Signal and Information Processing

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Contents

Chapter 1

Discrete Time Fourier Transforms

Now that we have discussed the Discrete Fourier Transform (DFT) and the Fourier Transform (FT), we will now cover the Discrete Time Fourier Transform (DTFT). Note that DTFT is different from the DFT, despite the similar names. We will be covering the same concepts such as inner product and energy from the DFT and FT for the DTFT, and this should seem very repetitive, and almost boring, intentionally.

1.1 Discrete Time Signals

1.1.1 Definition

When we used the DFT, we used it on *discrete* signals, which have a finite number of values (discrete) over a finite time period (i.e. $x(0) to x(N-1)$. When we talked about the continuous Fourier transform, we used *continuous time* signals, which were a continuous or analog signal from a time interval from $-\infty$ to ∞. When we are talking *discrete time* signals, which are discrete signals with an infinite time index (i.e. $x(-\infty)...x(-1), x(0), x(1)...x(\infty)$, noting that the indices of x are integers).

More formally, the discrete time signal x is a function mapping $\mathbb{Z}$ to complex value $x(n)$. As with discrete signals, we have a sampling time T_s, which is the time between samples n and $n+1$, and the sampling frequency $f_s = 1/T_s$.

For example, a shifted delta function $\delta(n - n_0)$ has a spike at time $n = n_0$. This signal continues to +∞ and -∞.

$$\delta(n - n_0) = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases} \tag{1.1}$$

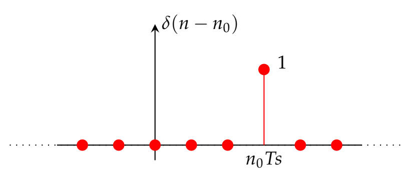


Figure 1.1. Shifted delta function $\delta(n - n_0)$

1.1.2 Inner Product and Energy

We define the inner product of discrete time signals x and y as the sum from $-\infty$ and $+\infty$ of $x(n)y*(n)$:

$$\langle x, y\rangle := \sum_{n=-\infty}^{\infty} x(n)y^*(n) \tag{1.2}$$

Like with the DFT and FT, the inner product tells how much x and y are like each other. If the inner product equals 0, then the signals are orthogonal and there is no relationship between the two. If it equals 1, when they are related and synchronous. If the inner product is -1, then x and y are anti-synchronous, meaning they are inversely related and move in the opposite direction.

The energy of a signal is defined as the inner product with itself.

$$\|x\|^2 := \langle x, y\rangle = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x_R(n)|^2 + \sum_{n=-\infty}^{\infty} |x_I(n)|^2 \tag{1.3}$$

Because the summation for the inner product extends to +∞ and -∞, the sum is more like a series than a sum. As a result, the inner product may not exist since the series may be infinite, and the energy may be infinite. This is in contrast to when we were studying continuous signals with the FT, where the inner product did not exist, and the energy may be infinite. With discrete signals and the DFT, the inner product does exist and the energy was finite. The difference here demonstrates that the DTFT has a sort of hybrid behavior between the DFT and FT.

As an example, we define a square pulse of odd length M+1 as a signal $\sqcap_{M+1}$ with values

$$\sqcap_{M+1}(n) = 1 \qquad \text{if } -\frac{M}{2} \le n \le \frac{M}{2} \tag{1.4}$$

$$\sqcap_{M+1}(n) = 0 \qquad \text{else } M \le n \tag{1.5}$$

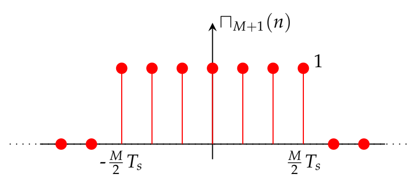


Figure 1.2. Square pulse $\sqcap_{M+1}$

To compute energy of the pulse we just evaluate the definition

$$\|\sqcap_{M+1\|^2} := \sum_{n=-\infty}^{\infty} |\sqcap_{M+1}(n)|^2 = \sum_{n=-M/2}^{M/2} (1)^2 = M+1 \tag{1.6}$$

With the square pulse $\sqcap_{M+1}(n)$ and the shifted pulse $\sqcap_{M+1}(n-K)$, we can calculate the inner product of the two. For shifts $0 \le K \le M+1$, the signals overlap for $K - M/2 \le n \le M/2$ and are 0 elsewhere (at any point where the pulses don't overlap, the product of the signals at that point is 0), so we focus our attention in the overlap region.

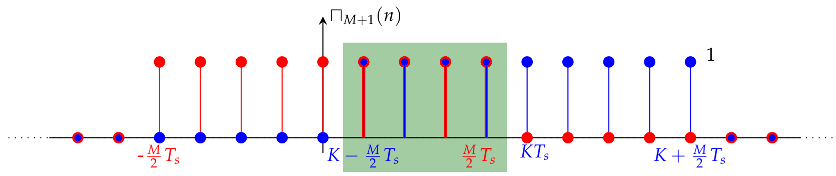


Figure 1.3. Overlapping square pulse $\sqcap_{M+1}$ (red) and shifted pulse $\sqcap_{M+1}(n-K)$ (blue)

$$\langle \sqcap_{M+1}(n), \sqcap_{M+1}(n-K)\rangle = \sum_{n=-\infty}^{\infty} \sqcap_{M+1}(n)\sqcap_{M+1}(n-K) = \sum_{n=K-M/2}^{M/2} (1)(1) = (M+1)-K \quad (1.7)$$

This inner product is proportional to the overlap between the pulses, which is another way of saying how much the pulses are similar to each other.

1.2 The Discrete Time Fourier Transform (DTFT)

1.2.1 Definition

Not surprisingly, as with the DFT and FT, the DTFT X is a sum of products of a discrete time signal x and a complex exponential. The argument f, or the frequency, is continuous. The DTFT also depends on the sampling time T_s.

$$X(f) := T_s \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi f n T_s} \quad (1.8)$$

As with the inner product of discrete time signals, this sum may not exist. As a result, not all signals have a DTFT, which is different than the DFT, for which all discrete signals have.

From this definition, we see that the DTFT has a discrete input but a continuous output, compared to the FT (continuous input and continuous output) and the DFT (discrete input and discrete output). This is a mismatch, showing that once again the DTFT is a hybrid of the DFT and FT. This is interesting and of little consequence to this fact, but it does have some philosophical significance.

