Stochastic Systems Analysis and Simulations

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Presentations

Class description and contents

Gambling
Who are us, where to find me, lecture times

- Alejandro Ribeiro, Luiz Chamon, Fernando Gama
- Walnut 3401 floor 4B. https://alelab.seas.upenn.edu/
- Teaching assistants: Vinicius Lima
- Class email: ese303@seas.upenn.edu.
- Don’t write to our personal addresses. Please.

- We also have a separate grader

- We meet on the Berger auditorium in Skirkanich Hall
- Mondays, Wednesdays, Fridays 10 am to 11 am
- My office hours, Wednesdays and Fridays from 11 am to 12 pm
- Anytime, as long as you have something interesting to tell me
- https://ese303.seas.upenn.edu/
Prerequisites

- **Probability theory**
- Stochastic processes are time-varying random entities
- If unknown, need to learn as we go
- Will cover in first seven lectures

- **Linear algebra**
- Vector matrix notation, systems of linear equations, eigenvalues

- **Programming in Matlab**
- Needed for homework.
- If you know programming you can learn Matlab in one afternoon
- But it has to be *this afternoon*

- **Differential equations, Fourier transforms**
- Appear here and there. Should not be a problem
Homework and grading

- 14 homework sets in 14 weeks
- Collaboration accepted, welcomed, and encouraged
- Sets graded as 0 (bad), 1 (good), 2 (very good) and 3 (outstanding)
- We’ll use the 3 sparingly. Goal is to earn 28 homework points

- **First Midterm** examination starts on Monday October 7, 36 points
- Take home due on Wednesday October 9
- Work independently. No collaboration, no discussion

- **Second midterm** is on Monday December 9 worth 36 points

- At least 60 points are required for passing.
- C requires at least 70 points. B at least 80. A at least 90
- Goal is for everyone to earn an A
Textbooks

- Textbook for the class is (older or newer editions acceptable)
- **Sheldon M. Ross ”Introduction to Probability Models”, Academic Press, whatever ed.**
- Same topics at advanced level (more rigor, includes proofs)
- **Sheldon Ross ”Stochastic Processes”, John Wiley & sons, 2nd ed.**
- Stochastic processes in systems biology
- **Darren J. Wilkinson ”Stochastic Modelling for Systems Biology”, Chapman & Hall/CRC, 1st ed.**
- Part on simulation of chemical reactions taken from here
- Use of stochastic processes in finance
- **Masaaki Kijima ”Stochastic Processes with Applications to Finance”, Chapman & Hall/CRC, 1st ed.**
This is not a programming class

- Just that. Not a programming class
This class has a reputation for being hard and demanding
⇒ I do not entirely agree but I take the point

On the other hand, the quality ratings are very good
⇒ S. Reid Warren Jr. Award (2012)
⇒ Lindback Award for Distinguished Teaching (2017)
⇒ Collected the two possible teaching awards in 8 years

This is, really, a great class. You will do things that look like magic

Also, the class is front loaded. It will become easier after the break.
⇒ Don’t drop it! You will enjoy it.
Some remarks on teaching methods

- I ask questions to individual students (cold calling).
  ⇒ Absolutely zero premium (penalty) on right (wrong) answer

- Do these questions serve any purpose?
  ⇒ I need to gauge what you understand of what I say
  ⇒ There are different ways of explaining ideas
  ⇒ Spatial memory associates parts of the room with concepts
  ⇒ People remember conversations better than lectures

- Can anyone explain to me why this is called cold calling?
Class contents

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Gambling
Stochastic systems

- Anything random that evolves in time
- Time can be discrete \((0, 1, \ldots)\) or continuous
- More formally, assign a function to a random event
- Compare with “random variable assigns a value to a random event”
- Generalizes concept of random vector to functions
- Or generalizes the concept of function to random settings
- Can interpret a stochastic process as a set of random variables
- Not always the most appropriate way of thinking
A voice recognition system

