Arbitrages, and pricing of stock options

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Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Arbitrage

- Bet on different events with each outcome paying a random return
- **Arbitrage**: It is possible to devise a betting strategy that **guarantees a positive return** no matter the combined outcome of the events
- Arbitrages often involve operating in two different markets
Example

- Booker 1  ⇒ Phillies win pay 1.5:1, Phillies loose pay 3:1
  - Bet $x$ on Phillies and $y$ against Phillies. Guaranteed Earnings?
    
    Phillies win: $0.5x - y > 0 \implies x > 2y$
    Phillies loose: $-x + 2y > 0 \implies x < 2y$

  - Arbitrage not possible. Notice that $1/(1.5) + 1/3 = 1$

- Booker 2  ⇒ Phillies win pay 1.4:1, Phillies loose pay 3.1:1
  - Bet $x$ on Phillies and $y$ against Phillies. Guaranteed Earnings?
    
    Phillies win: $0.4x - y > 0 \implies x > 2.5y$
    Phillies loose: $-x + 2.1y > 0 \implies x < 2.1y$

  - Arbitrage not possible. Notice that $1/(1.4) + 1/(3.1) > 1$
Example (continued)

- First condition on Booker 1 and second on Booker 2 are compatible
- Bet $x$ on Phillies on Booker 1, $y$ against Phillies on Booker 2
- Guaranteed earnings possible. Make $y = 1,000$, $x = 2,066$
  
  Phillies win: $0.5(2,066) - 1,000 = 33$
  Phillies loose: $-2066 + 2.1(1000) = 34$

- Notice that $1/(1.5) + 1/(3.1) < 1$

- If you plan on doing this, do it on, e.g., currency exchange markets
Let events on which bets are posted be \( k = 1, 2, \ldots, K \)

Let \( j = 1, 2, \ldots, J \) index possible joint outcomes

- Joint realizations, also called “world realization”, or “world outcome”

If world outcome is \( j \), event \( k \) yields return \( r_{jk} \) per unit invested (bet)

Do not define probability \( p_j \) of outcome \( j \)

Invest (bet) \( x_k \) in outcome \( k \) \( \Rightarrow \) return for world \( j \) is \( x_k r_{jk} \)

Bets \( x_k \) can be positive \((x_k > 0)\) or negative \((x_k < 0)\)

\( \Rightarrow \) Positive = regular bet. Negative = short bet

Total return \( \Rightarrow \sum_{k=1}^{K} x_k r_{jk} = x^T r_j \)

Vectors of returns for outcome \( j \) \( \Rightarrow \) \( r_j := [r_{j1}, \ldots, r_{jK}]^T \) (given)

Vector of bets \( \Rightarrow \) \( x_j := [x_{j1}, \ldots, x_{jK}]^T \) (controlled by gambler)
Arbitrage (clearly defined now)

- Arbitrage is possible if there exists investment strategy \( \mathbf{x} \) such that
  \[ \mathbf{x}^T \mathbf{r}_j > 0, \quad \text{for all } j = 1, \ldots, J \]

- Equivalently, arbitrage is possible if
  \[
  \max_{\mathbf{x}} \left( \min_j (\mathbf{x}^T \mathbf{r}_j) \right) > 0
  \]

- Portfolio \( \mathbf{x} \) and returns \( \mathbf{r}_j \) are vectors in \( \mathbb{R}^K \)

- Earnings \( \mathbf{x}^T \mathbf{r}_j \) are the inner product of \( \mathbf{x} \) and \( \mathbf{r}_j \)

- Earnings are positive if angle between \( \mathbf{x} \) and \( \mathbf{r}_j \) is less than \( \pi/2 \) (90°)
When is arbitrage possible?

