Stochastic Systems Analysis and Simulations

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Presentations

Class description and contents

Gambling
Who are us, where to find me, lecture times

- Alejandro Ribeiro
  - Assistant Professor, Dept. of Electrical and Systems Engineering
  - GRW 276, aribeiro@seas.upenn.edu,
  - http://alliance.seas.upenn.edu/~aribeiro/wiki

- Felicia Lin
  - Teaching assistant, life@seas.upenn.edu

- We also have a separate grader

- We meet on Moore 216
- Mondays, Wednesdays, Fridays 10 am to 11 am
- My office hours, Fridays at 2 pm
- Anytime, as long as you have something interesting to tell me
- http://alliance.seas.upenn.edu/~ese303/wiki
Prerequisites

- **Probability theory**
- Stochastic processes are time-varying random entities
- If unknown, need to learn as we go
- Will cover in first seven lectures

- **Linear algebra**
  - Vector matrix notation, systems of linear equations, eigenvalues

- Programming in **Matlab**
  - Needed for homework.
  - If you know programming you can learn Matlab in one afternoon
  - But it has to be **this afternoon**

- Differential equations, Fourier transforms
- Appear here and there. Should not be a problem
Homework and grading

- 14 homework sets in 14 weeks
- Collaboration accepted, welcomed, and encouraged
- Sets graded as 0 (bad), 1 (good), 2 (very good) and 3 (outstanding)
- We’ll use the 3 sparingly. Goal is to earn 28 homework points

- Midterm examination starts on Friday October 21 worth 36 points
- In class-piece + take home piece due on Monday October 24
- Work independently. No collaboration, no discussion
- If things are going well, no in-class piece

- Final examination on December 14-21 worth 36 points

- At least 60 points are required for passing.
- C requires at least 70 points. B at least 80. A at least 90
- Goal is for everyone to earn an A
Textbooks

- Textbook for the class is (older or newer editions acceptable)

- Same topics at advanced level (more rigor, includes proofs)

- Stochastic processes in systems biology

- Part on simulation of chemical reactions taken from here

- Use of stochastic processes in finance
I am a very emotional person. I will love most of you, despise a few
Seriously though, I work hard for this course, please do the same
Come to class, be on time, pay attention
Do all of your homework
Do not hand in as yours my own solution
Do not collaborate in the take-home midterm
A little bit of probability ...
Probability of getting an F in this class is 0.04
Probability of getting an F given you skip 4 homework sets is 0.7
I'll give you three notices, afterwards, I'll give up on you
Come and learn. Useful. Very good student ratings
Presentations

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Gambling
Stochastic systems

- Anything random that evolves in time
- Time can be discrete (0, 1, \ldots) or continuous
- More formally, assign a function to a random event
- Compare with “random variable assigns a value to a random event”
- Generalizes concept of random vector to functions
- Or generalizes the concept of function to random settings
- Can interpret a stochastic process as a set of random variables
- Not always the most appropriate way of thinking
A voice recognition system

- Random event $\sim$ word spoken. Stochastic process $\sim$ the waveform
  - Try the file speech_signals.m

![Waveform plots for "Hi", "Good", "Bye", and 'S']
Probability theory review (6 lectures)
  - Probability spaces
  - Conditional probability: time $n + 1$ given time $n$, future given past ...
  - Limits in probability, almost sure limits: behavior as $t \to \infty$ ...
  - Common probability distributions (binomial, exponential, Poisson, Gaussian)

Stochastic processes are complicated entities
Restrict attention to particular classes that are somewhat tractable

Markov chains (9 lectures)
Continuous time Markov chains (12 lectures)
Stationary random processes (9 lectures)
Midterm covers up to Markov chains
Markov chains

- A set of states 1, 2, ..., At time $n$, state is $X_n$
- **Memoryless** property
  - Probability of next state $X_{n+1}$ depends on current state $X_n$
  - But not on past states $X_{n-1}, X_{n-2}, \ldots$

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Happiness tomorrow affected by happiness today only
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

- Classification of states, ergodicity, limiting distributions
- Google's page rank, machine learning, virus propagation, queues ...
Continuous time Markov chains

- A set of states 1, 2, \ldots Continuous time index $t$
- Transition between states can happen at any time
- Future depends on present but is independent of the past

- Probability of changing state in an infinitesimal time $dt$

- Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions
- Chemical reactions, queues, communication networks, weather forecasting \ldots
Stationary random processes

- Continuous time $t$, continuous state $x(t)$, not necessarily memoryless
- System has a steady state in a random sense
- Prob. distribution of $x(t)$ constant or becomes constant as $t$ grows
- Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density.
- Black Scholes model for option pricing, speech, noise in electric circuits, filtering and equalization ...
Gambling

Presentations

Class description and contents

Gambling
An interesting betting game

- There is a certain game in a certain casino in which ...
  \[ \Rightarrow \text{your chances of winning are } p > 1/2 \]
- You place $b$ bets
  (a) With probability $p$ you gain $b$ and
  (b) With probability $(1 - p)$ you lose your $b$ bet
- The catch is that you either
  (a) Play until you go broke (lose all your money)
  (b) Keep playing forever
- You start with an initial wealth of $w_0$
- Shall you play this game?
Let $t$ be a time index (number of bets placed)

Denote as $x(t)$ the outcome of the bet at time $t$
- $x(t) = 1$ if bet is won (with probability $p$)
- $x(t) = 0$ if bet is lost (probability $(1 - p)$)

