Sampling

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Discrete time signals

Discrete time signals

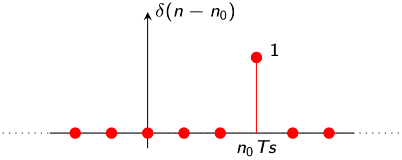
- To infinity, but no beyond $\Rightarrow$ Discrete but infinite time index $n \in \mathbb{Z}$.
- Discrete time signal x is a function mapping $\mathbb{Z}$ to complex value $x(n)$

$$x : \mathbb{Z} \to \mathbb{C} \qquad \text{(values } x(n) \text{ can be, often are, real)}$$

- Sampling time T_s is implicit. Time elapsed from sample n to $n+1$
- So is sampling frequency $f_s = 1/T_s$

- E.g., a shifted delta function $\delta(n - n_0)$ has a spike at time $n = n_0$

$$\delta(n-n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases}$$



- Signal continuous to plus and minus infinity (unlike discrete signals)

Inner product and energy

- Given two signals x and y define the inner product of x and y as

$$\langle x, y \rangle := \sum_{n=-\infty}^{\infty} x(n) y^*(n)$$

- Projection of x on y. How much of x falls in y direction.
- How much x and y are like each other $\Rightarrow$ orthogonality $\equiv$ unrelated
- Define the energy of the signal as the inner product with itself

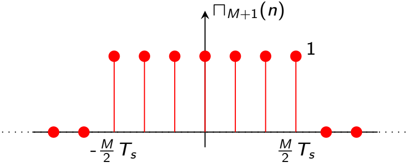
$$\|x\|^2 := \langle x, y \rangle = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x_R(n)|^2 + \sum_{n=-\infty}^{\infty} |x_I(n)|^2$$

- Sums extend to plus and minus infinity (they are series, not sums)
  $\Rightarrow$ Inner product may not exist. Energy may be infinite

Energy and inner products of pulses

- Define square pulse of odd length $M+1$ as signal $\sqcap_{M+1}$ with values

$$\sqcap_{M+1}(n) = 1 \qquad \text{if } -\frac{M}{2} \leq n \leq \frac{M}{2}$$
$$\sqcap_{M+1}(n) = 0 \qquad \text{else } M \leq n$$



- To compute energy of the pulse we just evaluate the definition

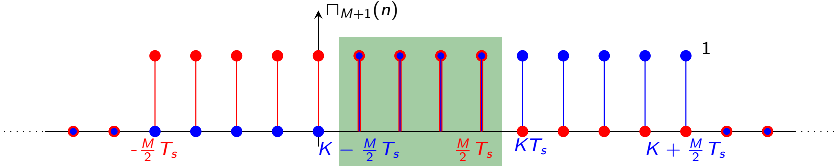
$$\|\sqcap_{M+1}\|^2 := \sum_{n=-\infty}^{\infty} |\sqcap_{M+1}(n)|^2 = \sum_{n=-M/2}^{M/2} (1)^2 = M+1$$

- Can normalize for unit energy as we did for discrete signal case
- But we rather not, as we did for continuous time (to let M grow)

Inner product of a pulse and a shifted pulse

- Inner product of pulse $\sqcap_{M+1}(n)$ and shifted pulse $\sqcap_{M+1}(n-K)$

$$\left\langle \sqcap_{M+1}(n), \sqcap_{M+1}(n-K) \right\rangle = \sum_{n=-\infty}^{\infty} \sqcap_{M+1}(n) \sqcap_{M+1}(n-K)$$



- For shifts $0 \leq K \leq M+1$, signals overlap for $K - M/2 \leq n \leq M/2$

$$\left\langle \sqcap_{M+1}(n), \sqcap_{M+1}(n-K) \right\rangle = \sum_{n=K-M/2}^{M/2} (1)(1) = (M+1) - K$$

- Proportional to overlap $\Rightarrow$ how much pulses "are like each other"

Discrete time Fourier transform

The discrete time Fourier transform (DTFT)

- The DTFT of discrete signal x is the function $X : \mathbb{R} \to \mathbb{C}$ with values

$$X(f) := T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$$

- Denote as $X = \mathcal{F}(x)$. Argument f is continuous and called frequency
- Sum need not exist $\Rightarrow$ Not all discrete time signals have a DTFT
- Definition depends on sampling time T_s. Facilitates connections later

- Fourier transform (FT) has continuous input and continuous output
- DFT is also well matched $\Rightarrow$ It has discrete input and discrete output
- DTFT is mismatched $\Rightarrow$ It has discrete input but continuous output
  - $\Rightarrow$ A little odd, but of little consequence

DTFT is also an inner product

- Define e_{fT_s} with values $e_{fT_s}(n) = T_s e^{j2\pi f n T_s}$. Write as inner product

$$X(f) = \langle x, e_{fT_s} \rangle = T_s \sum_{n=-\infty}^{\infty} x(n) e_{fT_s}^*(n)$$

- As in the case of the FT and the DFT, the DTFT value $X(f)$:
  - $\Rightarrow$ Is the projection of x onto discrete oscillation of freq. f
  - $\Rightarrow$ Measures how much $x(n)$ resembles discrete oscillation of freq. f
- Conceptually identical to FT & DFT $\Rightarrow$ Why a third definition?
  - $\Rightarrow$ All three, discrete time, discrete, and continuous signals exist
  - $\Rightarrow$ Deep connections between FT and DTFT and DTFT and DFT
- Analytical tool (as the FT). Not a computational tool (as the DFT)

Periodicity of DTFT

Theorem

The DFTF $X = \mathcal{F}(x)$ of discrete time signal x is periodic with period f_s

$$X(f + f_s) = X(f), \qquad \text{for all } f \in \mathbb{R}.$$

- Any frequency interval of length f_s contains all DTFT information
  $\Rightarrow$ We will use the canonical set $\Rightarrow f \in [-f_s/2, f_s/2]$
- For sampling time T_s, freqs. larger than $f_s/2$ have no physical meaning
  $\Rightarrow$ Frequency $-f$ is (more or less) the same as frequency f

Proof of periodicity property

Proof.

- Use the DTFT definition to write $X(f + f_s)$ as

$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi (f+f_s) n T_s}$$

- Separate the complex exponential in two factors

$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} e^{-j2\pi f_s n T_s}$$

- Use $f_s T_s = 1$ in last factor $\Rightarrow e^{-j2\pi f_s n T_s} = e^{-j2\pi n} = \left(e^{j2\pi}\right)^{-n} = 1$

- Substitute in previous expression and observe definition of DTFT

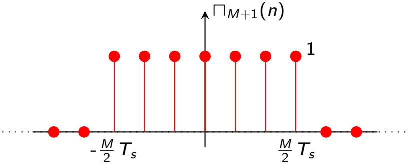
$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = X(f) \qquad \square$$

DTFT of a square pulse

- Consider square pulse of odd length $M+1$

$$\sqcap_{M+1}(n) = 1 \qquad \text{if } -\frac{M}{2} \leq n \leq \frac{M}{2}$$

$$\sqcap_{M+1}(n) = 0 \qquad \text{else } M \leq n$$



- To compute the pulse DTFT $X = \mathcal{F}(\sqcap_{M+1})$ evaluate the definition

$$X(f) = T_s \sum_{n=-\infty}^{\infty} \sqcap_{M+1}(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-M/2}^{M/2} e^{-j2\pi f n T_s}$$

- Write down the individual elements of the sum to express DTFT as

$$\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+1\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-1\right)T_s} + e^{j2\pi f\left(\frac{M}{2}\right)T_s}$$

DTFT of a square pulse (computation, 1 of 2)

- Multiply by $e^{j2\pi f\left(\frac{1}{2}\right)T_s}$ and $e^{j2\pi f\left(-\frac{1}{2}\right)T_s}$ to write the equalities

$$e^{j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{3}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s}$$

$$e^{-j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{3}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s}$$

- First term in first row = second term in second row
- Second term in first row = third term in second row (unseen)