1.2.2 DTFT as an Inner Product

We can define the DTFT as an inner product by substituting the complex exponential in the previous definition with the exponential e_{fT_s} with values $e_{fT_s}(n) = T_s e^{j2\pi f n T_s}$. We can then write the as inner product:

$$X(f) = \langle x, e_{fT_s}\rangle = T_s \sum_{n=-\infty}^{\infty} x(n)e^{*}_{fT_s}(n) \quad (1.9)$$

We did this for the DFT ($\langle x, e\rangle$) and FT ($\langle x, e_{kN}\rangle$), so why is this important with the DTFT? For one, the DTFT shows how much a discrete signal $x(n)$ resembles a discrete oscillation of freq. f. The other reason is that it shows the connection between the 3 transforms as all inner products with a complex exponential. However, they are all conceptually different because they have different input spaces. The FT exists in real life because signals are in continuous time in reality. The DTFT can be seen as an abstract of the FT. Lastly, the DFT does not exist in reality, but it does in the space we have created.

1.2.3 Periodicity

Theorem 1 *The DFTF $X = \mathcal{F}(x)$ of discrete time signal x is periodic with period f_s.*

$$X(f + f_s) = X(f), \qquad \text{for all } f \in \mathbb{R}. \tag{1.10}$$

From this theorem, we know that any frequency interval of length f_s contains all of the DTFT information. Therefore, we don't need to look at the whole signal, but rather just at the set of frequencies $f \in [-f_s/2, f_s/2]$. For sampling time T_s, frequencies larger than $f_s/2$ have no physical meaning. Additionally, the frequency $-f$ is (more or less) the same as frequency f.

Proof:
Use the DTFT definition to write $X(f + f_s)$ as

$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi(f+f_s)nT_s} \tag{1.11}$$

Separate the complex exponential in two factors

$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} e^{-j2\pi f_s n T_s} \tag{1.12}$$

Use $f_s T_s = 1$ in the last factor so that we get $e^{-j2\pi f_s n T_s} = e^{-j2\pi n} = \left(e^{j2\pi}\right)^{-n} = 1$. By substituting the previous expression, we observe the definition of the DTFT:

$$X(f + f_s) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = X(f) \tag{1.13}$$

■

1.2.4 DTFT of a Square Pulse

Computing the DTFT is painful, but we will give one example.

Consider square pulse of odd length $M + 1$.

$$\sqcap_{M+1}(n) = 1 \qquad \text{if } -\frac{M}{2} \leq n \leq \frac{M}{2}$$
$$\sqcap_{M+1}(n) = 0 \qquad \text{else } M \leq n$$

To compute the pulse DTFT $X = \mathcal{F}(\sqcap_{M+1})$, we evaluate the definition

$$X(f) = T_s \sum_{n=-\infty}^{\infty} \sqcap_{M+1}(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-M/2}^{M/2} x(n) e^{-j2\pi f n T_s} \tag{1.14}$$

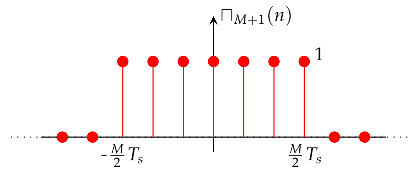


Figure 1.4. Square pulse $\sqcap_{M+1}$

Write down the individual elements of the sum to express DTFT as

$$\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+1\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-1\right)T_s} + e^{j2\pi f\left(\frac{M}{2}\right)T_s}$$

Multiply by $e^{j2\pi f\left(\frac{1}{2}\right)T_s}$ and $e^{j2\pi f\left(-\frac{1}{2}\right)T_s}$ to write the equalities

$$e^{j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{3}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s}$$
$$e^{-j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{3}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s}$$

Notice a pattern where the first term in first row equals the second term in the second row, the second term in first row equals the third term in second row, and so on up to the penultimate term in first row equalling the last term in the second row. By subtracting second row from first row, only two terms survive: the last term in the first row and the first term in the second row.

$$e^{j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{3}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s}$$
$$e^{-j2\pi f\left(\frac{1}{2}\right)T_s}\frac{X(f)}{T_s} = e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s} + e^{j2\pi f\left(-\frac{M}{2}+\frac{1}{2}\right)T_s} + \ldots + e^{j2\pi f\left(\frac{M}{2}-\frac{3}{2}\right)T_s} + e^{j2\pi f\left(\frac{M}{2}-\frac{1}{2}\right)T_s}$$

Implementing the subtraction results in the equality

$$\frac{X(f)}{T_s}\left[e^{j2\pi f\left(\frac{1}{2}\right)T_s} - e^{-j2\pi f\left(\frac{1}{2}\right)}T_s\right] = e^{j2\pi f\left(\frac{M}{2}+\frac{1}{2}\right)T_s} - e^{j2\pi f\left(-\frac{M}{2}-\frac{1}{2}\right)T_s} \tag{1.15}$$

Remembering that complex exponentials are conjugate, subtraction cancels the real parts, leaving us with the imaginary parts only, which are sines.

$$\frac{X(f)}{T_s}\left[2j\sin\left(2\pi f\left(\frac{1}{2}\right)T_s\right)\right] = 2j\sin\left(2\pi f\left(\frac{M+1}{2}\right)T_s\right) \tag{1.16}$$

We then solve for $X(f)$ and simplify the terms using the pulse length $T = (M+1)T_s$.

$$X(f) = T_s\frac{\sin\left(\pi f\left(M+1\right)T_s\right)}{\sin\left(\pi f T_s\right)} = T_s\frac{\sin\left(\pi f T\right)}{\sin\left(\pi f T_s\right)} \tag{1.17}$$

The final result is a ratio of a slow sine over a faster sine, which is very similar to the sinc function. Recall that the Fourier transform of a continuous pulse outputs a sinc function, so it is interesting that the DTFT of a sampled pulse is close to a sinc as well.

1.2.5 The FT, the DTFT, and the DFT

We would now like to connect the three concepts together and see how they are related to each other.

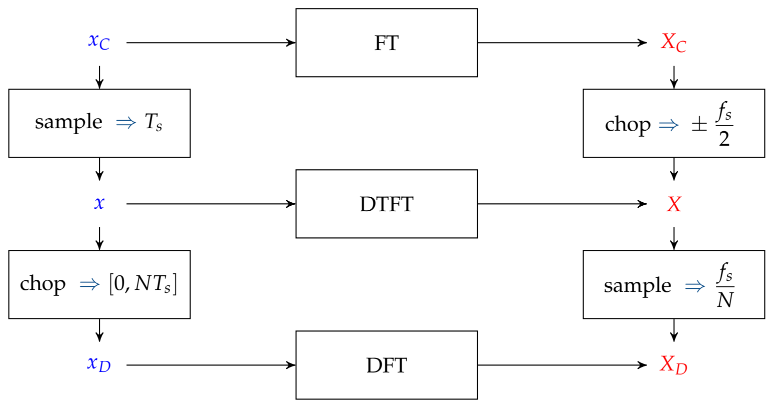


Figure 1.5. Relationship between the FT, DTFT, and DFT. We see that the DTFT acts as a bridge between the FT and the DFT via sampling and/or chopping.