- Random event $\sim$ word spoken. Stochastic process $\sim$ the waveform
  - Try the file speech_signals.m
Probability theory review (6 lectures)
- Probability spaces
- Conditional probability: time $n + 1$ given time $n$, future given past ...
- Limits in probability, almost sure limits: behavior as $t \to \infty$ ...
- Common probability distributions (binomial, exponential, Poisson, Gaussian)

Stochastic processes are complicated entities

Restrict attention to particular classes that are somewhat tractable

Markov chains (9 lectures)

Continuous time Markov chains (12 lectures)

Stationary random processes (9 lectures)

Midterm covers up to Markov chains
Markov chains

- A set of states 1, 2, ... At time $n$, state is $X_n$
- **Memoryless** property
  - Probability of next state $X_{n+1}$ depends on current state $X_n$
  - But not on past states $X_{n-1}, X_{n-2}, \ldots$

- Can be happy ($X_n = 0$) or sad ($X_n = 1$)
- Happiness tomorrow affected by happiness today only
- Whether happy or sad today, likely to be happy tomorrow
- But when sad, a little less likely so

- Classification of states, ergodicity, limiting distributions
- Google's page rank, machine learning, virus propagation, queues ...
Continuous time Markov chains

- A set of states 1, 2, ... Continuous time index $t$
- Transition between states can happen at any time
- Future depends on present but is independent of the past

- Probability of changing state in an infinitesimal time $dt$

- Poisson processes, exponential distributions, transition probabilities, Kolmogorov equations, limit distributions
- Chemical reactions, queues, communication networks, weather forecasting ...
Stationary random processes

- Continuous time $t$, continuous state $x(t)$, not necessarily memoryless
- System has a steady state in a random sense
- Prob. distribution of $x(t)$ constant or becomes constant as $t$ grows
- Brownian motion, white noise, Gaussian processes, autocorrelation, power spectral density.
- Black Scholes model for option pricing, speech, noise in electric circuits, filtering and equalization ...
Gambling

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Gambling
An interesting betting game

- There is a certain game in a certain casino in which ...

  ⇒ your chances of winning are $p > 1/2$

- You place $1$ bets
  
  (a) With probability $p$ you gain $1$ and
  
  (b) With probability $(1 - p)$ you lose your $1$ bet

- The catch is that you either
  
  (a) Play until you go broke (lose all your money)
  
  (b) Keep playing forever

- You start with an initial wealth of $w_0$

- Shall you play this game?
Let $t$ be a time index (number of bets placed)

Denote as $x(t)$ the outcome of the bet at time $t$

- $x(t) = 1$ if bet is won (with probability $p$)
- $x(t) = 0$ if bet is lost (probability $(1 - p)$)

$x(t)$ is called a Bernoulli random variable with parameter $p$

Denote as $w(t)$ the player’s wealth at time $t$

At time $t = 0$, $w(0) = w_0$

At times $t > 0$ wealth $w(t)$ depends on past wins and losses

More specifically we have

- When bet is won $w(t) = w(t - 1) + 1$
- When bet is lost $w(t) = w(t - 1) - 1$
\[ t = 0; \ w(t) = w_0; \ \text{max}_t = 10^3; \ // \text{Initialize variables} \]

% repeat while not broke up to time \text{max} _t

\begin{verbatim}
while (w(t) > 0) & (t < max_t) do
    x(t) = random('bino',1,p);  % Draw Bernoulli random variable
    if x(t) == 1 then
        w(t + 1) = w(t) + b;  % If x = 1 wealth increases by b
    else
        x(t + 1) = w(t) - b;  % If x = 0 wealth decreases by b
    end
end
\end{verbatim}

▶ Initial wealth \( w_0 = 20 \), bet \( b = 1 \), win probability \( p = 0.55 \)

▶ Shall we play?
One lucky player

- She didn’t go broke. After $t = 1000$ bets, her wealth is $w(t) = 109$
- Less likely to go broke now because wealth increased
Two lucky players