⋮

- Penultimate term in first row = last term in second row

- Subtracting second row from first row only two terms survive
  - ⇒ The last term in the first row and the first term in the second row

DTFT of a square pulse (computation, 1 of 2)

- Multiply by $e^{j2\pi f\left(\frac{1}{2}\right)T_s}$ and $e^{j2\pi f\left(-\frac{1}{2}\right)T_s}$ to write the equalities

$$e^{j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{3}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s}$$

$$e^{-j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{3}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s}$$

- First term in first row = second term in second row
- Second term in first row = third term in second row (unseen)

⋮

- Penultimate term in first row = last term in second row

- Subtracting second row from first row only two terms survive
  - ⇒ The last term in the first row and the first term in the second row

DTFT of a square pulse (computation, 2 of 2)

- Implementing the subtraction results in the equality

$$\frac{X(f)}{T_s}\left[e^{j2\pi f\left(\frac{1}{2}\right)T_s} - e^{-j2\pi f\left(\frac{1}{2}\right)}T_s\right] = e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s} - e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s}$$

- Complex exponentials are conjugate. Subtraction cancels real parts
- We keep imaginary parts only, which are sines

$$\frac{X(f)}{T_s}\left[2j\sin\left(2\pi f\left(\frac{1}{2}\right)T_s\right)\right] = 2j\sin\left(2\pi f\left(\frac{M+1}{2}\right)T_s\right)$$

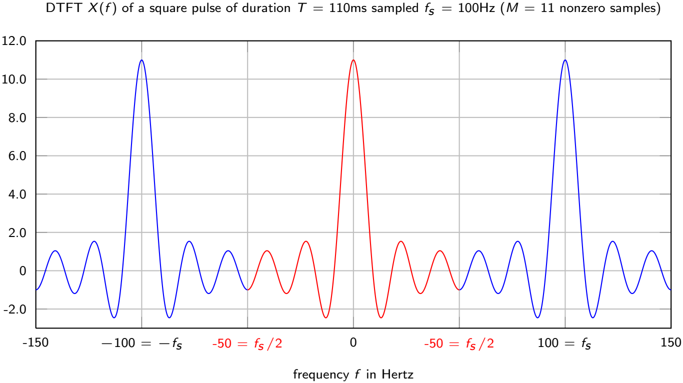
- Solve for $X(f)$ and simplify terms. Pulse length $T = (M+1)T_s$

$$X(f) = T_s\frac{\sin\left(\pi f\,(M+1)\,T_s\right)}{\sin\left(\pi f\,T_s\right)} = T_s\frac{\sin\left(\pi f\,T\right)}{\sin\left(\pi f\,T_s\right)}$$

- A slow sine over a fast sine $\Rightarrow$ not unlike a sinc pulse

Evaluation of the DTFT of a square pulse

- Sampling freq. $f_s = 100$Hz. Pulse length in time $T = 110$ms pulse
  $\Rightarrow$ Resulting in $M + 1 = 11$ nonzero samples

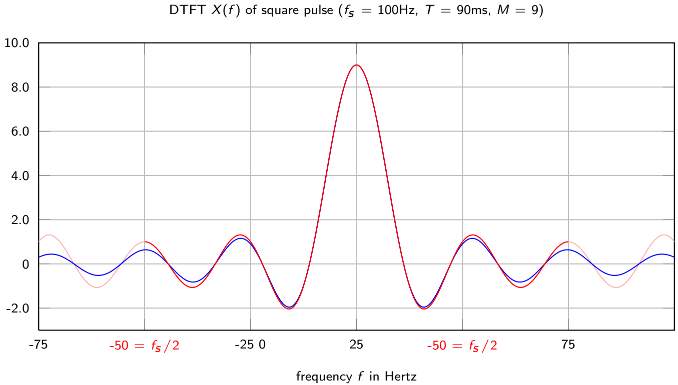


- DTFT is periodic, as we know it should. Focus on $f \in [-fs/2, f_s/2]$

The DTFT of a square pulse and the sinc pulse

- Similar to the sinc pulse $\Rightarrow T\frac{\sin(\pi f T)}{\pi f T} = T \text{sinc}(\pi f T)$

- Fourier transform of unsampled pulse

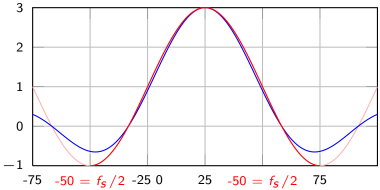


- Some difference for f close to $\pm f_2/2$. Also, sinc is not periodic

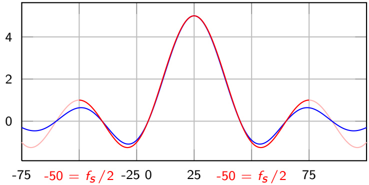
Pulses of different length

- As the pulse widens, the DTFT concentrates. Same as FT and DFT
- As pulse widens difference with FT of continuous time pulse diminishes

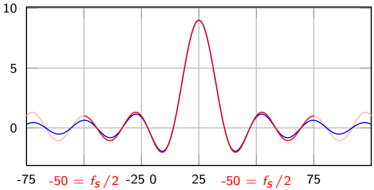
DTFT $X(f)$ of square pulse ($f_s = 100$Hz, $T = 30$ms, $M = 3$)



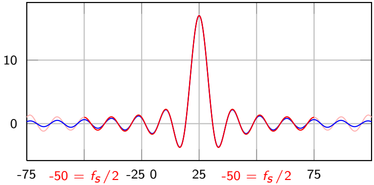
DTFT $X(f)$ of square pulse ($f_s = 100$Hz, $T = 50$ms, $M = 5$)



DTFT $X(f)$ of square pulse ($f_s = 100$Hz, $T = 90$ms, $M = 9$)



DTFT $X(f)$ of square pulse ($f_s = 100$Hz, $T = 170$ms, $M = 17$)



The FT and the DTFT

- Interpret signal $x(n)$ as samples $x_C(nT_s)$ of continuous signal $x_C(t)$
- DTFT $X = \mathcal{F}(x)$ is Riemann sum approximation of FT $X_C = \mathcal{F}(x_C)$

$$X_C(f) \; = \; \int_{-\infty}^{\infty} x_C(t) e^{-j2\pi ft} dt \approx T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fnT_s} \; = \; X(f)$$

- Only frequencies between $\pm f_s/2$ have meaning in DTFT $\Rightarrow$ Chop
- FT $X_C(f)$ $\Rightarrow$ sample in time, chop in frequency $\Rightarrow$ DTFT $X(f)$

The DTFT and the DFT

- Chop x to $n \in [0, N-1]$ $\Rightarrow$ Discrete signal x_D with DFT $X_D = \mathcal{F}(x_D)$

- If elements discarded from x are small

$$X(f) \;=\; T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} \approx T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi f n T_s}$$

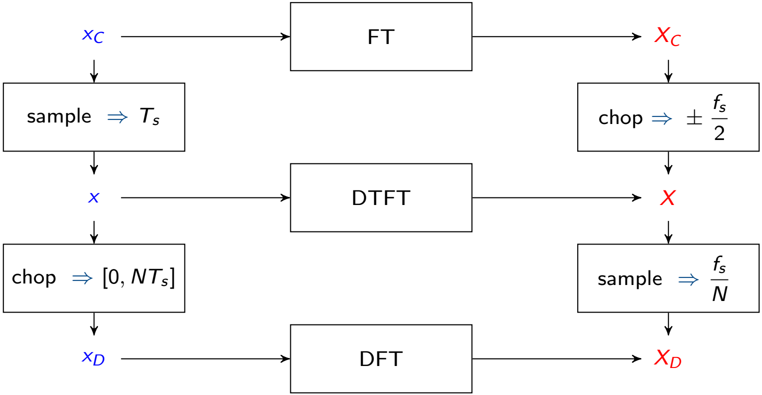
- True for all frequencies f. Sample in frequency at $f = (k/N)f_s$

$$X\left(\frac{k}{N} f_s\right) \approx T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi (k/N) f_s n T_s} \;=\; T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi kn/N} \;=\; T_s \sqrt{N}\, X_D(k)$$

- DTFT $\Rightarrow$ Chop in time, sample in frequency $\Rightarrow$ DFT

The FT, the DTFT, and the DFT

- The DTFT bridges FT and DFT by dual sample and chopping



- The argument was careless though $\Rightarrow$ We will probe deeper

The inverse (i)DTFT

- The iDTFT x of DTFT X, is the discrete time signal with elements

$$x(n) := \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df$$

- We denote $x = \mathcal{F}^{-1}(X)$. Sampling time T_s (freq. f_s) implicit in X
- Sign in exponent changes with respect to DTFT.