With the DTFT, we are interpreting the signal $x(n)$ as *samples* $x_C(nT_s)$ of a continuous signal $x_C(t)$. When we evaluate the DTFT $X_C(f) = \mathcal{F}(x)$ from the definition, we can also see that it is the Riemann sum approximation of the integral in the FT definition of $X_C = \mathcal{F}(x_C)$.

$$X_C(f) \; = \; \int_{-\infty}^{\infty} x_C(t) e^{-j2\pi f t} dt \approx T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} \; = \; X(f) \tag{1.18}$$

In addition to these observations, we know that only the frequencies between $\pm f_s/2$ have meaning in the DTFT. This means that in the frequency domain, we can *chop* off the higher frequencies. To sum it all together, we can achieve the DTFT from the FT by *sampling in time and chopping in frequency*.

If we compare the DTFT and the DFT, we see that we use a discrete signal x_D with the DFT $X_D = \mathcal{F}(x_D)$, which can be obtained by *chopping* the discrete time signal x to the range $n \in [0, N-1]$. In the definition of the DTFT, the summation is from $-\infty$ to ∞, but if the elements chopped away from x are small, we can at least make an approximation of the DTFT using $n \in [0, N-1]$.

$$X(f) \; = \; T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} \approx T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi f n T_s} \tag{1.19}$$

If we also *sample the DTFT in frequency* at $f = (k/N) f_s$, we can obtain the expression of the DFT:

$$X\left(\frac{k}{N} f_s\right) \approx T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi(k/N) f_s n T_s} \; = \; T_s \sum_{n=0}^{N-1} x_D(n) e^{-j2\pi k n/N} \; = \; T_s \sqrt{N} \, X_D(k) \tag{1.20}$$

Thus, we were able to obtain the DFT from the DTFT by *chopping x in time and sampling $X(f)$ in frequency*.

We have 3 transforms for 3 families of signals. We can go from continuous time signals to discrete time signals via sampling, and from discrete time signals to discrete signals by chopping. We can also observe an interesting duality across the center: sampling in time implicitly requires a chop in frequency and performing a chop in time implicitly performs a sampling in frequency.

1.3 The Inverse Discrete Time Fourier Transform (iDTFT)

Now that we have discussed the DTFT, it should come as no surprise that the inverse DTFT, or iDTFT, comes next. Once again, things should look very familiar from our past discussions with the DFT/iDFT and FT/iFT.

1.3.1 Definition

The iDTFT x of DTFT X is the discrete time signal with elements

$$x(n) := \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df \tag{1.21}$$

We denote the iDTFT as $x = \mathcal{F}^{-1}(X)$. The sign in the exponent changes with respect to DTFT. Notice that the DTFT is an indefinite sum but the iDTFT is a definite integral – an odd mismatch but doesn't mean much. Since the DTFT X is periodic, we can use any interval of width f_s to find the iDTFT. For example, we could use the intervals $[-f_s/2, f_s/2]$ or $[0, f_s]$ and get the same resulting iDTFT regardless.

$$x(n) = \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df = \int_{0}^{f_s} X(f) e^{j2\pi f n T_s} \, df \tag{1.22}$$

1.3.2 iDTFT as Inverse of the DTFT

We can't just say that the inverse DTFT is truly the inverse of the DTFT – this we must prove.

Theorem 2 *The iDTFT $\tilde{x}$ of the DTFT X of the discrete time signal x is the signal x*

$$\tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x. \tag{1.23}$$

This result shouldn't be a surprise since we have done this not just once, but twice already. However, we promise that this is the last one we will introduce.

This result implies that, as usual, discrete time signals can be written as sums of oscillations. This is useful because we can separate the fast- and slow-changing components of a signal.

$$x(n) = \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s} \, df \approx (\Delta f) \sum_{n=-N/2}^{N/2} X(f_k) e^{j2\pi f_k n T_s} \tag{1.24}$$

Proof: To prove that the iDTFT is, in fact, the inverse of the DFTFT, we want to show that $\tilde{x} = \mathcal{F}^{-1}(X) = \mathcal{F}^{-1}[\mathcal{F}(x)] = x$.

We start with the definition of inverse transform of X,

$$\tilde{x}(\tilde{n}) := \int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f \tilde{n} T_s}\, df$$

and the definition of the DTFT of x,

$$X(f) := T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s}.$$

By substituting the expression for $X(f)$ into the expression for $\tilde{x}(\tilde{n})$, we get

$$\tilde{x}(\tilde{n}) = \int_{-f_s/2}^{f_s/2} \left[T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} \right] e^{j2\pi f \tilde{n} T_s}\, df \tag{1.25}$$

We then exchange the integration with the sum so that we integrate first over f, then sum over n. You can pull out $x(n)$ from the integral because it doesn't depend on f.

$$\tilde{x}(\tilde{n}) = T_s \sum_{n=-\infty}^{\infty} x(n) \left[\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s}\, df \right] \tag{1.26}$$

Up until now we repeated the steps we already did for iDFT and iFT. There were steps in the iDFT that didn't work for the iFT, but luckily for us, they work here as well.