- There is a line that leaves all $r_j$ vectors to one side
- There is a line that leaves all $r_j$ vectors to one side

Arbitrage possible

- Prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$

- Think of $p_j$ as scaling factor

Arbitrage not possible

- There is prob. vector $p = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_p(r) = \sum_{j=1}^{J} p_j r_j = 0$$

- There is not a line that leaves all $r_j$ vectors to one side
Have “proved” following result, called arbitrage theorem

**Theorem**

*Given vectors of returns $r_j$, associated with random outcome $j = 1, \ldots, J$ an arbitrage is not possible if and only if there exist a probability vector $p$ such that $E_p(r) = 0$. Equivalently,*

$$\max_x \left( \min_j (x^T r_j) \right) \leq 0 \iff \sum_{j=1}^{J} p_j r_j = 0$$

*Prob. vector $p$ is NOT the prob. distribution of events $j = 1, \ldots, J$*
Consider a stock price $X(nh)$ that follows a geometric random walk

$$X((n + 1)h) = X(nh)e^{\sigma\sqrt{h}Y_n}$$

where $Y_n$ is a binary random variable with probability distribution

$$P[Y_n = 1] = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{h} \right), \quad P[Y_n = -1] = \frac{1}{2} \left( 1 - \frac{\mu}{\sigma} \sqrt{h} \right)$$

Recall that as $h \to 0$, $X(nh)$ becomes geometric Brownian motion

Are there arbitrage opportunities in the price of the stock?

⇒ Too general, let us consider a narrower problem
Consider the following investment strategy (stock flip):

**Buy:** Buy $1 in stock at time 0 for price $X(0)$ per unit of stock

**Sell:** Sell stock at time $h$ for price $X(h)$ for unit of stock

- Cost of transaction is $1$. Units of stock purchased are $1/X(0)$
- Cash after selling stock is $X(h)/X(0)$
- Return on investment is $X(h)/X(0) - 1$

There are two possible outcomes for the price of the stock at time $h$

- As per model we may have $Y_0 = 1$ or $Y_0 = -1$ respectively yielding

\[
X(h) = X(0)e^{\sigma\sqrt{h}}, \quad X(h) = X(0)e^{-\sigma\sqrt{h}}
\]

Possible returns are therefore

\[
r_1 = \frac{X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 = e^{\sigma\sqrt{h}} - 1, \quad r_2 = \frac{X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\sigma\sqrt{h}} - 1
\]
Present value of returns

- One dollar at time $h$ is not the same as 1 dollar at time 0
- Interest rate of a risk-free investment is $\alpha$ continuously compounded
- In practice, $\alpha$ is the money market rate
- Prices have to be compared at their present value

- The present value of $X(h)$ at time 0 is $X(h)e^{-\alpha h}$
- Then, return on investment is $e^{-\alpha h}X(h)/X(0) - 1$
- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are

$$r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{\sigma\sqrt{h}} - 1,$$

$$r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma\sqrt{h}} - 1.$$
Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1 - q)r_2 = 0$$

Arbitrage theorem in 1 dimension (only one bet, buy stock)

Substituting $r_1$ and $r_2$ for their respective values

$$q \left( e^{-\alpha h} e^{\sigma \sqrt{h}} - 1 \right) + (1 - q) \left( e^{-\alpha h} e^{-\sigma \sqrt{h}} - 1 \right) = 0$$

Can be easily solved for $q$. Expanding product and reordering terms

$$q e^{-\alpha h} e^{\sigma \sqrt{h}} + (1 - q) e^{-\alpha h} e^{-\sigma \sqrt{h}} = 1$$

Multiplying by $e^{\alpha h}$ and grouping terms with a $q$ factor

$$q \left( e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}} \right) = e^{\alpha h} - e^{-\sigma \sqrt{h}}$$
No arbitrage condition (continued)

- Solving for \( q \) finally yields
  \[ q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}} \]

- For small \( h \) we have \( e^{\alpha h} \approx 1 + \alpha h \) and \( e^{\pm \sigma \sqrt{h}} \approx 1 \pm \sigma \sqrt{h} + \sigma^2 h / 2 \)