- DTFT is an indefinite sum but iDTFT is a definite integral
  $\Rightarrow$ DTFT mismatch. Odd, but of little consequence

- Since DTFT X is periodic, any interval of width f_s does it. E.g.

$$x(n) = \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df = \int_{0}^{f_s} X(f) e^{j2\pi f n T_s} \, df$$

Indeed, the iDTFT is the inverse of the DTFT

Theorem

The iDTFT $\tilde{x}$ of the DTFT X of the discrete time signal x is the signal x

$$\tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x.$$

- What a surprise. It's getting tired. But this is the last one.
- As usual, discrete time signals can be written as sums of oscillations

$$x(n) = \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s}\, df \approx (\Delta f) \sum_{n=-N/2}^{N/2} X(f_k) e^{j2\pi f_k n T_s}$$

- Conceptual; cf. continuous signals. Not literal; cf. discrete signals.

Proof of inverse Fourier transform

Proof.

- We want to show $\Rightarrow \tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x$. Use definitions
- Definition of inverse transform of $X \Rightarrow \tilde{x}(\tilde{n}) := \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f \tilde{n} T_s}\, df$
- From definition of transform of $x \Rightarrow X(f) := T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$
- Substituting expression for $X(f)$ into expression for $\tilde{x}(\tilde{n})$ yields

$$\tilde{x}(\tilde{n}) = \int_{-f_s/2}^{f_s/2} \left[T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} \right] e^{j2\pi f \tilde{n} T_s}\, df$$

- Same as done for iDFT and iFT but with one integral and one sum

Proof of inverse Fourier transform

Proof.

- Exchange integration with sum $\Rightarrow$ Integrate first over f, then sum over n

$$\tilde{x}(\tilde{n}) = T_s \sum_{n=-\infty}^{\infty} x(n) \left[\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s} \, df \right]$$

- Pulled $x(n)$ out because it doesn't depend on f

- Up until now we repeated steps we already did for iDFT and iFT
  $\Rightarrow$ They worked for iDFT but didn't for iFT $\Rightarrow$ They work here.

- The innermost integral we have computed repeatedly $\Rightarrow$ It's a sinc

$$\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s} \, df = f_s \text{sinc}(\pi f_s (n - \tilde{n}) T_s) = f_s \text{sinc}(\pi (n - \tilde{n}))$$

- We used $f_s T_s = 1$ in second equality. Recall that n and $\tilde{n}$ are discrete

Proof of inverse Fourier transform

Proof.

- Evaluate sinc for $n = \tilde{n} \;\Rightarrow f_s \text{sinc}(\pi(n - \tilde{n})) = f_s$ because $\text{sinc}(0) = 1$
- Evaluate sinc for $n \neq \tilde{n} \;\Rightarrow f_s \text{sinc}(\pi(n - \tilde{n})) = 0$ because $\text{sinc}(k\pi) = 0$
- Lucky for us, the innermost integral was a delta function in disguise

$$\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s} \, df = f_s \delta(n - \tilde{n})$$

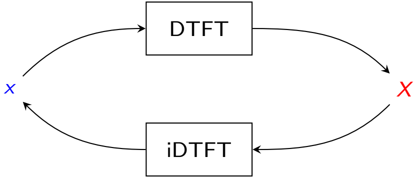
- Substituting in expression for $\tilde{x}(\tilde{n})$, only one term in sum is not null

$$\tilde{x}(\tilde{n}) = T_s f_s \sum_{n=-\infty}^{\infty} x(n) \delta(n - \tilde{n}) = x(\tilde{n})$$

- Also used $f_s T_s = 1$. Since we have $\tilde{x}(\tilde{n}) = x(\tilde{n})$ for all $\tilde{n} \;\Rightarrow \tilde{x} \equiv x$ □

From time to time frequency and back

- If a discrete signal x has a DTFT X, its DTFT has an iDTFT
  - $\Rightarrow$ The iDTFT of the DTFT X recovers original signal x
- The DTFT is a transformation without loss of information
  - $\Rightarrow$ Can always come back from frequency domain to time domain



- True of DFT–iDFT and FT–iFT as well. Hadn't need to mention yet

DTFT of a constant

The DTFT of a constant

- Discrete time constant x has value $x(n) = 1$ for all n. The DTFT is

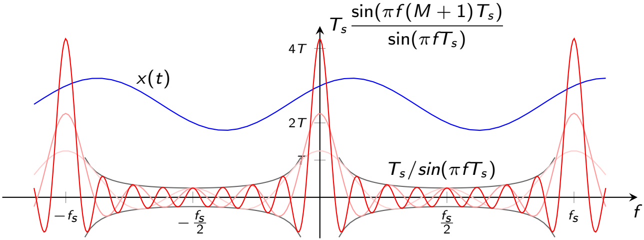
$$X(f) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s}$$

- It does not exist. For $n = 0$, $X(f) \to \infty$, for other n oscillates
- We know how to solve this problem $\Rightarrow$ Use delta function
- Write constant as pulse limit. DTFT of pulse we saw is ratio of sines
- Then, can think of writing DTFT of constant as the limit

$$X(f) = \lim_{M\to\infty} T_s \sum_{n=-M/2}^{M/2} e^{-j2\pi f n T_s} = \lim_{M\to\infty} T_s \frac{\sin(\pi f (M+1) T_s)}{\sin(\pi f T_s)}$$

- Except that it is this limit the one that does not exist

- As M grows, DTFT grows and narrows around $f = 0$. And $f = \pm kf_s$
  - $\Rightarrow$ But it doesn't decrease for other frequencies



- But when multiplying by $Y(f)$ and integrating we recover $Y(0)$

$$\lim_{M\to\infty} \int_{-f_s/2}^{f_s/2} Y(f)\, T_s \frac{\sin(\pi f(M+1)T_s)}{\sin(\pi f T_s)}\, df = Y(0)$$

- Define (already did) delta function as the entity with this property

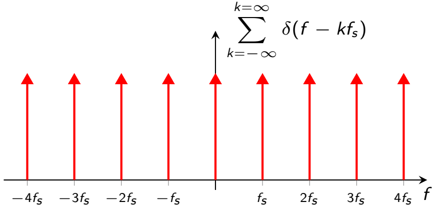
The Dirac train

- The delta function δ is a generalized function such that for all Y

$$\int_{-\infty}^{\infty} Y(f)\delta(f)\,df = Y(0)$$

- We can then *define* the DTFT of a constant as a delta function
- Almost correct, but observe that we also have peaks at $f = \pm k f_s$
- The DTFT of a constant is then defined as

$$X(f) = \sum_{k=-\infty}^{k=\infty} \delta(f - kf_s)$$



- We call this signal a train of deltas, a Dirac train, or a Dirac comb

What it means? Does it make sense?

