SYSTEMS 302
LECTURE 4

• COVARIANCE AND CORRELATION

• CONTINUOUS RANDOM VARIABLES
  • Uniform Distribution
    • Simulation of Random Variables
  • Normal Distribution
    • Standardized Random Variables
    • Applications

• For next time:
  • Devore, Sections 4.6, 5.3-5.5
CORRELATION COEFFICIENT

Normalize covariance to obtain:

\[
corr(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sigma(X_1)\sigma(X_2)}
\]

PROPERTIES:

(1) \( a_1, a_2 > 0 \Rightarrow corr(a_1X_1, a_2X_2) = corr(X_1, X_2) \)

(2) \( corr(X_1, X_2) \geq 0 \iff \text{cov}(X_1, X_2) \geq 0 \)

(3) \( -1 \leq corr(X_1, X_2) \leq 1 \)

(4) \( |corr(X_1, X_2)| = 1 \iff X_2 = aX_1 + b \)
**Uniform Random Variable**

\[ P(x_1 \leq X \leq x_2) = \frac{x_2 - x_1}{c} \]

**Density Representation**

\[ f(x) = \begin{cases} \frac{1}{c} , & 0 \leq x \leq c \\ 0 , & \text{otherwise} \end{cases} \]

\[ P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x) \, dx \]

\[ = \int_{x_1}^{x_2} \left( \frac{1}{c} \right) \, dx \]

**Cumulative Distribution Function (CDF)**

\[ F(x) = P(X \leq x) \]

\[ = \begin{cases} 0 , & x < 0 \\ \frac{x}{c} , & 0 \leq x \leq c \\ 1 , & x > c \end{cases} \]
• **Standard Normal Distribution**

\[ \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty \]

• **Moments:**

1. \( E(Z) = 0 \) (by symmetry)
2. \( \text{var}(Z) = 1 = \sigma^2(Z) \)

\[ \Rightarrow Z \sim N(0,1) \]

• **Standard Normal CDF**

\[ \Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \phi(x) \, dx \]

• **General Normal Distribution**

\[ X = \sigma Z + \mu \]

\( \sigma > 0, \quad Z \sim N(0,1) \)

\[ \begin{align*}
E(X) &= \mu \\
\sigma(X) &= \sigma
\end{align*} \]

\[ \Rightarrow X \sim N(\mu, \sigma^2) \]
**OIL PUMP PROBLEM**

(Production Efficiency)

**Example**

A Texas oil pump is set to fill 50 gal drums, but doesn't always fill exactly. Suppose fluctuations in filling amount, \( X \), are normally distributed as

\[
X \sim N(50, .04)
\]

**Question:** Given a capacity of 50.5 gal for standard oil drums, what is the likelihood of a spill?
OIL PUMP SOLUTION

Find: \( P(X > 50.5) \) given \( X \sim N(50, .04) \)

\[
Z = \frac{X - \mu}{\sigma} = \frac{X - 50}{.2}, \quad \sigma = \sqrt{.04}
\]

\( \Rightarrow Z \sim N(0, 1) \)

and:

\[
P(X > x) = P \left( \frac{X - \mu}{\sigma} > \frac{x - \mu}{\sigma} \right) = P \left( Z > \frac{x - \mu}{\sigma} \right)
\]

So:

\[
X = 50.5 \quad \Rightarrow \quad Z = \frac{50.5 - 50}{.2} = 2.5
\]

\( \Rightarrow P(X > 50.5) = P(Z > 2.5) \)

\[
= P(Z < -2.5)
\]

\[
= \Phi(-2.5) = .0062 \ [\text{Table A3}]
\]
Rockwell Hardness, $H$, (depth of surface penetration in thousandths of a millimeter) for a given metal alloy is known to be normally distributed with $\mu = 70$ and $\sigma = 3$:

$$H \sim N(70, 9)$$

Q1. If the acceptable hardness range for a specific application is $[67, 75]$, then what is the chance that a given metal sample will be accepted?

Q2. If the acceptance range is required to be symmetric about the mean, say $[\mu - c, \mu + c]$, then what value of $c$ would yield an acceptance probability of .95?
Q1 SOLUTION:

Find: \( P(67 \leq H \leq 75) \) given \( H \sim N(70, 9) \)

\[
P(67 \leq H \leq 75) = P(H \leq 75) - P(H \leq 67)
\]

\[
Z = \frac{H - \mu}{\sigma} = \frac{H - 70}{3} \sim N(0,1)
\]

\( H = 75 \Rightarrow Z = \frac{75 - 70}{3} = 1.67 \)

\[
\Rightarrow P(H \leq 75) = P(Z \leq 1.67) = \Phi(1.67) = .9525 \quad \text{[Table A3]}
\]

Similarly:

\[
P(H \leq 67) = P(Z \leq -1.00) = \Phi(-1.00) = .1587 \quad \text{[Table A3]}
\]

\[
\Rightarrow P(67 \leq H \leq 75) = .9525 - .1587 = .7938
\]

\[
\Rightarrow \text{About an 80\% acceptance rate.}
\]
Q2 SOLUTION

Find: $c$ such that $P(H \in [70 - c, 70 + c]) = .95$

$P(H \in [70 - c, 70 + c]) = P(70 - c \leq H \leq 70 + c)$

$$= 1 - 2 P(H \leq 70 - c) \quad \text{[by symmetry]}$$

So:

$P(H \in [70 - c, 70 + c]) = .95 \iff P(H \leq 70 - c) = .025$

$$\iff P\left(Z \leq \frac{(70 - c) - 70}{3} = -\frac{c}{3}\right) = .025$$

$$\iff \Phi\left(-\frac{c}{3}\right) = .025$$

$$\iff \Phi^{-1}[\Phi\left(-\frac{c}{3}\right)] = \Phi^{-1}(.025)$$

$$\iff -\frac{c}{3} = \Phi^{-1}(.025) = -1.96 \quad \text{[Table A3]}$$

$$\iff c = 3 \times (1.96) = 5.88 \text{ mm}$$