- Thus, the value of \( q \) as \( h \to 0 \) may be approximated as
  \[ q \approx \frac{1 + \alpha h - \left(1 - \sigma \sqrt{h} + \sigma^2 h / 2\right)}{1 + \sigma \sqrt{h} - \left(1 - \sigma \sqrt{h}\right)} = \frac{\sigma \sqrt{h} + (\alpha - \sigma^2 / 2) h}{2\sigma \sqrt{h}} \]

  \[ = \frac{1}{2} \left(1 + \frac{\alpha - \sigma^2 / 2}{\sigma} \sqrt{h}\right) \]

- Approximation proves that at least for small \( h \) \( 0 < q < 1 \)
  \( \Rightarrow \) Arbitrage not possible

- Also, suspiciously similar to probabilities of geometric random walk
  \( \Rightarrow \) Fundamental observation as we’ll see next
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Stock prices $X(t)$ follow geometric random walk (drift $\mu$, variance $\sigma^2$)

Risk free investment has return $\alpha$ (cost of money, money market)

Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

$$q = \frac{e^{\alpha h} - e^{-\sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}}$$

Notice that $q$ satisfies the equation (which we’ll use later on)

$$qe^{\sigma \sqrt{h}} + (1 - q)e^{-\sigma \sqrt{h}} = e^{\alpha h}$$

Can we have arbitrage using a more complex set of possible bets?
Consider the following general investment strategy:

**Observe:** Observe the stock price at times \( h, 2h, \ldots, nh \)

**Compare:** Is \( X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n \) ?

**Buy:** If above answer is yes, buy stock at price \( X(nh) \)

**Sell:** Sell stock at time \( mh \) for price \( X(mh) \)

- Possible bets are the observed values of the stock \( x_1, x_2, \ldots, x_l \)
  - \( \Rightarrow \) There are \( 2^n \) possible bets

- Possible outcomes are value at time \( mh \) and observed values
  - \( \Rightarrow \) There are \( 2^m \) possible outcomes
Explanation of general investment strategy

Bet 1 = \( n \) price increases, bet 2 = price increases in 1, \ldots, \( n-1 \) and price decrease in \( n \) ...

For each bet we have \( 2^{m-n} \) possible outcomes: \( m-n \) price increases, price increases in \( n+1, \ldots, m-1 \) and price decrease in \( m \) ...

<table>
<thead>
<tr>
<th>( \text{bet 1} )</th>
<th>( X(h) )</th>
<th>( X(2h) )</th>
<th>( X(3h) )</th>
<th>( X(nh) )</th>
<th>( X((n+1)h) )</th>
<th>( X((n+2)h) )</th>
<th>( X(mh) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{bet 1} )</td>
<td>( e^{\sigma \sqrt{h}} )</td>
<td>( e^{2\sigma \sqrt{h}} )</td>
<td>( e^{3\sigma \sqrt{h}} )</td>
<td>( e^{n\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{m\sigma \sqrt{h}} )</td>
</tr>
<tr>
<td>( \text{bet 2} )</td>
<td>( e^{\sigma \sqrt{h}} )</td>
<td>( e^{2\sigma \sqrt{h}} )</td>
<td>( e^{3\sigma \sqrt{h}} )</td>
<td>( e^{(n-2)\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{(m-2)\sigma \sqrt{h}} )</td>
</tr>
<tr>
<td>( \text{bet 2^n} )</td>
<td>( e^{-\sigma \sqrt{h}} )</td>
<td>( e^{-2\sigma \sqrt{h}} )</td>
<td>( e^{-3\sigma \sqrt{h}} )</td>
<td>( e^{-n\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-2\sigma \sqrt{h}} )</td>
<td>( X(nh)e^{-m\sigma \sqrt{h}} )</td>
</tr>
</tbody>
</table>

outcomes per each bet

Figure assumes \( X(0) = 1 \) for simplicity
Explanation of general investment strategy