- Informally $\Rightarrow \delta(f) = \infty$ for $f = 0$, $f = \pm f_s$, $f = \pm 2f_s$, $\ldots$
  $\Rightarrow \delta(f) = 0$ for all other f

- Mathematically, only has sense after multiplication and integration

$$\int_{-\infty}^{\infty} Y(f)X(f)\, df = \int_{-\infty}^{\infty} Y(f) \sum_{k=-\infty}^{k=\infty} \delta(f - kf_s)\, df = \sum_{k=-\infty}^{k=\infty} Y(f - kf_s)$$

- Recovers the values of $Y(f)$ at the points where the train has spikes

- In particular, the iDTFT recovers the constant

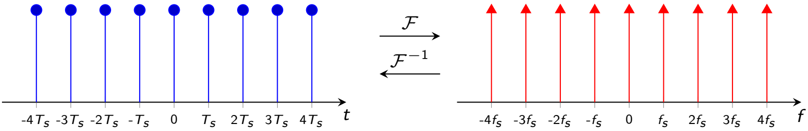
$$\int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s}\, df = \int_{-f_s/2}^{f_s/2} \sum_{k=-\infty}^{k=\infty} \delta(f - kf_s) e^{j2\pi f n T_s}\, df = e^{j2\pi 0 n T_s} = 1$$

- Definition makes sense $\Rightarrow$ Preserves consistency of DTFT analyses

Fourier transform of a Dirac train

The constant - Dirac train non-pair

- DTFT of a constant is a Dirac train $\Rightarrow$ suspiciously similar



- Can we use duality to say the FT of a train is another train?
  $\Rightarrow$ Not quite. Left signal is discrete. Right signal is continuous

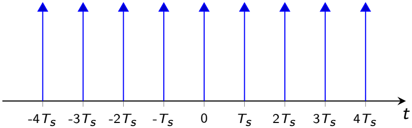
- Not a transform pair $\Rightarrow$ Can't define Dirac train in discrete time
  $\Rightarrow$ Definition of delta functions relies on integration

- But we are on to something

A Dirac train in the time domain

- For continuous time index t define continuous signal x as

$$x_C(t) = T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



- This signal is a Dirac train in time. Not a discrete time constant

- Being continuous, the Dirac train has a Fourier transform X_C

$$X_C(f) = \int_{-\infty}^{\infty} x_C(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \left[T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] e^{-j2\pi ft}\, dt$$

- Can be related to the DTFT of a discrete time constant

DTFT of a constant $\equiv$ FT of a Dirac train

- Exchange order of sum and integration, use delta function definition

$$X_C(f) = T_s \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j2\pi f t}\, dt \right] = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s}$$

- The sum on the right is the DTFT of a constant

$$X(f) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s}$$

- The DTFT of a constant and the FT of a Dirac train coincide

$$X_C(f) = X(f) = \sum_{k=-\infty}^{\infty} \delta(t - kf_s)$$

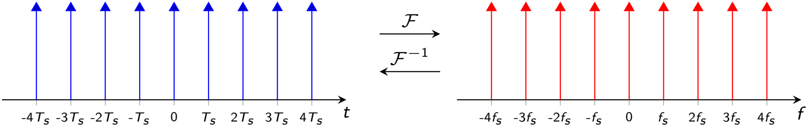
- Both are a Dirac trains in frequency with spacing f_s

The Dirac train - Dirac train FT pair

- FT of Dirac train with spacing T_s is a Dirac train with spacing f_s

$$x_C(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad \Longleftrightarrow \qquad X_C(f) = \sum_{k=-\infty}^{\infty} \delta(t - kf_s)$$

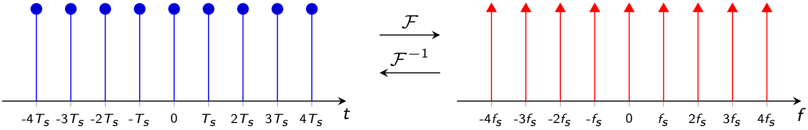
- The set of Dirac trains is an invariant class with respect to the FT



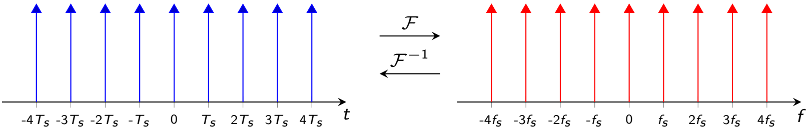
- This is a Fourier transform pair because both are continuous signals

Fundamentally different but equal

- Discrete time constant sampled at T_s $\Rightarrow$ DTFT $\Rightarrow$ Dirac train spaced f_s



- Dirac train spaced every T_s $\Rightarrow$ FT $\Rightarrow$ Dirac train spaced every f_s

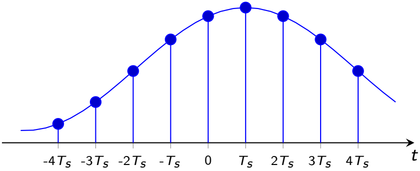


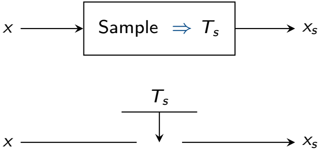
- Discrete time constant fundamentally different from continuous time train
- Thus, DFTF of constant fundamentally different from FT of Dirac train
- But they coincide $\Rightarrow$ Something deeper is at play here

- Consider continuous time signal x and sampling time T_s (freq. f_s)
- The sampled signal x_s is a discrete time signal with values

$$x_s(n) = x(nT_s)$$

- Creates discrete time signal x_s from continuous time signal x
- We've been doing this since first day. We want to understand it now
  $\Rightarrow$ Information lost from x when discarding all but samples $x(nT_s)$?





Sampling as multiplication by a Dirac train

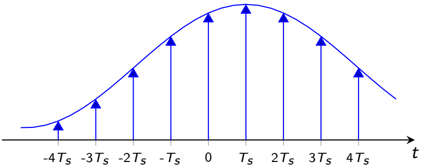
- Equivalently, we represent sampling as multiplication by a Dirac train

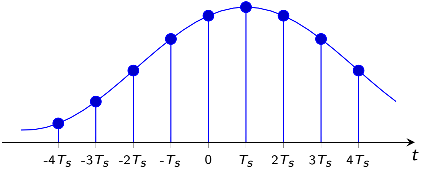
$$x_\delta(t) = x(t) \times T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- Indeed, since the only value that is relevant for $\delta(t - nT_s)$ is $x(nT_s)$

$$x_\delta(t) = T_s \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

- We can construct x_s if given x_δ and construct x_δ if given x_s





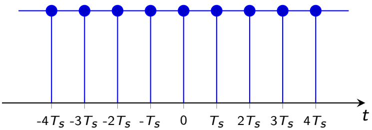
DTFT & FT of sampled signals coincide

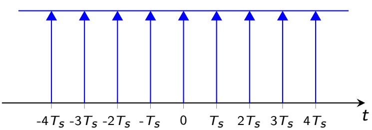
Theorem

The DTFT $X_s = \mathcal{F}(x_s)$ of the sampled signal x_s and the FT $X_\delta = \mathcal{F}(x_\delta)$ of the Dirac sampled signal x_δ coincide

$$X_\delta(f) = X_s(f)$$

- True for all freqs., not just between $\pm f_s/2$. FT $X_\delta(f)$ is periodic
- We already saw this property for sampling continuous time constants
  $\Rightarrow$ Discrete time constant and Dirac train





DTFT & FT of sampled signals coincide (proof)

Proof.

- Write the definition of the FT $X_\delta = \mathcal{F}(x_\delta)$ of Dirac sampled signal

$$X_\delta(f) = \int_{-\infty}^{\infty} \left[T_s \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)e^{-j2\pi f t} \right] df$$

- Exchange the order of summation and integration

$$X_\delta(f) = T_s \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(nT_s)\delta(t - nT_s)e^{-j2\pi f t}\, df \right]$$

- Multiplying by delta and integrating recovers value at spike. Thus,

$$X_\delta(f) = T_s \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j2\pi f nT_s} = T_s \sum_{n=-\infty}^{\infty} x_s(n)e^{-j2\pi f nT_s} = X_s(f)$$

- We use $x_s(n) = x(nT_s)$ and definition of DTFT in last two equalities □

Product in time $\Leftrightarrow$ convolution in frequency

- When we convolve signals in time we multiply their spectra
- Duality $\Rightarrow$ When we multiply them in time we convolve their spectra
  $\Rightarrow$ Don't need to prove. It has to be true because iFT is like an FT
- We obtain Dirac sampled signal x_δ by multiplying x with Dirac train

$$x_\delta(t) = x(t) \times T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- Spectrum X_δ is convolution of $X = \mathcal{F}(x)$ with the FT of Dirac train

$$X_\delta = X * \mathcal{F}\left[T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

