Week 1: Introduction – A lower bounded random walk

Consider a game in which players bet $1 and win $1 with probability $p$ or lose their bets with probability $q = 1 - p$. The wealth of a player as a function of time is a stochastic process. Indeed, if the player’s wealth at time $t$ is $w(t)$ the wealth at time $t + 1$ is either $w(t) + 1$ (with probability $p$) or $w(t) - 1$ (with probability $q$). Formally, we can write this as

$$
P[w(t + 1) = w(t) + 1 \mid w(t)] = p,\]
$$
$$
P[w(t + 1) = w(t) - 1 \mid w(t)] = q. \tag{1}$$

The first equation is read “the probability of $w(t + 1)$ being equal to $w(t) + 1$ given $w(t)$ is $p$.” The expressions in (1) are true as long as $w(t) \neq 0$. When $w(t) = 0$ the gambler is ruined and $w(t + 1) = 0$. A rather sophisticated, but sometimes useful way of expressing this fact is to write

$$
P[w(t + 1) = 0 \mid w(t) = 0] = 1. \tag{2}$$

This stochastic process is sometimes called a lower bounded random walk. That is because the wealth can be interpreted as the position on a line and wealth variations as steps taken randomly to the left or to the right. The origin is home, in that if the walker reaches 0 it no longer moves. We saw in class that if $p > 1/2$, then $w(t)$ is likely to diverge, making this a good game to play. But in this exercise, we let $p$ take any value.

A Simulation of a realization of the process. Write a function that takes as inputs the probability $p$, the initial wealth $w(0) = w_0$, and a maximum number of rounds $T$. The function must return a vector of length at most $T + 1$ containing the player’s wealth history $w(0), \ldots, w(T)$, computed according to the stochastic process described in (1) and (2). If the wealth is depleted at time $t < T$, i.e., if $w(t) = 0$ for some $t < T$, then the function should return a vector of length $t + 1$ with the player’s wealth history up to time $t$, i.e., $w(0), \ldots, w(t)$. The function must also return a boolean variable that distinguishes between runs that resulted in a broke player and those that did not. Show plots of simulated processes with $w_0 = 20$ and $T = 10^3$ for $p = 0.25$, $p = 0.5$, and $p = 0.75$.

B Probability of ruin. Fixing $p = 0.55$ and $w_0 = 10$, use your function from part ?? to estimate the probability $B(p, w_0)$ of eventually going broke (or the equivalent random walk eventually reaching home), i.e., the probability of having $w(t) = 0$ for some $t$. Because once the player’s wealth reaches zero it stays zero forever, this probability can be written as the limit

$$
B(p, w_0) = \lim_{t \to \infty} P[w(t) = 0 \mid w(0) = w_0]. \tag{3}
$$

Strictly speaking, you would need to run the simulation forever to make sure the gambler does not run out of money eventually. However, you can truncate simulations at time $T = 100$ for this exercise. Doing this, you are computing the probability of reaching home between times 0 and $T$, which we assume is a good approximation for the probability of reaching home between times 0
and $\infty$. Formally, we are assuming that $\mathbb{P}[w(100) = 0 \mid w(0) = w_0]$ is a good approximation for the limit in (??).

To estimate $\mathbb{P}[w(T) = 0 \mid w(0) = w_0]$, we must run the code from part ?? multiple times. Let the result of the $n$-th run be the wealth path $w_n(t)$ and define the function $\mathbb{I}[w_n(T) = 0]$ to be equal to 1 if the wealth at time $T$ is $w_n(T) = 0$ and 0 otherwise. The probability of the gambler going broke can then be estimated as

$$\hat{B}_N(p, w_0) = \frac{1}{N} \sum_{n=1}^{N} \mathbb{I}[w_n(T) = 0].$$

(4)

The expression in (??) is just the average number of times the gambler was ruined across all experiments. The function $\mathbb{I}[w_n(T) = 0]$ is called the indicator function of the event $w_n(T) = 0$ because it “indicates” whether the event occurred by taking the value 1.

To compute $\hat{B}_N(p, w_0)$ you need to decide on a number of experiments $N$. The more experiments you run, the more accurate your estimation. Alas, the more you need to wait. Report your probability estimate and the number of experiments $N$ used. Explain your criteria for selecting $N$.

C Probability of ruin as a function of initial wealth.

We want to study the probability of the gambler going broke as a function of the initial wealth. Fix $p = 0.55$ and vary the initial wealth between $w_0 = 1$ and $w_0 = 20$. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of the initial wealth. The number of experiments $N$ you use to compute probability estimates for different initial wealths need not be the same.

D Probability of ruin as a function of $p$.

The goal is to understand the variation of the probability of ruin for different values of the probability $p$. Fix $w_0 = 10$ and vary $p$ between 0.3 and 0.7—increments 0.02 should do. Show a plot of your probability estimates $\hat{B}_N(p, w_0)$ as a function of $p$. You should observe a fundamentally different behavior for $p < 1/2$ and $p > 1/2$. Comment on that.

E Time to ruin.

Fix $p = 0.4$. With this value of $p$, the gambler’s wealth will eventually deplete independently of the initial wealth $w_0$. This is something remarkable: despite the process being random, it is possible to say that $w(t)$ eventually becomes 0. This needs to be qualified though. Unlike as it may be, there is a chance of winning all hands. Of course, the probability of this happening becomes smaller as the gambler plays more hands. What we can say about a lower bounded random walk is that with probability one, the wealth $w(t)$ approaches 0 as $t$ grows. Formally, the limit $\lim_{t \to \infty} w(t)$ satisfies

$$\mathbb{P}\left[ \lim_{t \to \infty} w(t) = 0 \right] = 1.$$  

(5)

In words, we say that “$w(t) \to 0$ as $t \to \infty$ almost surely”. Different wealth paths are possible, but almost all of them result in a broke gambler. If we think of probabilities as measuring the likelihood of an event, the measure of the event $w(t) \neq 0$ is asymptotically null. An important quantity here is therefore the time $t$ at which $w(t) = 0$ for the first time, which we can write as

$$T_0 = \min_{t} \{w(t) = 0\}.$$  

(6)

Using your function from part ??, estimate the probability distribution of $T_0$ and its average value.