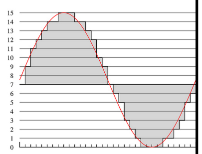


Lecture #09– Discrete Fourier Transform

# ESE 150 – Digital Audio Basics





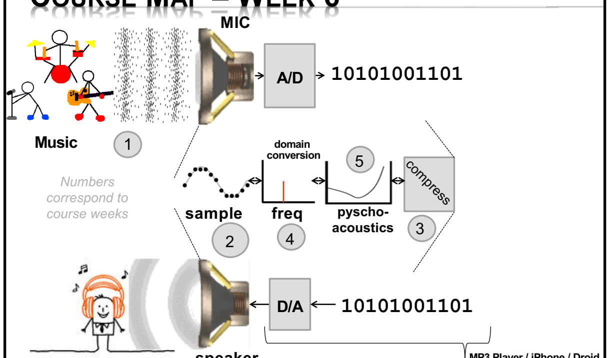


## Lecture Topics

- **Where are we on course map?**
- **Reminder: The Fourier Series**
  - can represent any signal in frequency domain
- **The Discrete Fourier Transform (DFT)**
  - can translate any signal between time and frequency domain
  - change of basis
- **References**



## Course Map – Week 6








The frequency domain &

## The Fourier Series

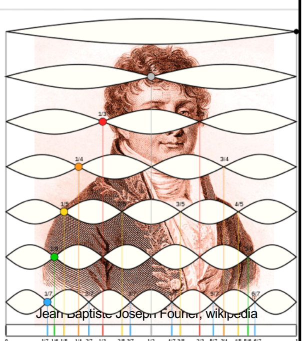


## History…

- **Fourier series:**
  - Any ***periodic*** signal can be represented as a sum of simple periodic functions: *sin* and *cos*

  sin(*nt*) and cos(*nt*)
  where *n* = 1, 2, 3, …

  These are called the ***harmonics*** of the signal







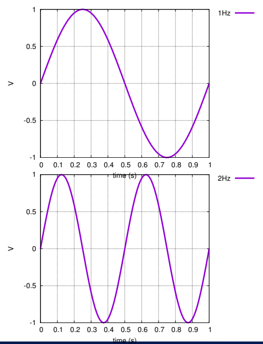

## Fourier Series – More Formally

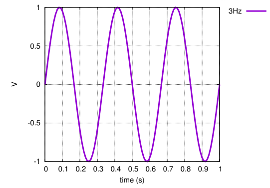
The Fourier Theorem states that any *periodic* function f(t) of period L can be cast in the form:

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

The constants: $a_0$, $a_n$ , and $b_n$ are called the Fourier coefficients of *f(t)*

## Fourier Series – Why does it work?

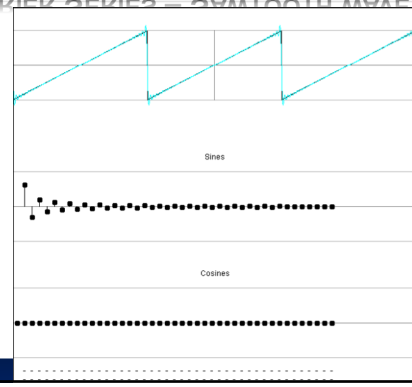



- The cos($nx$) and sin($nx$) functions form an <u>orthogonal basis:</u>
  - allow us to represent any periodic signal by taking a *linear combination*
  - of the basis functions without interfering with one another
  - AKA: superposition works!



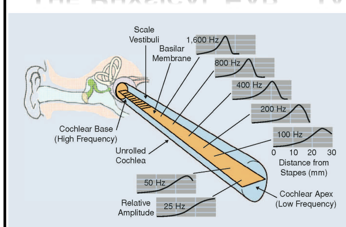
## Fourier Series – Sawtooth wave




(falstad.com/fourie



## The Physical Ear – Take-away




[FIG3] An illustration of an uncoiled cochlea. Due to the greater stiffness and smaller mass, the base of the basilar membrane is tuned to high frequencies while the apex resonates best with the low frequencies. The amplitude of the traveling waves across the membrane shows the frequency-to-place mapping.

