Week 13: Gaussian processes
Option pricing

The goal of this problem is to use one year of historic data of a company's stock to determine the price of a European-style option[1]. Consider the daily closing price of Cisco Systems (CSCO) between November 25, 2008 and November 24, 2009. The closing price of CSCO for this date range can be obtained from the class's site and is depicted in Fig. 1. This is actual data, as you can corroborate from information available in Google finance (choose the `1yr` option).

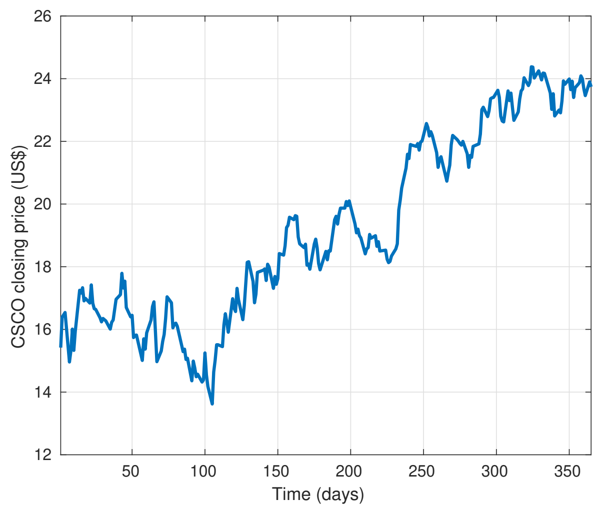

Figure 1: Daily closing price of Cisco Systems (CSCO) between November 25, 2008 and November 24, 2009.

To determine the option's price, we model the evolution of the stock price $X(t)$ using a geometric Brownian motion with drift. More specifically, we assume that the stock prices change according to

$$X(t+s) = X(t)e^{Y_t(s)}, \tag{1}$$

where $Y_t(s)$ is normally distributed with mean μs and variance $\sigma^2 s$ and is independent of t. Furthermore, we assume that the relative price changes $Y_t(s)$ over disjoint time intervals are independent, i.e., the random variables $(Y_t(s), Y_u(r))$ are independent if and only if $(t, t+s)$ and $(u, u+r)$ are disjoint. We will be interested in a discrete-time version of this process, sample at fixed intervals of duration h, say $h = 1$ day. Formally, we consider the stochastic process

$$Y_n = \log\left[X(nh)\right] - \log\left[X\big((n-1)h\big)\right] = Y_{(n-1)h}(h). \tag{2}$$

Notice from the model in (1) that the Y_n are independent identically distributed normals with mean μh and variance $\sigma^2 h$. Thus, we can infer the parameters μ and σ^2 from empirical data.

[1] A European-style option gives you the right to buy or sell an asset or stock at a specific price on a specific date. It is called an option because you are not required to do so: you have the *option* to buy or sell.

Indeed, the drift parameter μ is estimated by the sample mean

$$\hat{\mu} = \frac{1}{Nh} \sum_{n=1}^{N} Y_n \tag{3}$$

and the volatility parameter σ is estimated by the sample variance

$$\hat{\sigma}^2 = \frac{1}{(N-1)h} \sum_{n=1}^{N} \left(Y_n - \hat{\mu}h\right)^2. \tag{4}$$

Notice that we take the unbiased sample variance estimate by using the factor $1/(N-1)$, where N is the sample size. Still, for large samples (large N), diving by N yields almost the same estimate. In both (3) and (4), the Y_n are computed as in (2) using the data collected on the values of $X(nh)$.

Let us now formalize our option system. A European-style option on a stock is a contract that gives you the option to buy this stock at a predetermined price at a specific time in the future. This option is described by its strike price K, its strike time t, and its price c. Paying c to buy an option at time 0, gives you the opportunity to buy the stock for the strike price K at the strike time t. At that instant, the worth of the option depends on the value of the stock $X(t)$. If the stock has fallen below the strike price K, i.e., if $X(t) \leq K$, the option is worthless. On the contrary, if the price has risen above K, i.e., if $X(t) > K$, we can realize a gain $X(t) - K$ by exercising the option to buy the stock. Hence, we write the worth of the option w as

$$w = \left[X(t) - K\right]_+, \tag{5}$$

where $[x]_+ = \max(0, x)$ is projection onto the non-negative numbers.