$x(t)$ is called a Bernoulli random variable with parameter $p$

Denote as $w(t)$ the player’s wealth at time $t$
- At time $t = 0$, $w(0) = w_0$
- At times $t > 0$ wealth $w(t)$ depends on past wins and losses
- More specifically we have
  - When bet is won $w(t) = w(t - 1) + b$
  - When bet is lost $w(t) = w(t - 1) - b$
\( t = 0; \ w(t) = w_0; \ max_t = 10^3; \) // Initialize variables
\% repeat while not broke up to time \( max_t \)
\textbf{while} \ (w(t) > 0) \ & \ (t < max_t) \ \textbf{do}
\hspace{1em} x(t) = \text{random}(\text{`bino'},1,p); \ % \text{Draw Bernoulli random variable}
\hspace{1em} \textbf{if} \ x(t) == 1 \ \textbf{then}
\hspace{2em} w(t + 1) = w(t) + b; \ % \text{If } x = 1 \text{ wealth increases by } b
\hspace{1em} \textbf{else}
\hspace{2em} x(t + 1) = w(t) - b; \ % \text{If } x = 0 \text{ wealth decreases by } b
\hspace{1em} \textbf{end}
\hspace{1em} t = t + 1;
\textbf{end}

\begin{itemize}
  \item Initial wealth \( w_0 = 20 \), bet \( b = 1 \), win probability \( p = 0.55 \)
  \item Shall we play?
\end{itemize}
One lucky player

- She didn’t go broke. After $t = 1000$ bets, her wealth is $w(t) = 109$
- Less likely to go broke now because wealth increased
Two lucky players

- Wealths are $w_1(t) = 109$ and $w_2(t) = 139$
- Increasing wealth seems to be a pattern
Ten lucky players

- Weights $w_j(t)$ between 78 and 139
- Increasing wealth is definitely a pattern
One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke
But this does not mean that all players will turn out as winners
The twelfth player $j = 12$ goes broke
One hundred players

- Only one player \((j = 12)\) goes broke
- All other players end up with substantially more money
Average tendency

- It is not difficult to find a line estimating the average of $w(t)$
- $\bar{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t$

\[ w(t) = \frac{1}{2} \left( w_0 + w_{N+1} \right) \]
Where does the average tendency come from?

To discover average tendency \( \bar{w}(t) \) assume \( w(t - 1) > 0 \) and note

\[
\mathbb{E}[w(t) \mid w(t - 1)] = w(t - 1) + b \mathbb{P}[x(t) = 1] - b \mathbb{P}[x(t) = 0] \\
= w(t - 1) + bp - b(1 - p) \\
= w(t - 1) + (2p - 1)b
\]

Now, condition on \( w(t - 2) \) and use the above expression once more

\[
\mathbb{E}[w(t) \mid w(t - 2)] = \mathbb{E}[w(t - 1) \mid w(t - 2)] + (2p - 1)b \\
= w(t - 2) + (2p - 1)b + (2p - 1)b
\]

Proceeding recursively \( t \) times, yields

\[
\mathbb{E}[w(t) \mid w(0)] = w_0 + t(2p - 1)b
\]

This analysis is not entirely correct because \( w(t) \) might be zero.
Analysis of outcomes: mean

- For a more accurate analysis analyze simulation’s outcome
- Consider $J$ experiments
- For each experiment, there is a wealth history $w_j(t)$
- We can estimate the average outcome as

$$\bar{w}_J(t) = \frac{1}{J} \sum_{j=1}^{J} w_j(t)$$

- $\bar{w}_J(t)$ is called the sample average
- Do not confuse $\bar{w}_J(t)$ with $\mathbb{E}[w(t)]$
  - $\bar{w}_J(t)$ is computed from experiments, it is a random quantity in itself
  - $\mathbb{E}[w(t)]$ is a property of the random variable $w(t)$
  - We will see later that for large $J$, $\bar{w}_J(t) \to \mathbb{E}[w(t)]$
Analysis of outcomes: mean

- Expected value $\mathbb{E}[w(t)]$ in black (approximation)
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)
There is more information in the simulation’s output

Estimate the probability distribution function (pdf) ⇒ Histogram

Consider a set of points \( w^{(1)}, \ldots, w^{(N)} \)

Indicator function of the event \( w^{(n)} \leq w_j < w^{(n+1)} \)

\[ \mathbb{I} \left[ w^{(n)} \leq w_j < w^{(n+1)} \right] = 1 \text{ when } w^{(n)} \leq w_j < w^{(n+1)} \]

\[ \mathbb{I} \left[ w^{(n)} \leq w_j < w^{(n+1)} \right] = 0 \text{ else} \]

Histogram is then defined as

\[ H \left[ t; w^{(n)}, w^{(n+1)} \right] = \frac{1}{J} \sum_{j=1}^{J} \mathbb{I} \left[ w^{(n)} \leq w_j(t) < w^{(n+1)} \right] \]

Fraction of experiments with wealth \( w_j(t) \) between \( w^{(n)} \) and \( w^{(n+1)} \)
The pdf broadens and shifts to the right \((t = 10, 50, 100, 200)\)
What is this class about

- Analysis and simulation of **stochastic systems**
  - A system that **evolves in time with some randomness**
- They are usually quite **complex**  ⇒ Simulations
- We will learn how to **model** stochastic systems, e.g.,
  - $x(t)$ Bernoulli with parameter $p$
  - $w(t) = w(t - 1) + b$ when $x(t) = 1$
  - $w(t) = w(t - 1) - b$ when $x(t) = 0$
- ... how to **analyze**, e.g., $\mathbb{E} \left[ w(t) \mid w(0) \right] = w_0 + t(2p - 1)b$
- ... and how to **interpret** simulations and experiments, e.g,
  - Average tendency through sample average