- **Cochlea**
  - directly senses frequencies
  - Captures frequency domain
  - …not time domain
- **Frequency sensitive locations**
  - activated by sound waves
- **Neurons sense activation**

Picture above – uncoiled cochlea…
-- different stereovilli (Hairs) resonate at different frequencies
-- our ear work in Frequency Domain.





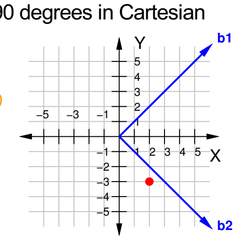
## Vector Background





## Change of Bases

- **There are more than one set of basis vectors that span a space**
  - For example, might rotate 90 degrees in Cartesian coordinates
    - $b1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, $b2 = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$
    - Preclass 3: dotproduct(b1,b2)

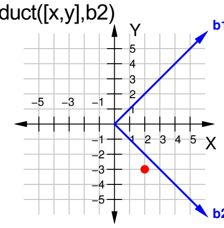







## Change of Bases

- **Can change basis by performing dot product**
  - $b1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, $b2 = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$
  - Represent points as linear combination: a*b1+c*b2
  - a=dotproduct([x,y],b1); c=dotproduct([x,y],b2)
    - What are a and c for case shown?
      - Preclass 3
    - Check correct by seeing that a*b1+c*b2 is what we expect



# The Fourier Transform





## Preclass 1

- **Compute Dot Products – what did we get?**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| time | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| Sample | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 |

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| B1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| B2 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 |
| B3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| B4 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 |
| B5 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |





## Preclass 1

$Sample[k] = \sin\left(\frac{2\pi n \times k}{T}\right)$

- **Note B1 to B5 were sample values from 1, 2, 3, 4, 5 Hz sine waves**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| time | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| Sample | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 |

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n=1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| n=2 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 |
| n=3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| n=4 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 |
| n=5 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |





## Observe

$Sample[k] = \sin\left(\frac{2\pi n \times k}{T}\right)$

- When we compute the dot-product with discrete frequency samples, the only non-zero was the frequency in the signal!

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| time | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| Sample | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0 |

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n=1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| n=2 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 |
| n=3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| n=4 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 |
| n=5 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |





## Observe

- **Can identify frequencies with dot product**
  - Identifying projection onto each basis vector in Fourier Series
- **Works because frequency sine waves are orthogonal**
- **Performing a change of basis**
  - From time-sample basis
  - To Fourier (sine, cosine) basis





## Time and Frequency Bases

- **Time Sample basis**
  - Also a multi-dimensional space
  - Dimension = # time samples
  - Vector [$t_0$,$t_1$,$t_2$,$t_3$, ….]
- **Frequency basis**
  - Multi-dimensional
  - Dimensions = Coefficients of sine and cosine components
  - $f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n2\pi t}{L} + b_n \sin\frac{n2\pi t}{L}\right)$
  - Vector [$a_0$,$a_1$,$b_1$,$a_2$,$b_2$,…]

## Discrete Fourier Transforms

- Fourier Transforms are nice,
  - but we want to store and process our signals with computers
- We extend Fourier Transforms into Discrete Fourier Transforms, or DFT
  - We know our music signal is now discrete: $x(t) \rightarrow f[k]$
  - The signal contains **N** samples: $0 \leq k \leq N-1$





## Warning

- Don't get lost in mathematical notation
- Math is not hard
  - …but compact (dense) with many variables
- k – sample – correspond to a time point
- n -- frequency component
- (note on p2 of preclass so can refer back to)





