Week 8: Continuous-time Markov chains
Insurance cash flow

The purpose of this exercise is to simulate and analyze the cash flow $X(t)$ of an insurance company whose initial capital is $X(t) = X_0$ (say thousand dollars). For simplicity, we assume that all customers are identical in that they pay the same premiums, submit claims for the same amount of money, and have the same risk. A more realistic model accounting for different types of customers with varying premiums, claims, and risk, that occur according to a probability distribution on their types is conceptually very similar, but computationally more involved.

The insurance company has a total of N customers that pay 1 (thousand dollars) per year to insure an asset of value c (thousand dollars). Each of these customers has an associated risk $0 \leq r \leq 1$, which denotes the probability of that costumer submitting a claim in any given year. The times at which premiums are paid, as well as the times at which claims are made, are modeled as Poisson processes. This assumption is reasonable as long as N is large and users re-insure after every claim. From the definition of Poisson processes, this mean that the time T_p between premium payments is exponentially distributed with parameter $\lambda = N$ premium payment per year, i.e.,

$$T_p \sim \exp(\lambda) = \exp(N). \tag{1}$$

Similarly, the time T_c between claims is exponentially distributed with parameter $\alpha = rN$ claims per year, i.e.,

$$T_c \sim \exp(\alpha) = \exp(rN). \tag{2}$$

Besides clients, the insurance company also has shareholders that expect dividends of d (thousand dollars) to be paid at a rate of β payoffs per year. These responsible shareholders, however, only expect payoffs if the company has sufficient reserve capital $X_r \gg c$. Consequently, the time T_d elapsed between dividend payoffs is modeled as an exponential with parameter β payoffs per year as long as the company capital exceeds X_r, i.e.,

$$\begin{cases} T_d \sim \exp(\beta), & \text{if } X(t) \geq X_r \\ T_d = 0, & \text{otherwise} \end{cases} \tag{3}$$

Finally, we complete the model by assuming that there exist a minimum (0) and maximum ($X_{\max}$) capital that the company can hold. Whenever $X(t) = 0$, no dividends or claims are paid, but the company is not bankrupt and may still receive premiums. If $X(t) = X_{\max}$, no more premium payments are accepted, but the company continues to pay claims and dividends. We also assume that if $X(t) < c$ and a claim is made then the capital of the company transitions to $X(t) = 0$. These three assumptions are unreasonable in general, unless we choose X_0 and $X_{\max}$ so that the probabilities of seeing $X(t) = 0$ or $X(t) = X_{\max}$ are negligible.

The goal of this exercise is to determine the probability of the insurance company paying dividends to its shareholder, an important metric to get investors on board. To do so, you will build a continuous time Markov chain (CTMC) model of $X(t)$, write code to simulate the process $X(t)$, and solve Kolmogorov's equations numerically to find the probability distribution of $X(t)$.

Before proceeding, we summarize the different events that may occur depending on the CTMC state, i.e., depending on the value of $X(t)$:

Table 1: Possible events for different values of $X(t)$

Reference	Capital	Possible events
A	$X(t) = 0$	premium payment
B	$0 < X(t) < c$	premium payment and "partial" claim payout (less than c)
C	$c \leq X(t) < X_r$	premium payment and claim payout
D	$X_r \leq X(t) < X_{\max}$	premium payment, claim payout, and dividend collection
E	$X(t) = X_{\max}$	claim payout and dividend collection

A CTMC states. Assume c and d are integers. Describe the states of the CTMC model, i.e., the possible values $X(t)$?

B Transition times. Given that the current state of the CTMC is $X(t) = x$, let T_x be the random time until the next transition out of x. Independent of the value of x, i.e., the reference range from Table 1, the probability distribution of T_x is exponential. Explain why and give the parameter ν_x for each event range described in Table 1. You may want to start with range (D).

C Possible state transitions. Consider that the current state of the CTMC $X(t) = x$ is in the reference range (D) from Table 1 and that a transition out of x occurs at time t. What are the possible values of $X(t)$ after the transition? Repeat the question for x in ranges (A), (B), (C), and (E).