- Fourier transform of the Dirac train (T_s) is another Dirac train (f_s)

The spectrum of the Dirac sampled signal

- Spectrum X_δ convolves X with a Dirac train with spacing f_s

$$X_\delta = X * \left[\sum_{k=-\infty}^{\infty} \delta(t - kf_s) \right]$$

- But convolution is a linear operation $\Rightarrow X_\delta = \sum_{k=-\infty}^{\infty} X * \delta(f - kf_s)$

- Convolving with shifted delta is a shift $\Rightarrow X_\delta(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$

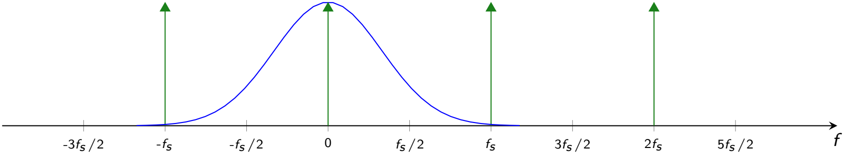
Theorem

Spectrum of sampled signal is a sum of shifted versions of original spectrum

$$X_s(f) = X_\delta(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

Spectrum periodization

- We start with the spectrum X of x and the Dirac train in frequency
- Sampling to create x_s $\Rightarrow$ Multiplication with time Dirac train (T_s)
- Which in frequency domain entails convolution with Dirac train (f_s)
- Which is equivalent to summing shifted copies of the spectrum X



- FT X of continuous time signal x

Spectrum periodization

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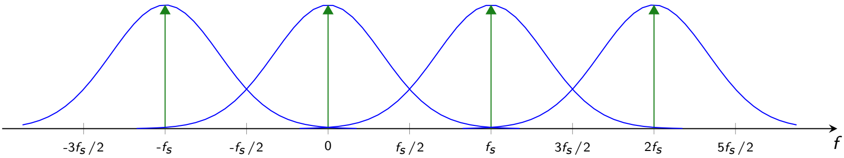

- First convolution step is to duplicate and shift spectrum to kf_s

Spectrum periodization

- We start with the spectrum X of x and the Dirac train in frequency
- Sampling to create x_s $\Rightarrow$ Multiplication with time Dirac train (T_s)
- Which in frequency domain entails convolution with Dirac train (f_s)
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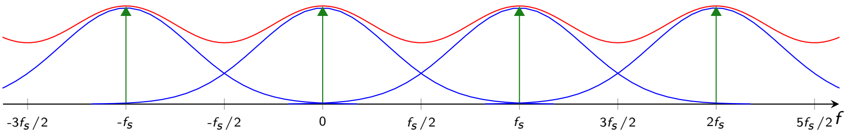
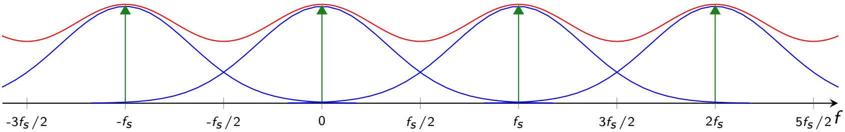


- Second convolution step is to sum all shifted copies

Information loss

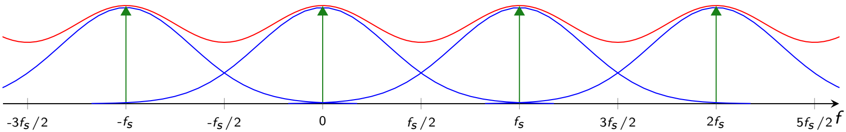
- When sampling x to x_s we lose information at high frequencies
  - $\Rightarrow$ Everything that happens above $f_s/2$ is lost
  - $\Rightarrow$ Freqs. close to $f_s/2$ distorted by superposition with freqs. above $f_s/2$



- We say that the sampling process results in spectral aliasing
  - $\Rightarrow$ When f_s is small, severe aliasing destroys all information

Increasing sampling time

- As we increase the sampling time, aliasing becomes less severe



- Aliasing eventually disappears $\Rightarrow$ Approximately true in general
- But exactly true for bandlimited signals.
  - $\Rightarrow$ Signals with $X(f) = 0$ for $f \notin [-W/2, W/2]$ (bandwidth W)

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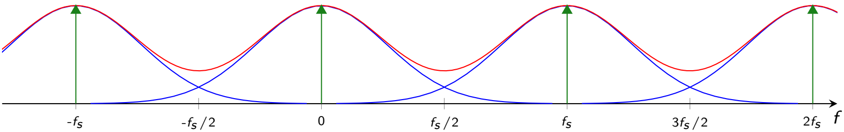

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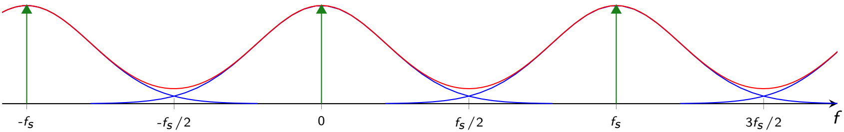

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- But exactly true for bandlimited signals.
  - $\Rightarrow$ Signals with $X(f) = 0$ for $f \notin [-W/2, W/2]$ (bandwidth W)

Sampling of bandlimited signals

- We have therefore proved the following theorem

Theorem

Let x be a signal of bandwidth W. If the signal is sampled at a frequency $f_s \geq W$ we have that

$$X_\delta(f) = X_s(f) = X(f)$$

for all frequencies $f \in [-W/2, W/2]$

- There is no loss of information $\Rightarrow$ We can recover x from x_δ
- Use low pass filter to remove all frequencies outside of $[-W/2, W/2]$

Discussions

Discrete time signals

Discrete time Fourier transform

Inverse discrete time Fourier transform

DTFT of a constant

Fourier transform of a Dirac train

Sampling

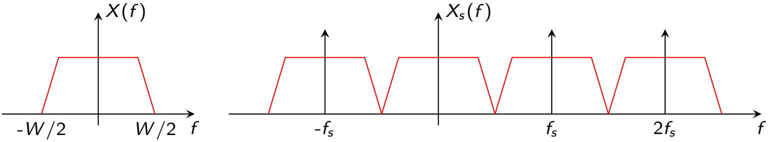
Discussions

Signal reconstruction

From the FT to the DFT

Sampling of bandlimited signals

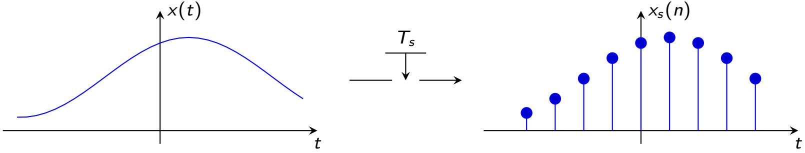
- Signal with bandwidth $W \Rightarrow X(f) = 0$ for all $f \notin [-W/2, W/2]$
- Upon sampling, spectrum is periodized but not aliased



- This means that sampling entails no loss of information
  - $\Rightarrow$ Can low pass x_s to recover x.