## DFT – Discrete Fourier Transform

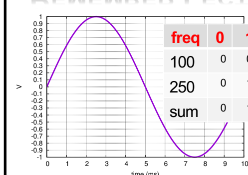
- Represent any sequence of time samples as

$$f[k] = a_0 + \sum_{n=1}^{N/2}\left(a_n \cos\frac{n2\pi k}{N} + b_n \sin\frac{n2\pi k}{N}\right)$$



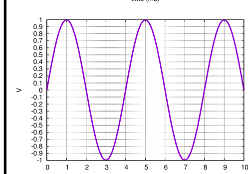


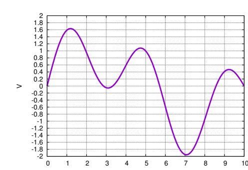
## Remember Lecture 5: Preclass 2



| freq | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 0 | 0.6 | 0.95 | 0.95 | 0.6 | 0 | -0.6 | -0.95 | -0.96 | -0.6 | 0 |
| 250 | 0 | 1.0 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 | 0 |
| sum | 0 | 1.6 | 0.95 | -0.05 | 0.6 | 1 | -0.6 | -1/95 | -0.95 | 0.4 | 0 |

$\sin(100\times2\pi\times t) + \sin(250\times2\pi\times t)$

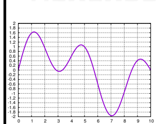








## Remember Week 2: Preclass 3



| freq | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 0 | 0.6 | 0.95 | 0.95 | 0.6 | 0 | -0.6 | -0.95 | -0.96 | -0.6 | 0 |
| 250 | 0 | 1.0 | 0 | -1 | 0 | 1 | 0 | -1 | 0 | 1 | 0 |
| sum | 0 | 1.6 | 0.95 | -0.05 | 0.6 | 1 | -0.6 | -1/95 | -0.95 | 0.4 | 0 |

n

k

$\sin(100\times2\pi\times t) + \sin(250\times2\pi\times t)$

$$f[k] = a_0 + \sum_{n=1}^{N/2}\left(a_n \cos\frac{n2\pi k}{N} + b_n \sin\frac{n2\pi k}{N}\right)$$

$b_{100} = 1$, $b_{500}=1$ … rest $a_i$, $b_i = 0$

[a0,a1,b1,a2,b2,….a500,b500] = [0,0,0,0,….0,1,0,0,0…..,0,1,0,0,…]





## DFT – Discrete Fourier Transform

- Represent any sequence of time samples as

$$f[k] = a_0 + \sum_{n=1}^{N/2}\left(a_n \cos\frac{n2\pi k}{N} + b_n \sin\frac{n2\pi k}{N}\right)$$

- Compute $a_n$, $b_n$ by dot product – preclass 1!
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$

## DFT – Discrete Fourier Transform

- **Compute $a_n$, $b_n$ by dot product – preclass 1!**
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$

k (→), n (↓)

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n=1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| n=2 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 |
| n=3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| n=4 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 | 0.59 | -0.95 | 0.95 | -0.59 | 0.00 |
| n=5 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |





## DFT – Discrete Fourier Transform

- **Represent any sequence of time samples as**

$$f[k] = a_0 + \sum_{n=1}^{N/2}\left(a_n \cos\frac{n2\pi k}{N} + b_n \sin\frac{n2\pi k}{N}\right)$$

- **Compute $a_n$, $b_n$ by dot product**
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$

Note: sum over different dimensions!





## Example Composite

- F(t)=(2/3)sin(3*2π t)+(1/3)sin(2π t)
- Sample[k]=(2/3)sin(3*2π (k/N))+(1/3)sin(2π(k/N))
- Frequency Coefficients
- {a0, b1, a1, b2, a2, b3, a3, b4, a4, b5, a5} = {0, (1/3), 0, 0, 0, (2/3), 0, 0, 0, 0, 0}

$$f[k] = a_0 + \sum_{n=1}^{N/2}\left(a_n \cos\frac{n2\pi k}{N} + b_n \sin\frac{n2\pi k}{N}\right)$$

| freq | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| 3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| sum | 0 | 0.83 | -0.077 | -0.077 | 0.83 | 0 | -0.83 | 0.077 | 0.077 | -0.83 | 0 |