We know the result of the innermost integral from something we've computed several times before – the sinc function. To simplify, we used $f_s T_s = 1$ to go from the first to the second expression.

$$\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s}\, df = f_s sinc(\pi f_s (n - \tilde{n}) T_s) = f_s sinc(\pi (n - \tilde{n})) \tag{1.27}$$

Recall that n and $\tilde{n}$ are discrete. For $n = \tilde{n}$, this expression evaluates to $f_s sinc(\pi(n - \tilde{n})) = f_s$ because $sinc(0) = 1$. For $n \neq \tilde{n}$, the expression evaluates to $f_s sinc(\pi(n - \tilde{n})) = 0$ because $sinc(k\pi) = 0$ for any integer k. This means that the sinc here acts as a delta function in disguise (it evaluates to 1 for one value and 0 for all other values), so we can rewrite the integral expression as:

$$\int_{-f_s/2}^{f_s/2} e^{j2\pi f \tilde{n} T_s} e^{-j2\pi f n T_s}\, df = f_s \delta(n - \tilde{n}) \tag{1.28}$$

We then substitute in the above expression for $\tilde{x}(\tilde{n})$, and because of the delta function, we get only one non-zero term. Here we also used $f_s T_s = 1$.

$$\tilde{x}(\tilde{n}) = T_s f_s \sum_{n=-\infty}^{\infty} x(n) \delta(n - \tilde{n}) = x(\tilde{n}) \tag{1.29}$$

Since we have $\tilde{x}(\tilde{n}) = x(\tilde{n})$ for all $\tilde{n}$, $\tilde{x} \equiv x$. ■

1.3.3 From Time to Frequency and Back

If a discrete signal x has a DTFT X, then that means its DTFT has an iDTFT. The iDTFT can be used to recover the original signal x. What does this imply? It means that the DTFT is a transformation without any loss of information, which allows us to move between the frequency domain and time domain without issues. This is also true for of the DFT-iDFT and FT-iFT paris as well.

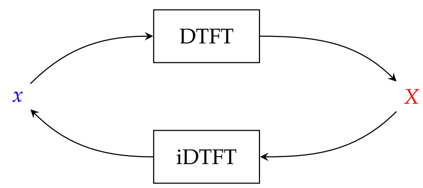


Figure 1.6. Moving between time and frequency domains using the DTFT and iDTFT without loss of information

1.4 The Dirac Train

1.4.1 DTFT of a Constant

Suppose we have a constant function $x_c(t) = 1$, sampled so that we get the discrete time constant $x(n) = x(nT_s) = 1$ for all n. The DTFT of x would be

$$X(f) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} \tag{1.30}$$

Unfortunately, this summation does not exist. If we set $n = 0$, $X(f)$ would be an infinite summation of 1. We can try to make the DTFT into a finite sum between $-M/2$ and $M/2$ like we did in continuous time. To do this, we write the constant as a pulse limit, whose DTFT we saw is a ratio of sines similar to a sinc. We can rewrite the DTFT of the constant function x as the limit

$$X(f) = \lim_{M\to\infty} T_s \sum_{n=-M/2}^{M/2} e^{-j2\pi f n T_s} = \lim_{M\to\infty} T_s \frac{\sin(\pi f (M+1) T_s)}{\sin(\pi f T_s)} \tag{1.31}$$

However, this limit also doesn't exist, because it's essentially saying that the summation is infinite. This makes sense, since we simply rewrapped the expression to look slightly different. To find this limit, we can look at the quantity inside the limit, the ratio of sines. We know that that has an iDTFT, which is the square pulse. We also can observe that there are peaks at $\pm k f_s$, demonstrating periodicity. If we multiply the limit by some arbitrary signal Y(f), we notice that after integrating, we recover Y(0).

$$\lim_{M\to\infty} \int_{-f_s/2}^{f_s/2} Y(f) T_s \frac{\sin(\pi f (M+1) T_s)}{\sin(\pi f T_s)} \, df = Y(0) \tag{1.32}$$

We were able to recover Y(0) by using the integral, which is what the delta function is

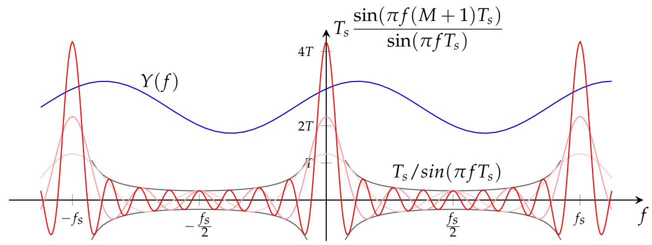


Figure 1.7. The limit of the DTFT of a square pulse

defined as. As a refresher, the delta function δ is a generalized function such that for all Y

$$\int_{-\infty}^{\infty} Y(f)\delta(f)\, df = Y(0) \tag{1.33}$$

We can therefore define the DTFT of a constant as a delta function. This relationship can't be *derived* because, as we saw earlier, the actual DTFT of a constant results in an infinite sum. So instead, we make a *definition* that the DTFT of a constant is the delta function. However, this isn't exactly right because of the periodic peaks at $\pm k f_s$. To correct our definition, we now say that the DTFT is defined as a train of deltas, also known as a Dirac train or a Dirac comb.

$$X(f) = \sum_{k=-\infty}^{k=\infty} \delta(f - k f_s) \tag{1.34}$$

Informally, $\delta(f) = \infty$ for $f = 0$, $f = \pm f_s$, $f = \pm 2 f_s$, ... and $\delta(f) = 0$ for all other f.

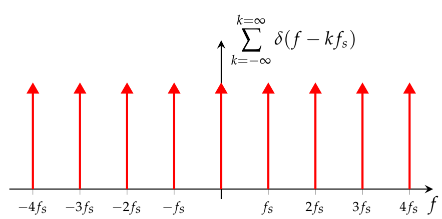


Figure 1.8. The DTFT of a constant: a Dirac train in frequency spaced every f_s

What does it mean exactly that the DTFT of a constant is the Dirac train? It means that the DTFT of a constant can't be observed in reality, and can only be observed once we pass it through an integral. What if we wanted to observe it without an integral? Simply put, you don't want to without it.

$$\int_{-\infty}^{\infty} Y(f)X(f)\, df = \int_{-\infty}^{\infty} Y(f) \sum_{k=-\infty}^{k=\infty} \delta(f - k f_s)\, df = \sum_{k=-\infty}^{k=\infty} Y(f - k f_s) \tag{1.35}$$

It also means that we are able to recover the values of $Y(f)$ at the points where the train has its peaks, at $\pm k f_s$. We can also recover the constant using the iDTFT:

$$\int_{-f_s/2}^{f_s/2} X(f) e^{j2\pi f n T_s}\, df = \int_{-f_s/2}^{f_s/2} \sum_{k=-\infty}^{k=\infty} \delta(f - k f_s) e^{j2\pi f n T_s}\, df = e^{j2\pi 0 n T_s} = 1 \tag{1.36}$$

1.4.2 The Dirac Train in the Time Domain

If we compare a constant to its DTFT, we see that they are suspiciously similar. Both look like Dirac trains.

