Week 1: Introduction – A lower bounded random walk

Consider a game in which players bet \$1 and win \$1 with probability p or lose their bets with probability $q = 1 - p$. The wealth of a player as a function of time is a stochastic process. Indeed, if the player's wealth at time t is $w(t)$ the wealth at time $t+1$ is either $w(t)+1$ (with probability p) or $w(t) - 1$ (with probability q). Formally, we can write this as

$$\begin{aligned} \mathbb{P}\left[w(t+1) = w(t) + 1 \mid w(t)\right] = p, \\ \mathbb{P}\left[w(t+1) = w(t) - 1 \mid w(t)\right] = q. \end{aligned} \tag{1}$$

The first equation is read "the probability of $w(t+1)$ being equal to $w(t) + 1$ given $w(t)$ is p." The expressions in (**??**) are true as long as $w(t) \neq 0$. When $w(t) = 0$ the gambler is ruined and $w(t+1) = 0$. A rather sophisticated, but sometimes useful way of expressing this fact is to write

$$\mathbb{P}\left[w(t+1) = 0 \mid w(t) = 0\right] = 1. \tag{2}$$

This stochastic process is sometimes called a *lower bounded random walk*. That is because the wealth can be interpreted as the position on a line and wealth variations as steps taken randomly to the left or to the right. The origin is home, in that if the walker reaches 0 it no longer moves. We saw in class that if $p > 1/2$, then $w(t)$ is likely to diverge, making this a good game to play. But in this exercise, we let p take any value.

A Simulation of a realization of the process. Write a function that takes as inputs the probability p, the initial wealth $w(0) = w_0$, and a maximum number of rounds T. The function must return a vector of length *at most* $T+1$ containing the player's wealth history $w(0), \ldots, w(T)$, computed according to the stochastic process described in (**??**) and (**??**). If the wealth is depleted at time $t < T$, i.e., if $w(t) = 0$ for some $t < T$, then the function should returns a vector of length $t + 1$ with the player's wealth history up to time t, i.e., $w(0), \ldots, w(t)$. The function must also return a boolean variable that distinguishes between runs that resulted in a broke player and those that did not. Show plots of simulated processes with $w_0 = 20$ and $T = 10^3$ for $p = 0.25$, $p = 0.5$, and $p = 0.75$.

B Probability of ruin. Fixing $p = 0.55$ and $w_0 = 10$, use your function from part **??** to estimate the probability $B(p, w_0)$ of eventually going broke (or the equivalent random walk eventually reaching home), i.e., the probability of having $w(t) = 0$ for some t. Because once the player's wealth reaches zero it stays zero forever, this probability can be written as the limit

$$B(p, w_0) = \lim_{t \to \infty} \mathbb{P}\left[w(t) = 0 \mid w(0) = w_0\right]. \tag{3}$$

Strictly speaking, you would need to run the simulation forever to make sure the gambler does not run out of money eventually. However, you can truncate simulations at time $T = 100$ for this exercise. Doing this, you are computing the probability of reaching home between times 0 and T, which we assume is a good approximation for the probability of reaching home between times 0 and ∞. Formally, we are assuming that $\mathbb{P}\left[w(100)=0 \mid w(0)=w_0\right]$ is a good approximation for the limit in (**??**).

To estimate $\mathbb{P}\left[w(T)=0 \mid w(0)=w_0\right]$, we must run the code from part **??** multiple times. Let the result of the n-th runs be the wealth path $w_n(t)$ and define the function $\mathbb{I}\left[w_n(T)=0\right]$ to be equal to 1 if the wealth at time T is $w_n(T)=0$ and 0 otherwise. The probability of the gambler going broke can then be estimated as

$$\hat{B}_N(p, w_0) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[w_n(T)=0\right]. \tag{4}$$

The expression in (**??**) is just the average number of times the gambler was ruined across all experiments. The function $\mathbb{I}\left[w_n(T)=0\right]$ is called the *indicator function* of the event $w_n(T)=0$ because it "indicates" whether the event occurred by taking the value 1.

To compute $\hat{B}_N(p, w_0)$ you need to decide on a number of experiments N. The more experiments you run, the more accurate your estimation. Alas, the more you need to wait. Report your probability estimate and the number of experiments N used. Explain your criteria for selecting N.

C Probability of ruin as a function of initial wealth. We want to study the probability of the gambler going broke as a function of the initial wealth. Fix $p=0.55$ and vary the initial wealth between $w_0=1$ and $w_0=20$. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of the initial wealth. The number of experiments N you use to compute probability estimates for different initial wealths need not be the same.

D Probability of ruin as a function of p. The goal is to understand the variation of the probability of ruin for different values of the probability p. Fix $w_0=10$ and vary p between 0.3 and 0.7—increments 0.02 should do. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of p. You should observe a fundamentally different behavior for $p<1/2$ and $p>1/2$. Comment on that.

E Time to ruin. Fix $p=0.4$. With this value of p, the gambler's wealth will eventually deplete independently of the initial wealth w_0. This is something remarkable: despite the process being random, it is possible to say that $w(t)$ eventually becomes 0. This needs to be qualified though. Unlikely as it may be, there is a chance of winning all hands. Of course, the probability of this happening becomes smaller as the gambler plays more hands. What we can say about a lower bounded random walk is that with probability one, the wealth $w(t)$ approaches 0 as t grows. Formally, the limit $\lim_{t\to\infty} w(t)$ satisfies

$$\mathbb{P}\left[\lim_{t\to\infty} w(t)=0\right]=1. \tag{5}$$

In words, we say that "$w(t)\to 0$ as $t\to\infty$ *almost surely*". Different wealth paths are possible, but almost all of them result in a broke gambler. If we think of probabilities as measuring the likelihood of an event, the measure of the event $w(t)\neq 0$ is asymptotically null. An important quantity here is therefore the time t at which $w(t)=0$ for the first time, which we can write as

$$T_0 = \min_t \left\{w(t)=0\right\}. \tag{6}$$

Using your function from part **??**, estimate the probability distribution of T_0 and its average value.