## Example Composite

- F(t)=(2/3)sin(3*2π t)+(1/3)sin(2π t)
- Sample[k]=(2/3)sin(3*2π (k/N))+(1/3)sin(2π(k/N))
- Frequency Coefficients
- {a0, b1, a1, b2, a2, b3, a3, b4, a4, b5, a5} = {0, (1/3), 0, 0, 0, (2/3), 0, 0, 0, 0, 0}

| freq | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| 3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| sum | 0 | 0.83 | -0.077 | -0.077 | 0.83 | 0 | -0.83 | 0.077 | 0.077 | -0.83 | 0 |

Note: Preclass 2 sample vector A





## Preclass 2

- **What did our dot products find?**

| B | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Dot prod | | | | | |

| freq | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.59 | 0.95 | 0.95 | 0.59 | 0.00 | -0.59 | -0.95 | -0.95 | -0.59 | 0.00 |
| 3 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 |
| sum | 0 | 0.83 | -0.077 | -0.077 | 0.83 | 0 | -0.83 | 0.077 | 0.077 | -0.83 | 0 |





## Preclass 2

- **What did our dot products find?**

| B | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Dot prod | | | | | |

- **Same as coefficients if multiply by 2/10**
- **Compute $a_n$, $b_n$ by dot product**
  - $a_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right)\sum_{k=0}^{k=N}\left(Sample[k]\times\sin\left(\frac{n2\pi k}{N}\right)\right)$

## Representations

- **Frequency Domain**
  - {0, (1/3), 0, 0, 0, (2/3), 0, 0, 0, 0, 0}
- **Time Domain**
  - {0,0.83,-0.077,-0.077,0.83,0,-0.83,0.077,0.077,-0.83,0}





## DFT – Discrete Fourier Transform

- **Represent any sequence of time samples as**

$$f[k] = a_0 + \sum_{n=1}^{N/2} \left( a_n \cos \frac{n2\pi k}{N} + b_n \sin \frac{n2\pi k}{N} \right)$$

- **Compute $a_n$, $b_n$ by dot product**
  - $a_n = \left(\frac{2}{N}\right) \sum_{k=0}^{k=N} \left( Sample[k] \times \cos\left(\frac{n2\pi k}{N}\right)\right)$
  - $b_n = \left(\frac{2}{N}\right) \sum_{k=0}^{k=N} \left( Sample[k] \times \sin\left(\frac{n2\pi k}{N}\right)\right)$





## Cosines

- **For simplicity – preclass 1 demonstrated with sine**
  - Show sine of different frequencies orthogonal
- **Also true for cosines**
  - Cosines of different frequencies orthogonal
  - Cosines orthogonal to sines





## Cosines orthogonal by frequency

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | dot product |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 1 | 0.31 | -0.81 | -0.81 | 0.31 | 1.00 | 0.31 | -0.81 | -0.81 | 0.31 | |
| B1 | 1 | 0.81 | 0.31 | -0.31 | -0.81 | -1.00 | -0.81 | -0.31 | 0.31 | 0.81 | |
| product | 1 | 0.25 | -0.25 | 0.25 | -0.25 | -1.00 | -0.25 | 0.25 | -0.25 | 0.25 | 0.00 |
| B2 | 1 | 0.31 | -0.81 | -0.81 | 0.31 | 1.00 | 0.31 | -0.81 | -0.81 | 0.31 | |
| product | 1 | 0.10 | 0.65 | 0.65 | 0.10 | 1.00 | 0.10 | 0.65 | 0.65 | 0.10 | **5.00** |
| B3 | 1 | -0.31 | -0.81 | 0.81 | 0.31 | -1.00 | 0.31 | 0.81 | -0.81 | -0.31 | |
| product | 1 | -0.10 | 0.65 | -0.65 | 0.10 | -1.00 | 0.10 | -0.65 | 0.65 | -0.10 | 0.00 |
| B4 | 1 | -0.81 | 0.31 | 0.31 | -0.81 | 1.00 | -0.81 | 0.31 | 0.31 | -0.81 | |
| product | 1 | -0.25 | -0.25 | -0.25 | -0.25 | 1.00 | -0.25 | -0.25 | -0.25 | -0.25 | 0.00 |
| B5 | 1 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | |
| product | 1 | -0.31 | -0.81 | 0.81 | 0.31 | -1.00 | 0.31 | 0.81 | -0.81 | -0.31 | 0.00 |