- Define the prob. distribution $q$ over possible outcomes as follows
- Start with a sequence of independent identically distributed $Y_n$
- Each element $Y_n$ is a binary random variable with probabilities

$$P[Y_n = 1] = q, \quad P[Y_n = -1] = 1 - q$$

- With $q = \left( e^{\alpha h} - e^{-\sigma \sqrt{h}} \right) / \left( e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}} \right)$ as in slide 16
- Joint prob. distribution $q$ on $X(h), X(2h), \ldots, X((n + m)h)$ outcomes obtained through transformation

$$X((n + 1)h) = X(nh)e^{\sigma \sqrt{h}Y_n}$$

- Notice once more that this is NOT the prob. distribution of $X(nh)$
- Will show that expected value of earnings with respect to $q$ is null

$\Rightarrow$ Thus, arbitrages are not possible
Return for given outcome

- Consider a time 0 unit investment in given arbitrary outcome
- Stock units purchased depend on the price \( X(nh) \) at buying time

\[
\text{Units bought} = \frac{1}{X(nh)e^{-\alpha nh}}
\]

- Have corrected \( X(nh) \) to express it in time 0 values
- Cash after selling stock given by price \( X(mh) \) at sell time \( m + n \)
- Expressed in time 0 values

\[
\text{Cash after sell} = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}
\]

- Return is then \( r(X(h), \ldots, X(mh)) = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \)
- Depends on \( X(mh) \) and \( X(nh) \) only
Consider expected value of all possible returns with respect to $q$

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$

Condition on observed values $X(h), \ldots, X(nh)$

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q(1:n) \left[ \mathbb{E}_q(n+1:m) \left[ \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \mid X(h), \ldots, X(nh) \right] \right]$$

In innermost expectation $X(nh)$ is given. Furthermore, process $X(t)$ is Markov, thus conditioning on $X(h), \ldots, X((n-1)h)$ is irrelevant. Thus

$$\mathbb{E}_q \left[ r(X(h), \ldots, X(mh)) \right] = \mathbb{E}_q(1:n) \left[ \frac{\mathbb{E}_q(n+1:m) \left[ X(mh) \mid X(nh) \right] e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \right]$$
Expected value of future values (measure $q$)

- Need to find expectation of future value $\mathbb{E}_{q(n+1:m)}[X(mh) \mid X(nh)]$

- From recursive relation for $X(nh)$ in terms of $Y_n$ sequence

  $$X(mh) = X((m - 1)h)e^{\sigma \sqrt{h}Y_{m-1}} = X((m - 2)h)e^{\sigma \sqrt{h}Y_{m-1}}e^{\sigma \sqrt{h}Y_{m-2}}$$

  $$\vdots$$

  $$= X(nh)e^{\sigma \sqrt{h}Y_{m-1}}e^{\sigma \sqrt{h}Y_{m-2}} \cdots e^{\sigma \sqrt{h}Y_{n+1}}$$

- All the $Y_n$ are independent. Then, upon taking expected value

  $$\mathbb{E}_{q(n+1:m)}[X(mh) \mid X(nh)] = X(nh)\mathbb{E}\left[e^{\sigma \sqrt{h}Y_{m-1}}\right] \mathbb{E}\left[e^{\sigma \sqrt{h}Y_{m-2}}\right] \cdots \mathbb{E}\left[e^{\sigma \sqrt{h}Y_{n+1}}\right]$$

- Need to determine expectation of relative price increase $\mathbb{E}\left[e^{\sigma \sqrt{h}Y_{n}}\right]$
The expected value of the relative price increase $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$ is

$$\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} \Pr [Y_n = 1] + e^{-\sigma \sqrt{h}} \Pr [Y_n = -1]$$