Non-vanishing sampling time

- That there is no loss of information is quite surprising
- We are discarding part of the signal, indeed, most of the signal



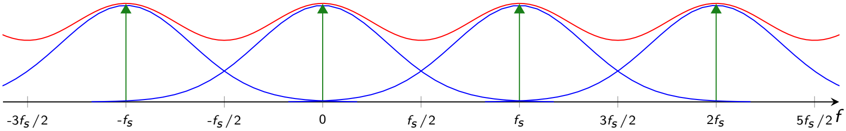
- It is reasonable to expect that we don't lose information as $T_s \to 0$

  $\Rightarrow$ But we don't have to let the sampling time vanish

- Any sampling time $T_s \leq \dfrac{1}{W}$ yields $f_s \geq W$ and no information loss

Sampling of non-bandlimited signals

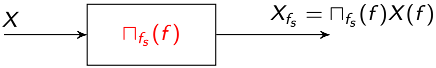
- Information in frequency components larger than $f_s/2$ is lost
  - $\Rightarrow$ Nothing we can do about that other than increasing f_s
- Can't capture variability faster than $f_s/2$ with sampling time T_s



- But aliasing is also distorting information in components below $f_s/2$

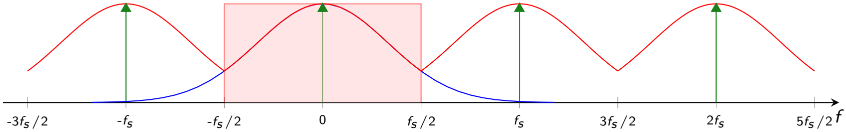
- To avoid aliasing distortion we preprocess x with a low pass filter
- I.e., we transform x into signal x_{f_s} with spectrum $X_{f_s} = \mathcal{F}(x_{f_s})$

$$X_{f_s}(f) = X(f)\sqcap_{f_s}(f)$$



- The signal x_{f_s} has bandwidth f_s and can be sampled without aliasing

  $\Rightarrow$ Frequency components below $f_s/2$ are retained with no distortion

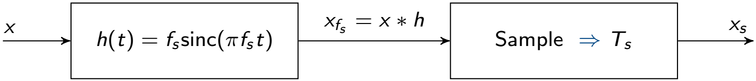


Prefiltering in time domain

- Prefiltering can be implemented as convolution in the time domain

$$x_{f_s} = x * h$$

- where h is iFT of low pass filter $X(f)\sqcap_{f_s}$ $\Rightarrow$ $h(t) = f_s \text{sinc}(\pi f_s t)$

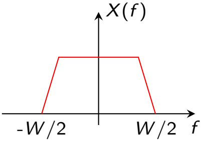


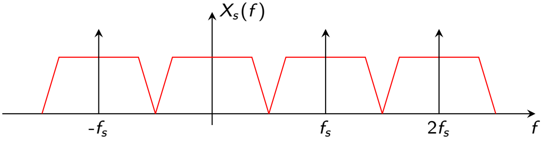
- Convolution has to be implemented in continuous time (circuits)

Signal reconstruction

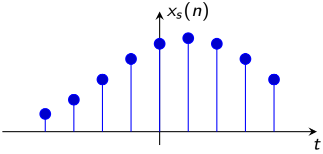
Low pass filter recovery

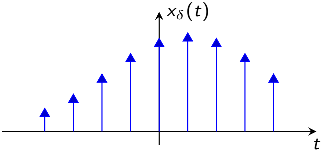
- Bandwidth W ($X(f) = 0$ for all $f \notin [-W/2, W/2]$). Sample at $f_s \geq W$
- Can recover signal x from sampled signal x_s with low pass filter
  $\Rightarrow$ What does exactly mean that "we use a low pass filter"?





- Can't filter discrete time signal and have continuous time magically appear

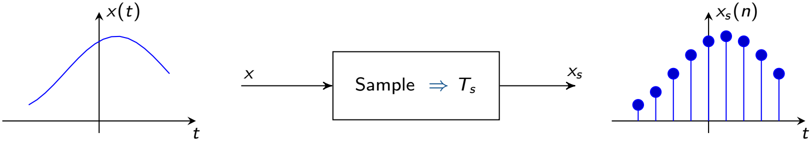




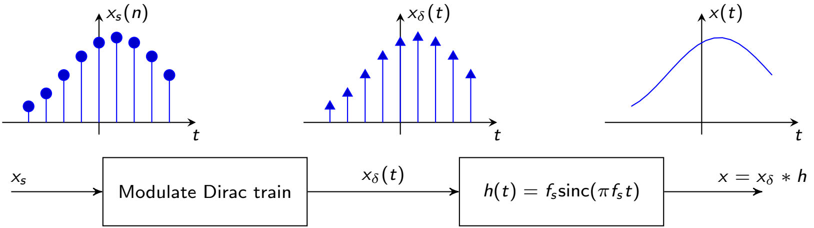
- But we can filter the continuous time Dirac sampled signal $x_\delta(t)$

Ideal sampling – reconstruction system

- We sample by keeping observations at nT_s $\Rightarrow x_s(n) = x(nT_s)$



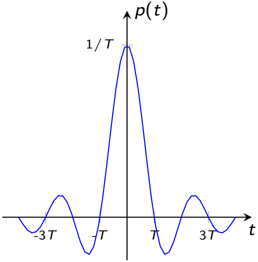
- To reconstruct we modulate Dirac train $\Rightarrow x_\delta(t) = T_s \sum_{n=-\infty}^{\infty} x_s(n)\delta(t - nT_s)$
- And low pass filter Dirac train x_δ $\Rightarrow x = x_\delta * \left[f_s \text{sinc}(\pi f_s t)\right]$



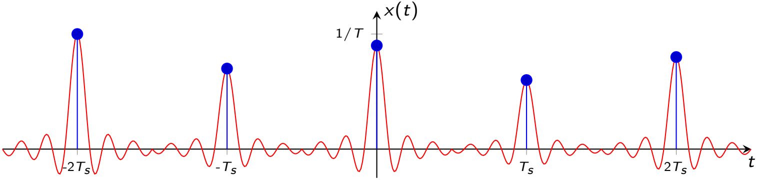
- Dirac train is an abstract representation $\Rightarrow$ Can't be generated
- Modulate train of (narrow) pulses

$$x_p(t) = T_s \sum_{n=-\infty}^{\infty} x_s(n) p(t - nT_s)$$

- If pulse is sufficiently narrow $\Rightarrow x_p \approx x_\delta$
- E.g. $p(t) = \frac{1}{T} \text{sinc}\left(\pi \frac{t}{T}\right)$ with $T \ll T_s$



- Scale pulse by $x(n)$, shift to $t = nT_s$, sum all copies $\Rightarrow$ convolution?



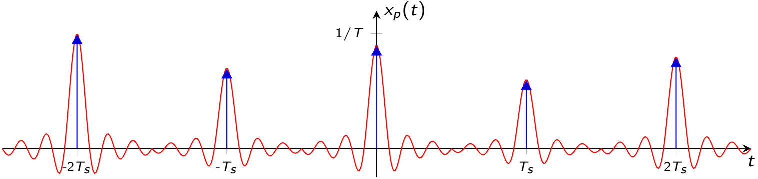
- Pulse train modulation can be represented as convolution with x_δ

$$x_p = p * x_\delta$$

- Indeed use definition of x_δ and convolution linearity to write $p * x_\delta$ as

$$x_p = p * \left[T_s \sum_{n=-\infty}^{\infty} x_s(n)\delta(t - nT_s) \right] = T_s \sum_{n=-\infty}^{\infty} x_s(n) \Big[p * \delta(t - nT_s) \Big]$$

- Convolving with shifted delta is a shift $\Rightarrow x_p(t) = T_s \sum_{n=-\infty}^{\infty} x_s(n) p(t - nT_s)$



Spectrum of modulated pulse train

- Convolution in time is equivalent to multiplication in frequency
- Then, the spectrum of $X_p = \mathcal{F}(x_p)$ is the product of $P = \mathcal{F}(p)$ and X_δ

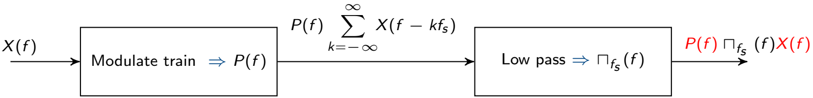
$$X_p(f) = P(f)X_\delta(f) = P(f)\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

- Reconstructed signal x_r obtained by low pass filtering. FT $X_r = \mathcal{F}(x_r)$ is

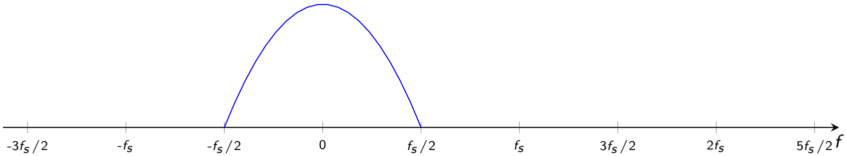
$$X_r(f) = P(f)X_\delta(f)\sqcap_{f_s}(f) = P(f)\sqcap_{f_s}(f)\sum_{k=-\infty}^{\infty} X(f - kf_s)$$