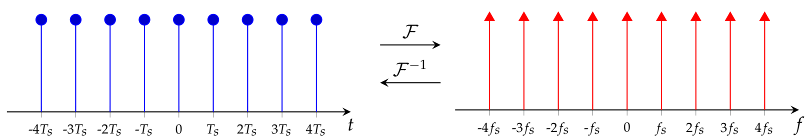


Figure 1.9. The discrete time constant in time and its DTFT, the Dirac train in frequency

However, we cannot say that the Fourier transform of a train is another train because the constant (on the left) is discrete, whereas the Fourier transform (on the right) is continuous. We cannot define the Dirac train in discrete time because the definition of delta functions uses integration.

In continuous time (and not as discrete time constant), a Dirac train $x_C(t)$ can be defined as

$$x_C(t) = T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{1.37}$$

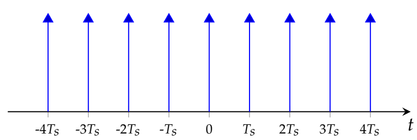


Figure 1.10. A Dirac train in the continuous time domain

Because this Dirac train is continuous, it also has a Fourier transform X_C.

$$X_C(f) = \int_{-\infty}^{\infty} x_C(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \left[T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] e^{-j2\pi ft}\, dt \tag{1.38}$$

We can use this definition of the continuous-time Dirac train to relate it to the DTFT of a discrete time constant. From the above expression, we can exchange the order of the sum and integration and use the delta function definition to obtain

$$X_C(f) = T_s \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j2\pi ft}\, dt \right] = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} \tag{1.39}$$

The resulting summation is the DTFT of a constant.

$$X(f) = T_s \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n T_s} = T_s \sum_{n=-\infty}^{\infty} e^{-j2\pi f n T_s} \tag{1.40}$$

From this, it can be seen that the DTFT of a constant and the FT of a Dirac train coincide. Both are, in fact, Dirac trains with spacing f_s.

$$X_C(f) = X(f) = \sum_{k=-\infty}^{\infty} \delta(t - kf_s) \tag{1.41}$$

The FT of a Dirac train with spacing T_s is the same as a Dirac train with spacing f_s.

$$x_C(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{1.42}$$

We can say that the set of Dirac trains is invariant with respect to the FT because it looks the same in the time and frequency domains with the related spacing. We can also call the Dirac train and the FT of the Dirac train as a Fourier transform pair because both are continuous signals.

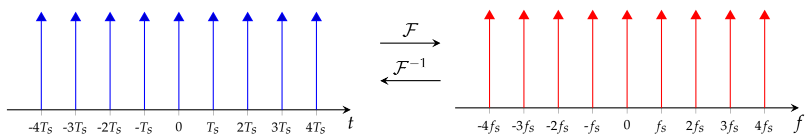


Figure 1.11. A Dirac train in the continuous time domain and its FT, a Dirac train in the frequency domain

In summary, the DTFT of a discrete time constant sampled at T_s is the Dirac train in frequency spaced every f_s.

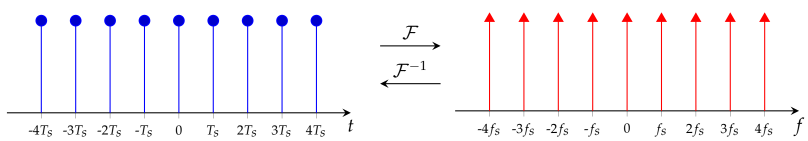


Figure 1.12. The discrete time constant in time and its DTFT, the Dirac train in frequency

The Fourier transform of the Dirac train in continuous time spaced every T_s is the Dirac train in frequency spaced every f_s.

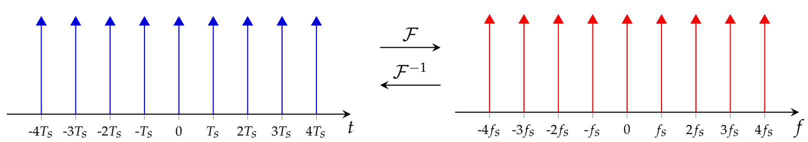


Figure 1.13. A Dirac train in the continuous time domain and its FT, a Dirac train in the frequency domain

The discrete time constant is fundamentally different from the continuous time train, so the DTFT of the constant is fundamentally different from the FT of the Dirac train. However, we do see that the DTFT of the constant and FT of the Dirac train coincide, revealing something deeper at play. This will be saved for a future discussion.

1.5 Sampling

Now that we understand the DTFT and Dirac Train, we can start to understand **Sampling**. Sampling is an extremely important concept to signal processing, and leads to some interesting conclusions.

1.5.1 What is a Sampled Signal?

So what exactly is a sampled signal? The answer to this question is rather simple. A sampled signal, which we denote as x_s, is a discretized version of the continuous signal x. In order to get from x to x_s, we retain specific values of x spaced by a sampling time T_s. As a formula, this can be represented by

$$x_s(n) = x(nT_s). \tag{1.43}$$

As you can see, this sampling of a signal is something we've done since the beginning of the semester. The only difference is that this representation x_s extends past some value N and in fact, has no limit. Thus, the signal is not limited within the window of $[0, N-1]$. So why is this representation important? Well, we live in a continuous world. In order to read in data or information as a signal, we have to convert it into a discrete domain. This implies the need for sampling to collect, manipulate, and understand information. However, this sampling creates a loss in information as all values of $x(t)$ not equal to $x(nT_s)$ for any n are lost. This causes another question to arise: how much information is lost? This is the question we will attempt to answer as we continue our learning.

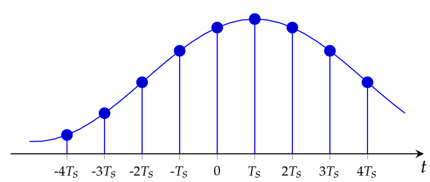


Figure 1.14. Signal x sampled with sampling time T_s

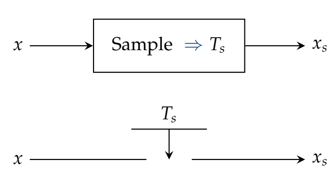


Figure 1.15. Representation of the loss in information between multiples of T_s

1.5.2 New Representation of Sampling

We can simply represent a sampled signal x_s using $x_s(n) = x(nT_s)$. However, there is a slight problem in doing so. The representation is dependent on manually taking values within the continuous function $x(t)$, making it difficult to establish a direct relationship between x and x_s. One solution to find some function of $x(t)$ that results in x_s. One such function is the multiplication of x with a Dirac Train.

$$x_\delta(t) = x(t) \times T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{1.44}$$

This representation is quite easy to understand visually. By multiplying the two, you essentially scale the Dirac Train by whatever value the corresponding $x(t)$ is at times nT_s. This essentially modifies the Dirac Train to take on the shape of the continuous signal $x(t)$. Now, this resulting **Modulated Dirac Train** or x_δ is not exactly the same as a sampled signal x_s as spikes still go to infinity.