According to definition of measure $q$, it holds

$$\Pr [Y_n = 1] = q, \quad \Pr [Y_n = -1] = 1 - q$$

Substituting in expression for $\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right]$

$$\mathbb{E} \left[ e^{\sigma \sqrt{h} Y_n} \right] = e^{\sigma \sqrt{h}} q + e^{-\sigma \sqrt{h}} (1 - q) = e^{\alpha h}$$

where last equality follows from definition of probability $q$

Reweave the quilt $\Rightarrow$ use expected relative price increase to compute expected future value to find expected return
Reweave the quilt

► Substitute expected relative price increase into expression for expected future value

\[ \mathbb{E}_{q(n+1:m)} [X(mh) \mid X(nh)] = X(nh) e^{\alpha h} e^{\alpha h} \cdots e^{\alpha h} = X(nh) e^{\alpha (m-n)h} \]

► Substitute result into expression for expected return

\[ \mathbb{E}_q [r(X(h), \ldots, X(mh))] = \mathbb{E}_{q(1:n)} \left[ \frac{X(nh) e^{\alpha (m-n)h} e^{-\alpha mh}}{X(nh) e^{-\alpha nh}} - 1 \right] \]

► Exponentials cancel each other, finally yielding

\[ \mathbb{E}_q [r(X(h), \ldots, X(mh))] = \mathbb{E}_{q(1:n)} [1 - 1] = 0 \]

► Arbitrage not possible in any trading strategy if \( 0 \leq q \leq 1 \) exists
If prices follow geometric Brownian motion

- Stock prices follow a geometric Brownian motion, i.e.,
  \[ X(t) = X(0)e^{Y(t)} \]
- with \( Y(t) \) Brownian motion with drift \( \mu \) and variance \( \sigma^2 \)
- What is the no arbitrage condition?
- Approximate geometric Brownian motion by geometric random walk
- No arbitrage measure \( q \) exists for geometric random walk
  - This requires \( h \) sufficiently small
  - Notice that prob. distribution \( q = q(h) \) is a function of \( h \)
- Approximation arbitrarily accurate by letting \( h \to 0 \)
- Existence of the prob. distribution \( q := \lim_{h \to 0} q(h) \) proves that arbitrages are not possible in stock trading
Recall that as $h \to 0 \Rightarrow q \approx \frac{1}{2} \left( 1 + \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h} \right)$

And consequently $\Rightarrow (1 - q) \approx \frac{1}{2} \left( 1 - \frac{\alpha - \sigma^2/2}{\sigma} \sqrt{h} \right)$

Thus, measure $q := \lim_{h \to 0} q(h)$ is geometric Brownian motion

$\Rightarrow$ Variance $\Rightarrow \sigma^2$ (same as stock price)

$\Rightarrow$ Drift $\Rightarrow \alpha - \sigma^2 / 2$

Measure showing arbitrage not possible is a geometric random walk

Which is also the way stock prices evolve

Furthermore, the variance is the same as that of stock prices

The drifts are different $\Rightarrow \mu$ for stocks and $\alpha - \sigma^2 / 2$ for no arbitrage
Compute expected return on an investment on stock $X(t)$

Buy 1 share of stock at time 0. Cash invested $\Rightarrow X(0)$

Sell stock at time $t$. Cash after sell $\Rightarrow X(t)$

Expected value of cash after sell given $X(0)$ is

$$\mathbb{E} [X(t) \mid X(0)] = X(0)e^{(\mu + \sigma^2/2)t}$$

Alternatively, invest $X(0)$ risk free in the money market

Guaranteed cash at time $t$ is $X(0)e^{\alpha t}$

Invest in stock only if $\mu + \sigma^2/2 > \alpha \Rightarrow$ risk premium
Compute expected return as if $q$ were the actual distribution

- And recall that $q$ is NOT the actual distribution
- As before, cash invested is $X(0)$ and cash after sale is $X(t)$
- Expected cash value is different because prob. distribution is different

\[
\mathbb{E}_q \left[ X(t) \mid X(0) \right] = X(0)e^{(\alpha-\sigma^2/2+\sigma^2/2)t} = X(0)e^{\alpha t}
\]