- Low pass filter eliminates all frequencies outside of $[-f_s/2, f_s/2]$

$$X_r(f) = P(f)\sqcap_{f_s}(f)X(f)$$



- We start with a bandlimited signal that we sample at $f_s = W$
- Spectrum is $\Rightarrow X(f)$



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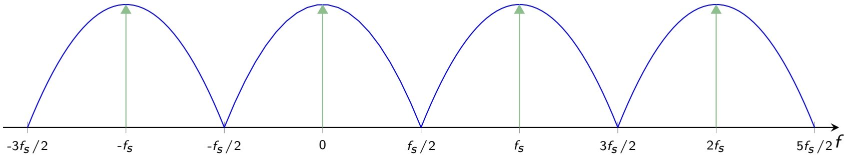
- The spectrum X_s of the sampled signal is periodization of X

$$\Rightarrow X_s(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$$



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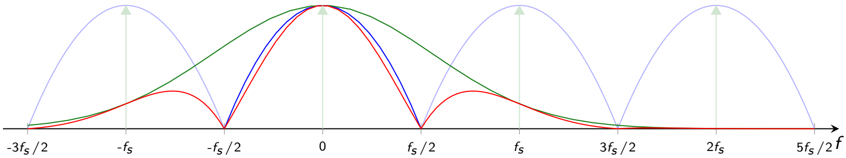
More on the spectrum of sampling and recovery

- To recover the signal we modulate a pulse train. Pulse FT is $P(f)$

$$\Rightarrow X_p(f) = P(f) \times \sum_{k=-\infty}^{\infty} X(f - kf_s)$$



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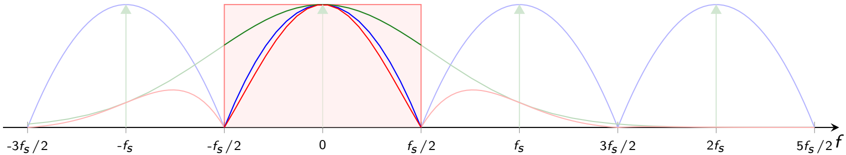
- We finalize recovery with a low pass filter of bandwidth f_s

$$\Rightarrow X_r(f) = \sqcap_{f_s}(f) P(f) X(f - kf_s)$$



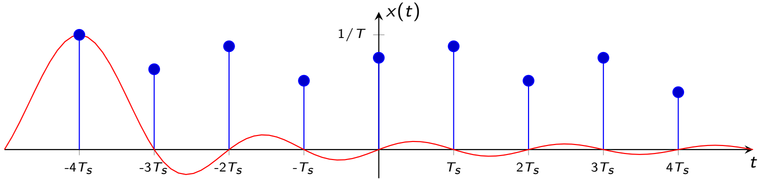
- Good pulse for recovery $\Rightarrow X(f) = 1$ for $f \in [-f_s/2, f_s/2]$

- Do we know a pulse with $X(f) = 1$ for $f \in [-f_s/2, f_s/2]$?
  $\Rightarrow$ We do! $\Rightarrow$ The sinc pulse $f_s \text{sinc}(\pi f_s t)$
- Don't even need to use low pass filter $\Rightarrow$ sinc pulse already lowpass

Theorem

A signal of bandwidth $W \leq f_s$ can be recovered from samples $x(nT_s)$ as

$$x(t) = f_s T_s \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}\big(\pi f_s(t - nT_s)\big)$$



- Reconstruction without a Dirac train $\Rightarrow$ (mostly) implementable

Modulation of a sinc train

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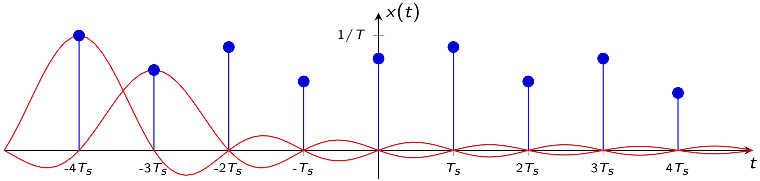

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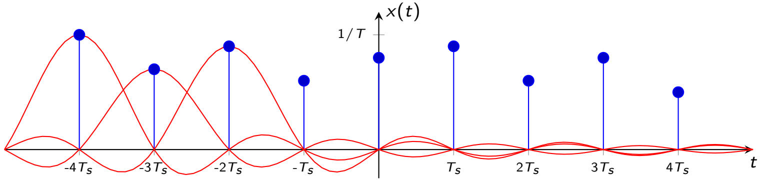

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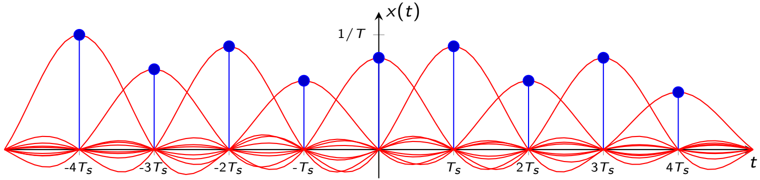

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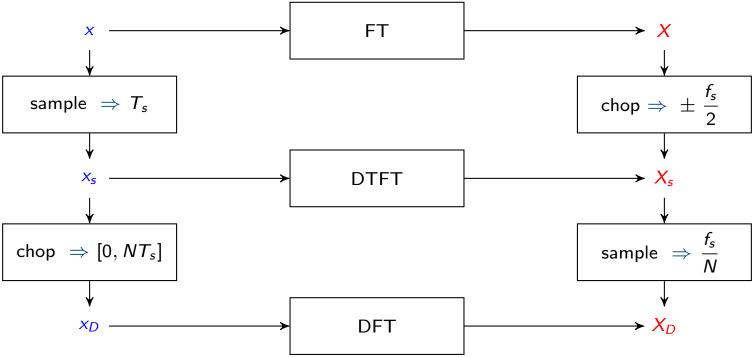
From the FT to the DFT

The DFT as a proxy for the FT

- We use the DFT for frequency analysis of continuous time signals
- Justifiable $\Rightarrow$ They're approximately equal for small T_s and large N

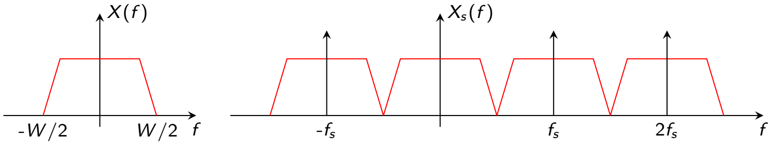


- Sampling $\Rightarrow$ Can understand what is lost in the approximation

- Sampling in time ≡ periodization (not "chop") in frequency

$$x_s(n) = x(nT_s) \quad \Longleftrightarrow \quad X_s(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

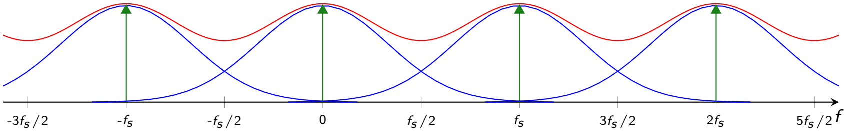
- Replicate. Shift to recenter at $f = kf_s$. Add all shifted copies
- If signal is bandlimited $\Rightarrow X_s(f) = X(f)$ for all $f \in [-f_s/2, f_s/2]$
  $\Rightarrow$ Spectra coincide perfectly $\Rightarrow$ No approximation



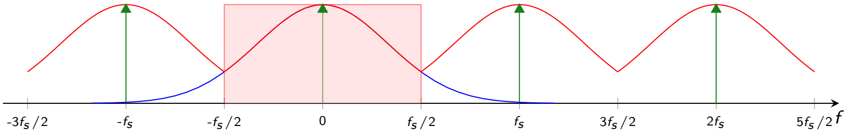
- In general, signals are not bandlimited and we expect some distortion