Since the only value that is relevant for $\delta(t - nT_s)$ is $x(nT_s)$,

$$x_\delta(t) = T_s \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \tag{1.45}$$

or

$$x_\delta(t) = T_s \sum_{n=-\infty}^{\infty} x_s(n)\delta(t - nT_s) \tag{1.46}$$

Thus, we can construct x_s if given x_δ and construct x_δ if given x_s by using either of the formulas established. Now, we will continue to use the Modulated Dirac Train x_δ to represent x_s. The reason for this is in their equivalence (proved in the next section). The visual Dirac Train representation is shown below:

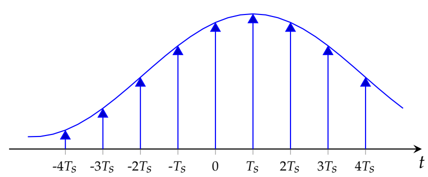


Figure 1.16. Representation of the Modulated Dirac Train x_δ

1.5.3 DTFT and FT of sampled signals

The DTFT $X_s = \mathcal{F}(x_s)$ of the sampled signal x_s and the FT $X_\delta = \mathcal{F}(x_\delta)$ of the Dirac sampled signal x_δ coincide and

$$X_\delta(f) = X_s(f). \tag{1.47}$$

Why does this happen? Well, as we saw before, x_s and x_δ are essentially equivalent. The only difference is that the spikes in x_δ are infinite in value, but that doesn't make a huge difference conceptually. Thus, the FT of two equivalent functions are equivalent. The reason we take the DTFT of x_s and the FT of x_δ is because x_δ is technically in the continuous domain (a function of two continuous functions). However, that is not too important. What is important is that you understand $X_\delta(f) = X_s(f)$.

So how do we prove that $X_\delta(f) = X_s(f)$? We can begin by first writing the equation for X_δ, the FT of x_δ.

$$X_\delta(f) = \int_{-\infty}^{\infty} \left[T_s \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)e^{-j2\pi f t} \right] df \tag{1.48}$$

Next we can exchange the order of summation and integration.

$$X_\delta(f) = T_s \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(nT_s)\delta(t - nT_s)e^{-j2\pi ft}\, df \right] \tag{1.49}$$

Then, by multiplying by the delta function and integrating, you will recover the value at the spikes.

$$X_\delta(f) = T_s \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j2\pi fnT_s} = T_s \sum_{n=-\infty}^{\infty} x_s(n)e^{-j2\pi fnT_s} = X_s(f) \tag{1.50}$$

Note that for the last step, we used the fact that $x_s(n) = x(nT_s)$ and the definition of the DTFT.

So what is X_δ? In order to answer this, we must first be aware of a simple property. A multiplication in time of a signal is equal to a convolution in frequency. And when we convolve a signal in time with some function, we multiply the spectrum of the signal by the spectrum of that other function. Now since the signal x_δ is the represented as the multiplication of $x(t)$ with the Dirac Train,

$$x_\delta(t) = x(t) \times T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{1.51}$$

the Fourier Transform of x_δ or X_δ is equal to the convolution of $X(f)$ and the FT of the Dirac Train,

$$X_\delta = X * \mathcal{F}\left[T_s \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right] \tag{1.52}$$

Now, remember that the Fourier Transform of a Dirac Train (T_s) is another Dirac Train (f_s). Because of this, we can represent X_δ as

$$X_\delta = X * \left[\sum_{k=-\infty}^{\infty} \delta(t - kf_s) \right] \tag{1.53}$$

But since convolution is a linear operator, we can move the summation sign to the outside of X.

$$X_\delta = \sum_{k=-\infty}^{\infty} X * \delta(f - kf_s) \tag{1.54}$$

And since the inside of the summation is now a convolution between X and a shifted delta, the spectrum of a sampled signal is the sum of shifted versions of the original spectrum.

$$X_\delta(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{1.55}$$

$$X_s(f) = X_\delta(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{1.56}$$

So now that we know this, let's revisit the question of "how much information do we lose from sampling?" Well, we first know that the amount of information lost depends on how fast a signal changes. The faster a signal changes, the more information that is lost given a constant sampling time. but to really understand how much information is lost, we have to go into the frequency domain (where we can see how fast a signal changes). Thus, the amount of information lost can be determined by the differences between the spectra X and X_δ. Through our calculations and determination of a formula relating X and X_δ, we can now move forward with better answering the question.

1.5.4 Spectrum Periodization

As we have seen, the spectrum of X is equal to a convolution of X with a Dirac Train. Thus, we determine the effect of sampling by seeing the effect a convolution with a Dirac Train has. We first start with the spectrum X. From there, we understand that a convolution is equivalent to summing shifted copies of the spectrum X. Thus, the next step is to duplicate X shift spectrum to each kf_s where k represents all integers. From this resulting spectrum, we can obtain the spectrum X_δ by summing all the replicated spectra.

As seen, this resulting spectrum X_δ or X_s looks rather different from the initial spectrum X_δ. Frequencies above $f_s/2$ and below $-f_s/2$ are lost as they are overtaken by the repeated spectra. The frequencies around $f_s/2$ and $-f_s/2$ are also distorted by the summation with the tails of other replicated and shifted spectra. We refer to this distortion and loss of information as **Spectral Aliasing**.

We can see that the smaller f_s is, the more that the replicated spectra will distort/overlap each other. Thus, if we choose a f_s large enough (or T_s large enough), we can avoid more aliasing. Alternatively, if f_s is sufficiently small, all of the information within the signal could be lost through sever aliasing. Now, if we keep increasing f_s, we will continue to decrease aliasing. But can we completely remove it? Generally no, as most spectra of signals run across all frequencies. However, if the signal does not run for all frequencies, and if it only has values in its spectrum between some value $[-W/2, W/2]$, then that signal is band-limited and aliasing can be completely removed. That is, aliasing can be completely removed if $X(f) = 0$ for $f \notin [-W/2, W/2]$ (bandwidth W). How do we remove aliasing? By simply increasing the f_s, the spectra will no longer overlap, removing aliasing. Thus, we arrive at a theorem:

Theorem 3 *Let x be a signal of bandwidth W. If the signal is sampled at a frequency $f_s \geq W$ we have that*

$$X_\delta(f) = X_s(f) = X(f) \tag{1.57}$$

for all frequencies $f \in [-W/2, W/2]$.