Finally, we will need a benchmark to analyze our investment strategies. Throughout this exercise, we will use correct our gains by the return of a risk-free market investment. If we denote by α the money market rate of return, then the value of a gain r at time t is

$$v = re^{-\alpha t}. \tag{6}$$

A Deriving (3) and (4). When proposing the estimators (3) and (4), we made some claims on the random variables Y_n. Explain why it is true that the Y_n are independent identically distributed normals with mean μh and variance $\sigma^2 h$. Using this fact, explain why μ and σ^2 can be estimated using (3) and (4).

B Determine drift and volatility for CSCO. Use the data provided for CSCO to determine the drift and volatility parameters of $X(t)$. In that data, the interval between successive stock values vary depending on whether it is a weekday ($h = 1$, e.g., Monday to Tuesday), weekend ($h = 3$ for Friday to Monday), or holidays. Although it is not very hard to account for this nuisance, you can neglect this in your analysis and assume that all values in the data are separated by one day. **Caveat**: your year now has only 252 days.

C Is geometric Brownian motion a good model? If a geometric Brownian motion with drift $\hat{\mu}$ and volatility $\hat{\sigma}^2$ is a good model for the evolution of the CSCO stock price, then $Y_n \sim \mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h\right)$. Estimate the pdf of Y_n using a histogram and compare it with the pdf of $\mathcal{N}\left(\hat{\mu}h, \hat{\sigma}^2 h\right)$. Perform this comparison for values of Y_n in $[-0.1, 0.1]$ with a bin size of 0.01. Are you in awe that the model coincides with reality or are you disappointed to see it is a useless abstraction? If you are interested, you can obtain other historical stock series and compound your amazement (or disappointment).

By the way, as we have discussed before, comparing cdfs is usually a better idea than comparing pdfs. If you wish, you can compare the cdfs instead of the pdfs and comment on why comparing cdfs is a better alternative.

D Expected return. Obtain an expression for the expected return of an investment on CSCO as a function of time t and the parameters $\hat{\mu}$ and $\hat{\sigma}^2$. Discount your return by the money market rate α. Evaluate the expected return for $\alpha = 0.1\%$ a day. What is the probability that investing on CSCO yields a discounted rate of return of at least 5% after one year?

E Risk neutral measure. Determine the risk neutral measure for the CSCO stock price.

F Expected return for the risk neutral measure. Assume you are leaving in an alternative reality where the stock price evolves according to the risk neutral measure. What is the expected discounted rate of return for an investment in CSCO in this alternative reality? What is the non-discounted rate of return?

G Deriving the Black-Scholes formula. The Black-Scholes formula used to price an option is obtained by determining the price c that yields zero expected return with respect to the risk neutral measure, i.e., c is chosen as the solution of

$$\mathbb{E}_q\left[e^{-\alpha t}\left[X(t) - K\right]_+ - c\right] = 0, \tag{7}$$

where the expected value is taken with respect to the risk neutral measure q and not the geometric Brownian motion followed by the stock price $X(t)$. Explain why the expression in (7) yields zero expected return with respect to the risk neutral measure. Obtain a closed form expression for the price $c(t)$ as a function of the time t in terms of α, σ^2, K, and the current price $X(0)$.

H Determining the option price. Determine the option price c as a function of t when the strike price coincides with the expected value of the stock, i.e., $K = \mathbb{E}\left[X(t)\right]$. Repeat the calculation when $K = 1.2\,\mathbb{E}\left[X(t)\right]$ and $K = 0.8\,\mathbb{E}\left[X(t)\right]$. What is the use of buying options with strike prices $K = 1.2\,\mathbb{E}\left[X(t)\right]$ and $K = 0.8\,\mathbb{E}\left[X(t)\right]$?

Planning: Parts A, B, C and D can be solved using the information provided. To solve Parts E and F you need to know what the risk neutral measure is, something that we are studying on Monday. To answer G and H you can proceed from the risk neutral measure to find the no-arbitrage price of an option or you can wait until we cover the Black-Scholes formula on Wednesday.

The estimated 5.5 hours required to complete this homework are broken down below. If any part takes you significantly longer than estimated, you should seek help from your TAs.

Homework part	Estimated time	Homework part	Estimated time
A	30 mins	F	30 mins
B	30 mins	G	60 mins
C	60 mins	H	30 mins
D	30 mins	Problem comprehension	30 mins
E	30 mins		