Lost in approximation

- Signal is not bandlimited $\Rightarrow$ freqs. above $f_s/2$ not seen in DTFT
- Without prefiltering $\Rightarrow$ aliasing distorts freqs. close to $f_s/2$



- With prefiltering $\Rightarrow$ all freqs. below $f_s/2$ approximated correctly



- Which means that we do use a low pass filter prior to sampling

The DTFT as proxy for the FT (1 of 3)

- Filter $\Rightarrow$ multiply in frequency by H $\Rightarrow$ convolve in time with h

$$X_f = HX \quad \Longleftrightarrow \quad x_f = x * h$$

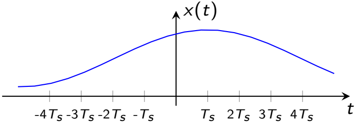
- Sample filtered signal X_f $\Rightarrow$ Periodize filtered spectrum X_f

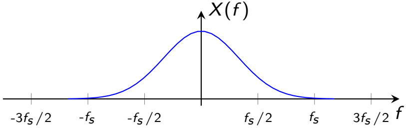
$$x_s(n) = x_f(nT_s) \quad \Longleftrightarrow \quad X_s(f) = \sum_{k=-\infty}^{\infty} X_f(f - kf_s)$$

- Distortion (information loss) occurs during filtering step
  - $\Rightarrow$ Frequency $\Rightarrow$ Loss above $f_s/2$ + some distortion if H not perfect
  - $\Rightarrow$ Time $\Rightarrow$ Convolution with h

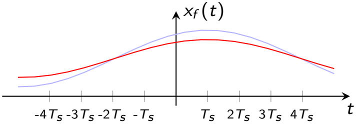
The DTFT as proxy for the FT (2 of 3)

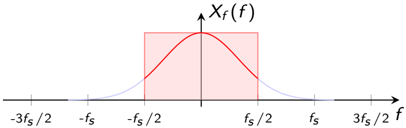
- Continuous time signal x with FT X $\Rightarrow$ Not necessarily bandlimited



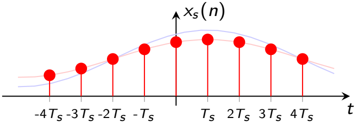


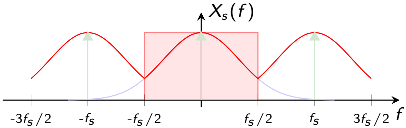
- Continuous time filtered signal x_f $\Rightarrow$ filtering smoothes (distorts) x





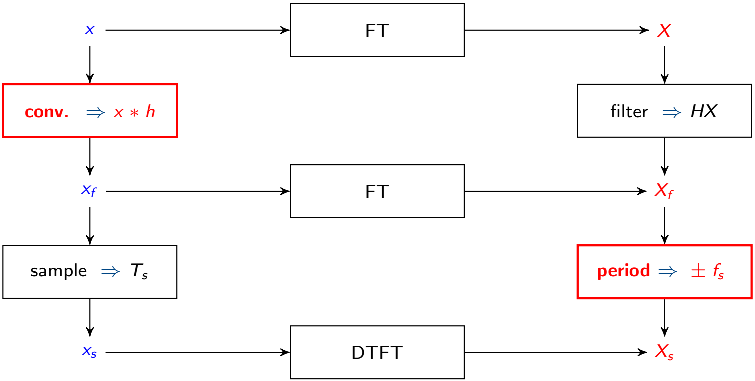
- Sampled signal x_s obtained from filtered x_f $\Rightarrow$ No further distortion





The DTFT as proxy for the FT (3 of 3)

- Filtering (chop) induces convolution. Sampling induces periodization



- Small distortion ⇒ Make f_s so that $X(f) \approx 0$ for $f \notin [-f_s/2, f_s/2]$

Windowing ⇒ From the DTFT to the DFT

- DTFT of sampled signal x_s is $\Rightarrow X_s(f) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}$

- Windowed signal ⇒ Nullify signal values outside of interval $[0, N-1]$

$$x_w(n) = x_s(n), \qquad \text{for all } \; n \in [0, N-1]$$

- Windowed signal is $x_w(n) = 0$ outside of window (all $n \notin [0, N-1]$)

- DTFT of windowed signal x_w is $\Rightarrow X_s(f) = T_s \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n T_s}$

Spectrum after windowing

- Windowing equivalent to multiplication with square pulse
- More generically $\Rightarrow$ define a window signal w_N as one for which

$$w_N(n) = 0 \qquad \text{for all} \quad n \notin [0, N-1]$$

- Rewrite discrete time windowed signal as $\Rightarrow x_w(n) = x(n) \times w_N(n)$
- Since multiplication in time is equivalent to convolution in frequency

$$X_w(f) = X_s(f) * W_N(f)$$

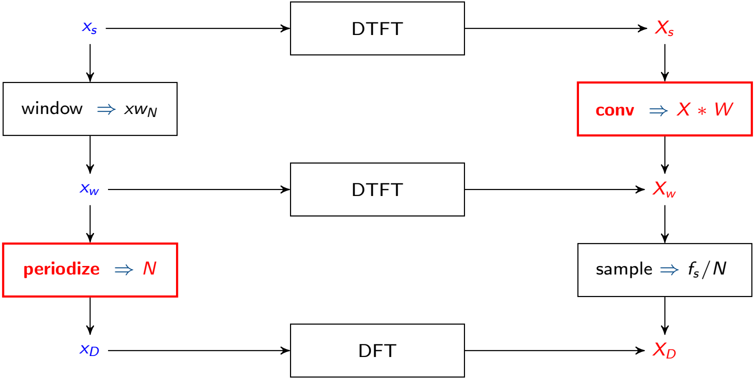
- Multiplicative distortion given by DTFT of window function
- If x_s is already finite $\Rightarrow$ No distortion (dual of bandlimited)

Frequency sampling

- DTFT of windowed signal x_w is $\Rightarrow X_s(f) = T_s \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n T_s}$
- Reinterpret x_w as discrete signal x_D (null vs undefined outside $[0, N-1]$)
- Signal x_D has a DFT (finite) $\Rightarrow X_D(f) \; = \; \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi kn/N}$
- Comparing expressions $\Rightarrow X_s\left(\frac{k}{N} f_s\right) = T_s \sqrt{N}\, X_D(k)$
- Sample in time $\equiv$ periodize in frequency $\Rightarrow$ Dual property holds?
  - $\Rightarrow$ Yes. The iDFT is a periodic operation
  - $\Rightarrow$ We have $x_D(n+N) = x_D(N)$ because $e^{j2\pi k(n+N)/N} = e^{j2\pi kn/N}$

The DFT as proxy for the DTFT (1 of 2)

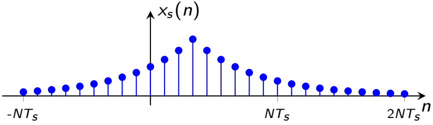
- Window (chop) induces convolution. Sampling induces periodization

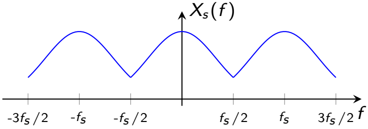


- Small distortion $\Rightarrow$ Make N so that $x(n) \approx 0$ for $n \notin [0, N-1]$

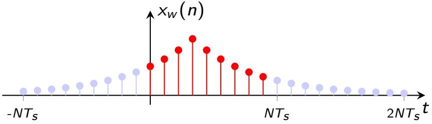
The DFT as proxy for the DTFT (2 of 2)

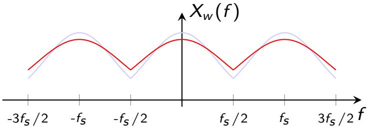
- Discrete time signal x_s with DTFT X_s $\Rightarrow$ Not necessarily finite





- Discrete time windowed signal x_w $\Rightarrow$ windowing smoothes (distorts) X_s





- Discrete DFT X_D samples windowed DTFT X_w $\Rightarrow$ No further distortion