With $f_s \geq W$, we end up with no loss of information, and we can recover x completely from x_δ by using a low pass filter to remove all frequencies outside of $[-W/2, W/2]$. That is, by retaining one of the replicated spectra and removing the rest with the low pass filter, we now have the exact same spectrum as X.

1.6 Signal Reconstruction

A non-bandlimited signal is one where f_s is below W. Thus, given the f_s, sampling will result in some sort of distortion. Now, we have answered how information is lost in sam-

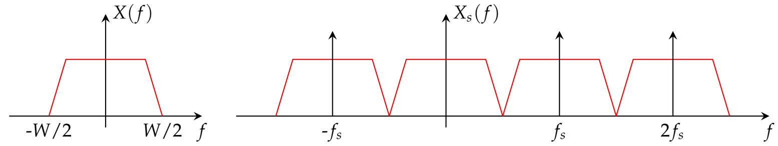


Figure 1.17. Representation of a sampled signal with bandwidth W and $f_s \geq W$. As seen, none of the replicated spectra overlap, meaning there is no aliasing.

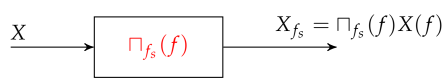
pling, but that creates another question: How can we recover x from a non-bandlimited signal with minimal lost information? One answer is **Prefiltering**.

1.6.1 Prefiltering

We can remove aliasing distortion by adjusting x before sampling to make it bandwidth f_s. What exactly does this mean? Well, by using a low pass filter, we can remove frequencies above $f_s/2$ and below $-f_s/2$, essentially creating a signal with bandwidth f_s. This means that frequencies in $[-f_s/2, f_s/2]$ are retained. Now, there is still a loss of information as frequencies outside that range are removed. However, those within the range are still there to make up the bulk of the signal. This transformed x is notated as x_{fs}.

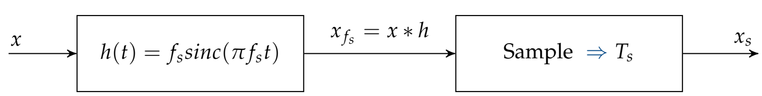
We can transform x into x_{fs} using a low pass filter. Specifically, we multiply the spectrum X by the low pass filter $\sqcap_{fs}(f)$.

$$X_{f_s}(f) = X(f) \sqcap_{f_s}(f) \tag{1.58}$$



As stated, the resulting signal x_{fs} has bandwidth f_s and can be sampled without aliasing. This prefiltering can also be represented as a convolution in the time domain between x and the iFT of the square pulse ($f_s sinc(\pi f_s t)$):

$$x_{f_s} = x * f_s sinc(\pi f_s t) \tag{1.59}$$



1.6.2 Low Pass Filter Recovery

Now when we use a low pass filter to recover x a discrete time signal x_s, we can't recover a continuous signal (can't go from discrete to continuous domain). However, we can recover the continuous x from the Dirac Train representation of x_s, or x_δ, as it is technically continuous. This means that using a low pass filter to recover x on x_δ will actually get the continuous signal x. Thus, this reconstruction is:

$$x = x_\delta * \left[f_s sinc(\pi f_s t) \right] \tag{1.60}$$

And in order to reconstruct x from x_s, we can first change x_s to x_δ by creating a modulated Dirac Train. This reconstruction is shown:

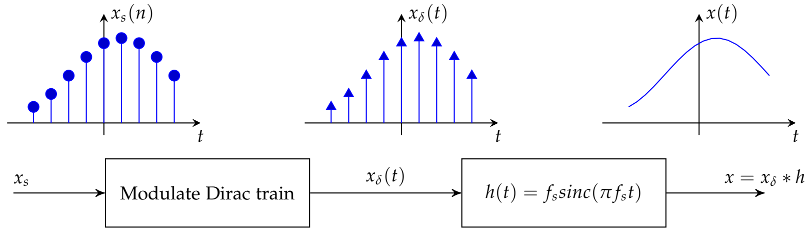


Figure 1.18. Procedure for reconstruction of x from x_s using a low pass filter. x_s is first changed to x_δ by changing x_s to a modulated Dirac Train

1.7 From the FT to the DFT

We typically use the DFT to approximate the FT. This is generally ok as long as T_s is sufficiently small and N large. Through the analysis of sampling, we can understand what is lost in the approximation.

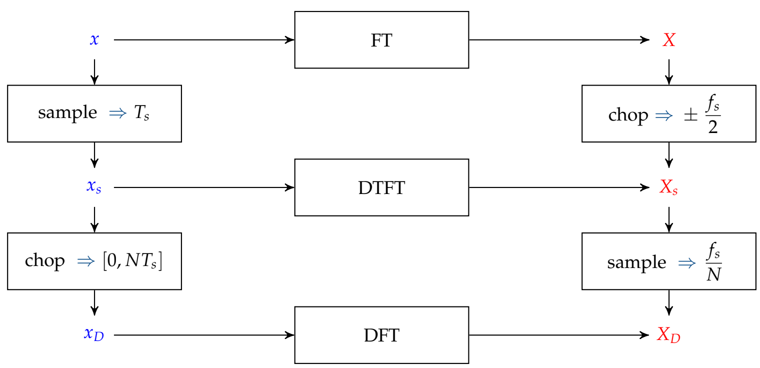


Figure 1.19. Diagram showcasing relationship between FT, DTFT, and DFT

1.7.1 First the FT to the DTFT

In order to go from the FT to the DFT, we first have to understand the transition between the FT and the DTFT. The continuous signal x undergoes a FT to get continuous spectrum X. The sampled signal x_s undergoes a DTFT to get to spectrum X_s. Both signals and Spectra aren't windowed or limited, but x only has values in intervals of T_s.

