
1.1	0.1
1.7	-0.3
2.3	-0.9
2.9	-0.7

DFT

f	A
0	-1.3
1s	-1.5
1c	1.9
2s	0.1
2c	0.9
3s	1.1
3c	-0.5
4s	0.6
4c	-0.2
5s	0.3
5c	-0.1

t	v
-2.9	0.1
-2.5	-0.3
-2.1	-0.2
-1.7	0.1
-1.3	0.3
-0.8	0.3
-0.4	0.1
0	0
0.4	0
0.8	0.1
1.3	0
1.7	-0.3
2.1	-0.7
2.5	-0.9
2.9	-0.7

DFT

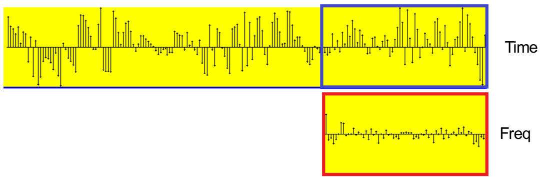
f	A
0	-2
1s	-2
1c	2.3
2s	-0
2c	1.1
3s	1.5
3c	-0
4s	0.8
4c	-0
5s	0.6
5c	0
6s	0.5
6c	-0
7s	0.5
7c	0.1

SAVING RESOURCES

- **However, N can get very large**
  - e.g., with a sampling rate of 48,000Hz
  - How big is N for a 4 minute song?
- **How many operations does this translate to?**
  - To compute one frequency component?
  - To compute all N frequency components?
- **This is not practical. Instead, we use a window of values to which we apply the transform**
  - Typical size: …, 512, 1024, 2048, …

A WINDOW OPERATION

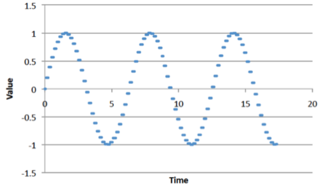
- **In the example below, we traverse the signal but only look at 64 samples at a time**



(sepwww.stanford.edu/oldsep/hale/FftLab.html)

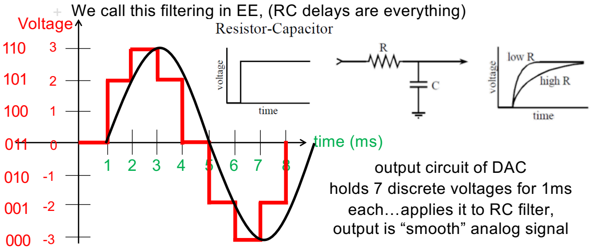
CONNECT THE DOTS

- **Intuition, with enough dots, not hard to "connect-the-dots" to reconstruct (understand) the continuous signal.**
  - What is the continuous signal here? (preclass 3)
  - Assumes certain regularity conditions
  - What is enough?



DAC - FILTERING

- **Digital-to-Analog (DAC) Conversion**
  - What a capacitor does?
  - What happens when apply voltage across a resistor/capacitor?
  - We call this filtering in EE, (RC delays are everything)

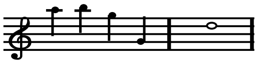


output circuit of DAC holds 7 discrete voltages for 1ms each…applies it to RC filter, output is "smooth" analog signal

RECONSTRUCTION

- **Not really connect-the-dots in time**
  - (previous explanation was oversimplified)
- **Recall near Nyquist rate**
  - Could often miss the peak
  - Get poor sine waves
    - …look like peak moves around even if sampled above Nyquist rate
- **Better reconstruction**
  - Convert to frequency
    - Which can perfectly represent up to half sampling rate
  - Reconstruct from frequency basis

BIG IDEAS

- **Can represent signals in frequency domain**
  - Different basis – basis vectors of sines and cosines
- **Often more convenient and efficient than time domain**
  - Remember musical staff

- **Can convert between time and frequency domain**
  - Using a dot-product to calculate time or frequency components

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{N}[a_n\cos(nt)+b_n\sin(nt)]$$

THIS WEEK IN LAB

- **Lab 4:**
  - You will identify Frequency components using FFT in Matlab
    - Bring headphones
- **Remember:**
  - Lab 3 report is due on Friday

LEARN MORE

- **ESE325 – whole course on Fourier Analysis**
- **ESE224 – signal processing**
- **ESE215, 319, 419 – reason about behavior of circuits in time and frequency domains**

REFERENCES

- **S. Smith, "The Scientists and Engineer's Guide to Digital Signal Processing," 1997.**
- https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/