## Sine orthogonal cosines

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | dot product |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A (sin) | 0 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | 0.95 | 0.59 | -0.59 | -0.95 | 0.00 | |
| B1 cos | 1 | 0.81 | 0.31 | -0.31 | -0.81 | -1.00 | -0.81 | -0.31 | 0.31 | 0.81 | 1.00 | |
| product | 0 | 0.77 | 0.18 | 0.18 | 0.77 | 0.00 | -0.77 | -0.18 | -0.18 | -0.77 | 0.00 | 0.00 |
| B2 cos | 1 | 0.31 | -0.81 | -0.81 | 0.31 | 1.00 | 0.31 | -0.81 | -0.81 | 0.31 | 1.00 | |
| product | 0 | 0.29 | -0.48 | 0.48 | -0.29 | 0.00 | 0.29 | -0.48 | 0.48 | -0.29 | 0.00 | 0.00 |
| B3 cos | 1 | -0.31 | -0.81 | 0.81 | 0.31 | -1.00 | 0.31 | 0.81 | -0.81 | -0.31 | 1.00 | |
| product | 0 | -0.29 | -0.48 | -0.48 | -0.29 | 0.00 | 0.29 | 0.48 | 0.48 | 0.29 | 0.00 | 0.00 |
| B4 cos | 1 | -0.81 | 0.31 | 0.31 | -0.81 | 1.00 | -0.81 | 0.31 | 0.31 | -0.81 | 1.00 | |
| product | 0 | -0.77 | 0.18 | -0.18 | 0.77 | 0.00 | -0.77 | 0.18 | -0.18 | 0.77 | 0.00 | 0.00 |
| B5 cos | 1 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | 1.00 | -1.00 | 1.00 | |
| product | 0 | -0.95 | 0.59 | 0.59 | -0.95 | 0.00 | 0.95 | -0.59 | -0.59 | 0.95 | 0.00 | 0.00 |





## Complex Numbers

- **Cartesian version:** a+b*i*
- **Polar (Magnitude, angle) version:** $M \times e^{i\theta}$
- Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

## Magnitude

- **If only care about magnitude, we can mix the sine and cosine coefficients together**
  - Take their magnitude: $M=\sqrt{a_n^2+b_n^2}$

$$f[k] = a_0 + \sum_{n=1}^{N/2} \left( a_n \cos \frac{n2\pi k}{N} + b_n \sin \frac{n2\pi k}{N} \right)$$

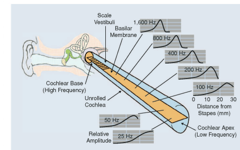
- **That's what we're getting from fft in Matlab when we take magnitude of Complex result**





## Phase / Angle

- **What are we giving up when drop the angle?**
- **Phase – when it starts**
  - Sine vs. Cosine are just shifted in time from each other
    - $\sin(\theta+\pi/2)=\cos(\theta)$
  - The angle is just saying how much shift is between sine and cosine
  - Humans cannot hear phase
  - Ear/Choclea just determining magnitude at frequency, not exactly when it starts




[FIG3] An illustration of an uncoiled cochlea. Due to the greater stiffness and smaller mass, the base of the basilar membrane is tuned to high frequencies while the apex resonates best with the low frequencies. The amplitude of the traveling waves across the membrane shows the frequency-to-place mapping.