In order to get from x to x_s, x must be sampled at some sampling time T_s. The equivalent to this in the frequency domain is a periodization (not a "chop"). Specifically:

$$x_s(n) = x(nT_s) \quad \Longleftrightarrow \quad X_s(f) = \sum_{k=-\infty}^{\infty} X(f - kf_s) \tag{1.61}$$

As can be seen from the equation relating X_s and X, there is a replication, shift, and sum of the X spectrum. Therefore, if the spectrum of X is bandlimited such as in this example:

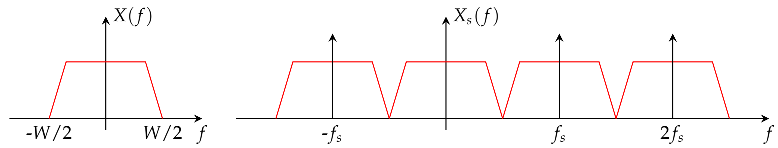


Figure 1.20. Replication and summation of a bandlimited sample when going from X to X_s

then there is no loss in information when approximating. However, generally signals are not bandlimited, and some distortion is expected. In the case of a non-bandlimited signal, we use a low pass filter, denoted h in the time domain and H in the frequency domain, to reduce the loss in information. Specifically, we use an H that retains frequencies between $f_s/2$ and $-f_s/2$, reducing the distortion to just a smoothing of the signal. This smoothing is due to the removal of frequencies above $f_s/2$ and below $-f_s/2$. However this resulting signal does not suffer from distortion caused by summation of overlapping replicated *spectra* within X_s.

To use this filter H in a process called **prefiltering**, we multiply X by H. Equivalently, this is a convolution between x and h ($x * h$). This resulting signal after the filtering is x_f and its spectrum is X_f. The equations are denoted:

$$X_f = HX \iff x_f = x * h \tag{1.62}$$

Also note that if H is not a perfect filter, there could be some additional distortion.

Thus, after taking into account prefiltering, the original diagram can be altered. The top portion representing the relationship between FT and DTFT is shown:

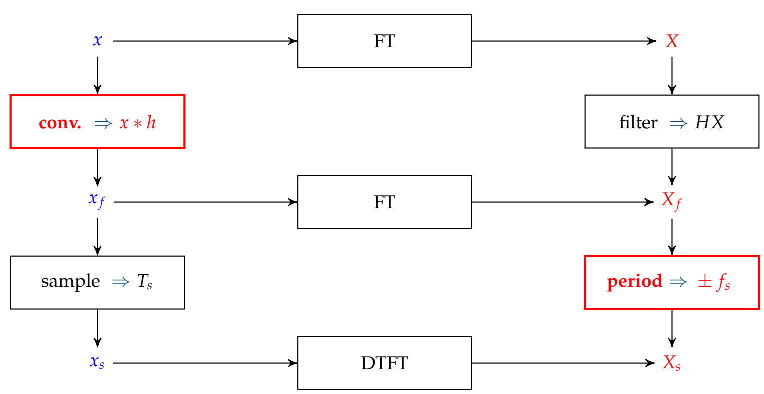


Figure 1.21. Relationship between FT and DTFT with prefiltering

Note that filtering in the frequency domain induces sampling in the time domain. And that sampling in the time domain induces periodization in the frequency domain.

1.7.2 From DTFT to DFT

Recall from the beginning of the semester that the discretized version of x is a signal with information only at intervals of T_s (similar to x_s) and is restricted in a window between values 0 and $N-1$. Let us denote this discrete signal as x_d. Since we know how to get from x to x_s, if we can determine how to get from x_s to x_d, we will understand the relationship between x and x_d fully. However, how exactly do we get from x_s to x_d? The answer is in the differences between x_s and x_d. x_s has no restriction in time as it is not windowed. However, x_d. Thus, if we can window or restrict x_s to values between 0 and $N-1$, then we can get x_d.

So how do we window? Easy: we can use the same method we used in prefiltering. Except this time, we do it in the time domain. Define the window w_N as:

$$w_N(n) = 0 \qquad \text{for all } \; n \notin [0, N-1] \tag{1.63}$$

and the windowed signal x_w as:

$$x_w(n) = x_s(n), \qquad \text{for all } \; n \in [0, N-1] \tag{1.64}$$

By multiplying that window with signal x_s, we will get x_d. Equivalently, a multiplication in the time domain is a convolution in the frequency domain. Thus, knowing that the FT of the window w_N is W_N,

$$x_w(n) = x(n) \times w_N(n) \tag{1.65}$$

and

$$X_w(f) = X_s(f) * W_N(f) \tag{1.66}$$

So is x_w and xd the same? The answer is no. x_w still uses time as an input. That is, the signal is $x_w(t)$ where t is in seconds. However, the only times with information are at intervals of T_s. x_d on the other hand takes in some integer between $[0, N-1]$ as an input. So how do we correct this? Well, we can first start by looking at the frequency domains of x_w and x_s. Taking the DTFT of x_w, we get:

$$X_w(f) = T_s \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n T_s} \tag{1.67}$$

and DFT of x_d is

$$X_d(f) \;=\; \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_d(n) e^{-j2\pi k n / N} \tag{1.68}$$

Comparing expressions, we get:

$$X_w\left(\frac{k}{N} f_s\right) = T_s \sqrt{N}\, X_d(k) \tag{1.69}$$

From this, we can see that there is a need for sampling in the frequency domain. As stated before, this leads to periodization in the opposite domain (time). Thus, the relationship between the DTFT and the DFT can be shown as below:

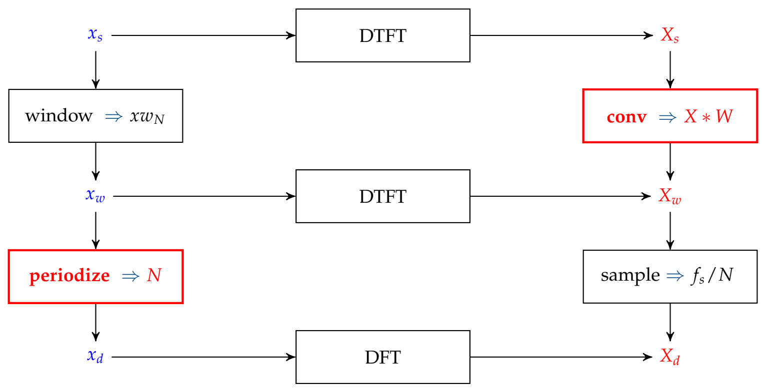


Figure 1.22. Relationship between DTFT and DFT with windowing

1.8 Conclusion

From all the work done, we've come to some interesting and important conclusions. First, we saw how sampling, though important in reading in information, leads to a loss in information. We then analyzed what exactly was lost, and determined that the lost rised from the periodization of a signal's spectrum. From there we further analyzed the process, and came with a complete relationship between FT, DTFT, and DFT. Additionally, we determined how to get from x to x_s and ultimately to x_d.