- Wealths are $w_1(t) = 109$ and $w_2(t) = 139$
- Increasing wealth seems to be a pattern
Ten lucky players

- Wealths $w_j(t)$ between 78 and 139
- Increasing wealth is definitely a pattern
One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke
One unlucky player

- But this does not mean that all players will turn out as winners
- The twelfth player $j = 12$ goes broke
One hundred players

- Only one player \( (j = 12) \) goes broke
- All other players end up with substantially more money
Average tendency

- It is not difficult to find a line estimating the average of $w(t)$
- $\bar{w}(t) \approx w_0 + (2p - 1)t \approx w_0 + 0.1t$
Where does the average tendency come from?

- To discover average tendency notice that for all times $t$ we can write

$$W(t + 1) = W(t) + (2X(t) - 1)$$

- Taking expectation on both sides and using linearity of expectations

$$E[W(t + 1)] = E[W(t)] + (2E[X(t)] - 1)$$

- The expected value of $X(t)$ is

$$E[X(t)] = 1 \times P(X(t) = 1) + 0 \times P(X(t) = 1) = p$$

- Which yields

$$E[W(t + 1)] = E[W(t)] + (2p - 1)$$

- Applying recursively

$$E[W(t + 1)] = w_0 + (2p - 1)t$$
For a more accurate analysis analyze simulation’s outcome

Consider \( J \) experiments

For each experiment, there is a wealth history \( w_j(t) \)

We can estimate the average outcome as

\[
\bar{w}_J(t) = \frac{1}{J} \sum_{j=1}^{J} w_j(t)
\]

\( \bar{w}_J(t) \) is called the sample average

Do not confuse \( \bar{w}_J(t) \) with \( \mathbb{E}[w(t)] \)

- \( \bar{w}_J(t) \) is computed from experiments, it is a random quantity in itself
- \( \mathbb{E}[w(t)] \) is a property of the random variable \( w(t) \)
- We will see later that for large \( J \), \( \bar{w}_J(t) \to \mathbb{E}[w(t)] \)
Analysis of outcomes: mean

- Expected value $\mathbb{E}[w(t)]$ in black (approximation)
- Sample average for $J = 10$ (blue), $J = 20$ (red), and $J = 100$ (magenta)
There is more information in the simulation’s output

Estimate the probability distribution function (pdf) \( \Rightarrow \) Histogram

Consider a set of points \( w^{(0)}, \ldots, w^{(N)} \)

Indicator function of the event \( w^{(n)} \leq w_j < w^{(n+1)} \)

\( I[ w^{(n)} \leq w_j < w^{(n+1)} ] = 1 \) when \( w^{(n)} \leq w_j < w^{(n+1)} \)

\( I[ w^{(n)} \leq w_j < w^{(n+1)} ] = 0 \) else

Histogram is then defined as

\[
H[t; w^{(n)}, w^{(n+1)}] = \frac{1}{J} \sum_{j=1}^{J} I[ w^{(n)} \leq w_j(t) < w^{(n+1)} ]
\]

Fraction of experiments with wealth \( w_j(t) \) between \( w^{(n)} \) and \( w^{(n+1)} \)
The pdf broadens and shifts to the right ($t = 10, 50, 100, 200$)
What is this class about

- Analysis and simulation of stochastic systems
  ⇒ A system that evolves in time with some randomness

- They are usually quite complex ⇒ Simulations

- We will learn how to model stochastic systems, e.g.,
  - \( x(t) \) Bernoulli with parameter \( p \)
  - \( w(t) = w(t - 1) + 1 \) when \( x(t) = 1 \)
  - \( w(t) = w(t - 1) - 1 \) when \( x(t) = 0 \)

- ... how to analyze, e.g., \( \mathbb{E}[W(t)] = w_0 + t(2p - 1) \)

- ... and how to interpret simulations and experiments, e.g,
  - Average tendency through sample average