- Same return as risk free investment regardless of parameters’ values
- Measure $q$ is called risk neutral measure
- Risky stock investments yield same return as risk free investments
- “Alternate universe” in which investors do not demand risk premiums

- Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)
- Basis for Black-Scholes. More later
Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing
Options

- An option is a contract to buy shares of a stock at a future time
- Strike time $t = $ Convened time for stock purchase
- Strike price $K = $ Price at which stock is purchased at strike time
- At time $t$, option holder may decide to
  - Buy a stock at strike price $K = $ exercise the option
  - Do not exercise the option
- May buy option at time 0 for price $c$
- How do we **determine the option’s worth**, i.e., price $c$, at time 0?
- Answer given by Black-Scholes formula for option pricing
Let $e^{\alpha t}$ be the compounding of a risk free investment

Let $X(t)$ be the stock’s price at time $t$

Price modeled as geometric Brownian motion, drift $\mu$, variance $\sigma^2$

Risk neutral measure $q$ is also a geometric Brownian motion

$\Rightarrow$ Variance $\sigma^2$ and drift $\alpha - \sigma^2/2$
At time $t$, the option’s worth depends on the stock’s price $X(t)$.

- If stock’s price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  $\Rightarrow$ Option is worthless (better to buy stock at current price)
- Since had paid $c$ for the option at time 0, lost $c$ on this investment
  $\Rightarrow$ return on investment is $r = -c$

If stock’s price larger than strike price $\Rightarrow X(t) > K$
  $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$

To obtain return express as time 0 values and subtract $c$

$$r = e^{-\alpha t}(X(t) - K) - c$$

May combine both in single equation $\Rightarrow r = e^{-\alpha t}(X(t) - K)^+ - c$

$(\cdot)^+$ denotes projection on positive reals
Consider mixed positions on stocks and options

Is there a position guaranteeing positive return, i.e., an arbitrage?

Assume expected return under risk neutral measure is nonzero

\[
\mathbb{E}_q[r] = \mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] \neq 0
\]

Then, an arbitrage is possible according to arbitrage theorem

If expected return under risk neutral measure is zero

\[
\mathbb{E}_q[r] = \mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] = 0
\]

Then, no arbitrage is possible according to arbitrage theorem

Select options price \( c \) to prevent arbitrage opportunities
To have no arbitrage, must select option’s price $c$ so that

$$
\mathbb{E}_q \left[ e^{-\alpha t} (X(t) - K)^+ - c \right] = 0
$$

where expectation is with respect to risk neutral measure.

From above condition, the no-arbitrage price of the option is

$$
c = e^{-\alpha t} \mathbb{E}_q \left[ (X(t) - K)^+ \right]
$$

Source of Black-Scholes formula for option valuation.

Rest of derivation is just evaluation of expected value.

Same argument used to price any derivative of the stock’s price.
Use fact that prices are a geometric random walk

- Let us evaluate expectation to compute option’s price $c$
- Prices follow a geometric random walk $\Rightarrow X(t) = X_0 e^{Y(t)}$
- $X_0 =$ price at time 0,
- $Y(t)$ random walk with drift parameter $\mu$ and variance parameter $\sigma^2$
- Can rewrite no arbitrage condition as

$$c = e^{-\alpha t} \mathbb{E}_q \left[ \left( X_0 e^{Y(t)} - K \right)^+ \right]$$

- $Y(t)$ random walk. Then, in particular, $Y(t) \sim \mathcal{N}(\mu t, t\sigma^2)$

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t\sigma^2}} \int_{-\infty}^{\infty} (X_0 e^y - K)^+ e^{-(y-\mu t)^2/(2t\sigma^2)} \, dy$$
Evaluation of the integral

- Note that \((X_0e^{Y(t)} - K)^+ = 0\) for all values \(Y(t) \leq \log(K/X_0)\)
- Because integrand is null for \(Y(t) \leq \log(K/X_0)\) can write