D Transition probabilities. Describe the transition probabilities, assuming a transition occurs at time t, from a state $X(t) = x$ to the possible states described in part C, when x is in each reference range from Table 1. Again, you may want to start with range (D).

E System simulation. Write a function to simulate the cash flow of the insurance company. It should take as inputs the initial capital X_0; rates λ, α, and β; claim and dividend costs c and d, respectively; the capital thresholds X_r and $X_{\max}$; and a maximum simulation time $T_{\max}$. The output should be a vector of times t at which state transitions occurred and a vector of states $X(t)$ associated with those times. To guide you in writing this code consider the following:

1. Given the current state $X(t) = x$, draw the time of the next transition T_x. Recall that T_x is exponentially distributed with parameter computed in part B. Note that this parameter is different at different times.

2. Given that a transition occurs at time $t + T_x$, there are only a few possible destination states that you described in part C. The transitions into those states occur with probabilities you derived in part D. Draw the next state accordingly.

3. Update the state $X(t)$ according to the transition drawn.

Run your code for $X_0 = 200$; number of clients $N = 200$; client risk $r = 4\%$; claim payout $c = 20$; dividend costs $d = 30$, payed quarterly; capital thresholds $X_r = 200$ and $X_{\max} = 300$; and maximum simulation time $T_{\max} = 5$ years.

F Kolmogorov's forward equations. Kolmogorov's forward equations are a set of coupled linear differential equations whose solution is the transition probability function $P_{xy}(t)$ that describes the probability of transitioning from state x to state y between times s and $s+t$. Write down the transition rates q_{xy} from state x into state y for the five different capital ranges in Table 1. As before, it is a good idea to start with range (D). Use these transition rates to write Kolmogorov's forward equation for the capital ranges (A)–(E).

G Kolmogorov's backward equations. Kolmogorov's backward equations are an alternative set of coupled linear differential equations whose solution is also the transition probability function $P_xy(t)$. Write Kolmogorov's backward equation for the capital ranges (A)–(E).

H Solving Kolmogorov's equations. Let the matrix $P(t)$ collect the functions $P_{xy}(t)$ for the different ranges from Table 1 and define the vector $p(t)$ whose x-th element describes the probability $\mathbb{P}[X(t) = x]$. Build a matrix R with off-diagonal elements $r_{xy} = q_{xy}$ and diagonal elements $r_{xx} = -\nu_x$. The solution of Kolmogorov's (forward or backward) equations is then given by the matrix exponential $P(t) = e^{Rt}$ and the probability $p(t)$ can be obtained as $p(t) = P^T(t)p(0)$.

For the same parameters of Part E, use Kolmogorov's equations to determine the probability distribution $p(t)$ for years 0 to 5 in quarterly increments. Plot these quarterly probabilities in a single plot.

Hint: the MATLAB function to compute matrix exponentials is `expm` (use `help expm` to get more details).

I Probability of paying dividends. Use your results from Part H to approximate the probability of the insurance company paying dividends in quarters 1 through 20. Discuss your method. This is a typical engineering question and there is therefore more than one correct way to answer.

Planning: solving this exercise is time consuming and you should therefore plan accordingly. You must be able to answer parts A–D based on our discussion of the exponential distributions, although we will get into more details on Monday. You should wait to start part E until after Monday's class. The concepts used in parts F and G will be covered on Wednesday's lecture. Although part H gives you all the details you need, you can solve parts H and I any time after Wednesday, although we will cover matrix exponential on Friday.

The estimated 7 hours required to complete this homework are broken down below. If any part takes you significantly longer than estimated, you should seek help from your TAs.

Homework part	**Estimated time**	**Homework part**	**Estimated time**
A	15 mins	F	30 mins
B	30 mins	G	15 mins
C	15 mins	H	120 mins
D	15 mins	I	30 mins
E	120 mins	Problem comprehension	30 mins