## Discrete Fourier Transforms

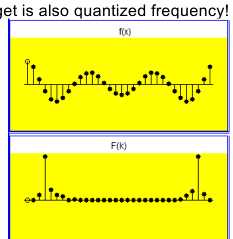
- **A DFT transforms N samples of a signal in time domain**
  - into a (periodic) frequency representation with N samples
  - So we don't have to deal with real signals anymore
- **We work with sampled signals (quantized in time),**
  - and the frequency representation we get is also quantized frequency!

Music Signal in Discrete Time →

$$a_n = \left(\frac{2}{N}\right) \sum_{k=0}^{k=N} \left( Sample[k] \times \cos\left(\frac{n2\pi k}{N}\right) \right)$$

$$b_n = \left(\frac{2}{N}\right) \sum_{k=0}^{k=N} \left( Sample[k] \times \sin\left(\frac{n2\pi k}{N}\right) \right)$$

Music Signal "Transformed" To Frequency Domain →




(sepwww.stanford.edu/oldsep/hale/FftLab.html)



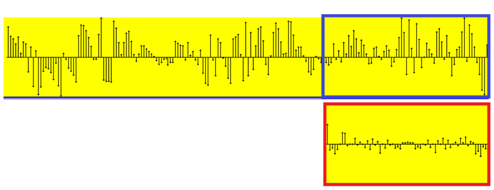
## Saving Resources

- **However, N can get very large**
  - e.g., with a sampling rate of 44,000Hz
  - How big is N for a 4 minute song?
- **How many operations does this translate to?**
  - To compute one frequency component?
  - To compute all N frequency components?
- **This is not practical. Instead, we use a window of values to which we apply the transform**
  - Typical size: …, 512, 1024, 2048, …



## A Window Operation

- **In the example below, we traverse the signal but only look at 64 samples at a time**



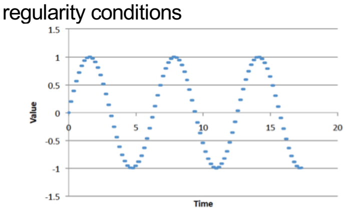

(sepwww.stanford.edu/oldsep/hale/FftLab.html)





## Connect the Dots

- **Intuition, with enough dots, not hard to "connect-the-dots" to reconstruct (understand) the continuous signal.**
  - What is the continuous signal here? (Lecture 1: preclass 3)
  - Assumes certain regularity conditions
  - What is enough?



## Reconstruction

- **Not really connect-the-dots in time**
  - (previous explanation was oversimplified)
- **Recall near Nyquist rate**
  - Could often miss the peak
  - Get poor sine waves
    - …look like peak moves around even if sampled above Nyquist rate
- **Better reconstruction**
  - Convert to frequency
    - Which can perfectly represent up to half sampling rate
  - Reconstruct from frequency basis





## Take-Away

- **Two, complementary ways to represent signals**
  - Time domain, Frequency Domain
- **Can convert between them**
  - There is math too do this
- **Frequencies (sines, cosines) form an orthogonal basis set**
  - Can perform dot products to extract frequency components





## Big Ideas

- **Can represent signals in frequency domain**
  - Different basis – basis vectors of sines and cosines
- **Often more convenient and efficient than time domain**
  - Remember musical staff

- **Human hearing mechanism directly operates on frequencies**
- **Can convert between time and frequency domain**
  - Using a dot-product to calculate time or frequency components

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$$





## Learn More

- **ESE325 – whole course on Fourier Analysis**
- **ESE224 – signal processing**
- **ESE215, 319, 419 – reason about behavior of circuits in time and frequency domains**





## Admin

- **Feedback**
- **Lab 4 writeup due today**
- **Lab 5**
  - Posted
  - In Lab on Monday
- **Wednesday lecture: Masking**





## References


- **S. Smith, "The Scientists and Engineer's Guide to Digital Signal Processing," 1997.**
- [https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/](https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/)