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t\sigma^2}} \int_{\log(K/X_0)}^\infty (X_0e^y - K) e^{-(y-\mu t)^2/(2t\sigma^2)} \, dy
\]

- Change of variables \(z = (y - \mu t)/\sqrt{t\sigma^2}\). Associated replacements

  Variable: \(y \Rightarrow \sqrt{t\sigma^2}z + \mu t\)

  Differential: \(dy \Rightarrow \sqrt{t\sigma^2} \, dz\)

  Integration limit: \(\log(K/X_0) \Rightarrow a := \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}}\)

- Option price then given by

\[
c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_a^\infty \left(X_0e^{\sqrt{t\sigma^2}z + \mu t} - K\right) e^{-z^2/2} \, dz
\]
Separate in two integrals $c = e^{-\alpha t}(I_1 - I_2)$ where

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{t\sigma^2}z + \mu t} e^{-z^2/2} \, dz$$

$$I_2 := \frac{K}{\sqrt{2\pi}} \int_a^\infty e^{-z^2/2} \, dz$$

Gaussian Q function (ccdf of normal RV with mean 0 and variance 1)

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} \, dz$$

Comparing last two equations we have $I_2 = KQ(a)$

$I_1$ requires some more work
 Evaluation of the integral (continued)

▶ Reorder terms in integral $I_2$

\[
I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{t\sigma^2}z + \mu t} e^{-z^2/2} \, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{\sqrt{t\sigma^2}z - z^2/2} \, dz
\]

▶ The exponent can be written as a square minus a "constant" (no $z$)

\[
-z^2/2 + \sqrt{t\sigma^2}z - t\sigma^2/2 + t\sigma^2/2
\]

▶ Substituting the latter into $I_1$ yields

\[
I_1 = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{-(z-\sqrt{t\sigma^2})^2/2 + t\sigma^2/2} \, dz = \frac{X_0 e^{\mu t + t\sigma^2/2}}{\sqrt{2\pi}} \int_a^\infty e^{-(z-\sqrt{t\sigma^2})^2/2} \, dz
\]
Evaluation of the integral (continued)

- Change of variables $u = z - \sqrt{t\sigma^2} \Rightarrow du = dz$ and integration limit

  $$a \Rightarrow b := a - \sqrt{t\sigma^2} = \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}} - \sqrt{t\sigma^2}$$

- Implementing change of variables in $I_1$

  $$I_1 = \frac{X_0e^{\mu t + t\sigma^2/2}}{\sqrt{2\pi}} \int_{b}^{\infty} e^{u^2/2} du = X_0e^{\mu t + t\sigma^2/2} Q(b)$$

- Putting together results for $I_1$ and $I_2$

  $$c = e^{-\alpha t}(I_1 - I_2) = e^{-\alpha t} X_0e^{\mu t + t\sigma^2/2} Q(b) - e^{-\alpha t} KQ(a)$$

- For non-arbitrage stock prices $\Rightarrow \alpha = \mu + \sigma^2/2$

- Substitute to obtain Black-Scholes formula
Black-Scholes formula for option pricing

\[ c = X_0 Q(b) - e^{-\alpha t} K Q(a) \]

Where \( a := \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}} \) and \( b := a - \sqrt{t\sigma^2} \)

Note further that \( \mu = \alpha - \sigma^2/2 \). Can then write \( a \) as

\[ a = \frac{\log(K/X_0) - (\alpha - \sigma^2/2) t}{\sqrt{t\sigma^2}} \]

- \( X_0 \) = stock price at time 0, \( c \) = option cost at time 0,
- \( K \) = option’s strike price, \( t \) = option’s strike time
- \( \alpha \) = benchmark risk-free rate of return (cost of money)
- \( \sigma^2 \) = volatility of stock
- Black-Scholes formula independent of stock’s mean tendency \( \mu \)