Arbitrages, and pricing of stock options

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Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

Arbitrage

- Bet on different events with each outcome paying a random return
- Arbitrage: It is possible to devise a betting strategy that guarantees a positive return no matter the combined outcome of the events
- Arbitrages often involve operating in two different markets

Example

- Booker 1 $\Rightarrow$ Phillies win pay 1.5:1, Phillies loose pay 3:1
- Bet x on Phillies and y against Phillies. Guaranteed Earnings?

$$\begin{aligned} \text{Phillies win:} \quad & 0.5x - y > 0 \;\Rightarrow x > 2y \\ \text{Phillies loose:} \quad & -x + 2y > 0 \Rightarrow x < 2y \end{aligned}$$

- Arbitrage not possible. Notice that $1/(1.5) + 1/3 = 1$
- Booker 2 $\Rightarrow$ Phillies win pay 1.4:1, Phillies loose pay 3.1:1
- Bet x on Phillies and y against Phillies. Guaranteed Earnings?

$$\begin{aligned} \text{Phillies win:} \quad & 0.4x - y > 0 \quad\;\; \Rightarrow x > 2.5y \\ \text{Phillies loose:} \quad & -x + 2.1y > 0 \Rightarrow x < 2.1y \end{aligned}$$

- Arbitrage not possible. Notice that $1/(1.4) + 1/(3.1) > 1$

Example (continued)

- First condition on Booker 1 and second on Booker 2 are compatible
- Bet x on Phillies on Booker 1, y against Phillies on Booker 2
- Guaranteed earnings possible. Make $y = 1,000$, $x = 2,066$

$$\begin{aligned} \text{Phillies win:} \quad & 0.5(2,066) - 1,000 &= 33 \\ \text{Phillies loose:} \quad & -2066 + 2.1(1000) &= 34 \end{aligned}$$

- Notice that $1/(1.5) + 1/(3.1) < 1$
- If you plan on doing this, do it on, e.g., currency exchange markets

Events, returns, and investment strategy

- Let events on which bets are posted be $k = 1, 2, \ldots, K$
- Let $j = 1, 2, \ldots, J$ index possible joint outcomes
  - Joint realizations, also called "world realization", or "world outcome"
- If world outcome is j, event k yields return r_{jk} per unit invested (bet)
- Do not define probability p_j of outcome j

- Invest (bet) x_k in outcome k $\Rightarrow$ return for world j is $x_k r_{jk}$
- Bets x_k can be positive ($x_k > 0$) or negative ($x_k < 0$)

  $\Rightarrow$ Positive = regular bet. Negative = short bet

- Total return $\Rightarrow \displaystyle\sum_{k=1}^{K} x_k r_{jk} = \mathbf{x}^T \mathbf{r}_j$
- Vectors of returns for outcome j $\Rightarrow \mathbf{r}_j := [r_{j1}, \ldots, r_{jK}]^T$ (given)
- Vector of bets $\Rightarrow \mathbf{x}_j := [x_{j1}, \ldots, x_{jK}]^T$ (controlled by gambler)

Arbitrage (clearly defined now)

- Arbitrage is possible if there exists investment strategy $\mathbf{x}$ such that

$$\mathbf{x}^T \mathbf{r}_j > 0, \quad \text{for all } j = 1, \ldots, J$$

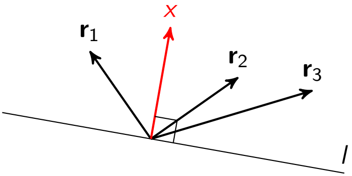
- Equivalently, arbitrage is possible if

$$\max_{\mathbf{x}} \left(\min_j \left(\mathbf{x}^T \mathbf{r}_j \right) \right) > 0$$

- Portfolio $\mathbf{x}$ and returns $\mathbf{r}_j$ are vectors in $\mathbb{R}^K$
- Earnings $\mathbf{x}^T \mathbf{r}_j$ are the inner product of $\mathbf{x}$ and $\mathbf{r}_j$

- Earnings are positive if angle between $\mathbf{x}$ and $\mathbf{r}_j$ is less than $\pi/2$ (90°)

- There is a line that leaves all $\mathbf{r}_j$ vectors to one side

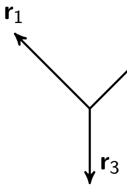


- Arbitrage possible
- Prob. vector $\mathbf{p} = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

- does not exist

- There is not a line that leaves all $\mathbf{r}_j$ vectors to one side



- Arbitrage not possible
- There is prob. vector $\mathbf{p} = [p_1, \ldots, p_J]^T$ on world outcomes such that

$$\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

- Think of p_j as scaling factor

Arbitrage theorem

- Have "proved" following result, called arbitrage theorem

Theorem

Given vectors of returns $\mathbf{r}_j$*, associated with random outcome* $j = 1, \ldots, J$ *an arbitrage is not possible if and only if there exist a probability vector* $\mathbf{p}$ *such that* $\mathbb{E}_{\mathbf{p}}(\mathbf{r}) = \mathbf{0}$*. Equivalently,*

$$\max_{\mathbf{x}} \left(\min_{j} \left(\mathbf{x}^T \mathbf{r}_j \right) \right) \leq 0 \quad \Leftrightarrow \quad \sum_{j=1}^{J} p_j \mathbf{r}_j = \mathbf{0}$$

- Prob. vector $\mathbf{p}$ is **NOT** the prob. distribution of events $j = 1, \ldots, J$

Example: Arbitrages in geometric random walk

- Consider a stock price $X(nh)$ that follows a geometric random walk

$$X\big((n+1)h\big) = X(nh)e^{\sigma\sqrt{h}Y_n}$$

- where Y_n is a binary random variable with probability distribution

$$\mathrm{P}\left[Y_n = 1\right] = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{h}\right), \quad \mathrm{P}\left[Y_n = -1\right] = \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{h}\right)$$

- Recall that as $h \to 0$, $X(nh)$ becomes geometric Brownian motion
- Are there arbitrage opportunities in the price of the stock?

  $\Rightarrow$ Too general, let us consider a narrower problem

- Consider the following investment strategy (stock flip):

  Buy: Buy \$1 in stock at time 0 for price $X(0)$ per unit of stock

  Sell: Sell stock at time h for price $X(h)$ for unit of stock

- Cost of transaction is \$1. Units of stock purchased are $1/X(0)$
- Cash after selling stock is $X(h)/X(0)$
- Return on investment is $X(h)/X(0) - 1$

- There are two possible outcomes for the price of the stock at time h
- As per model we may have $Y_0 = 1$ or $Y_0 = -1$ respectively yielding

$$X(h) = X(0)e^{\sigma\sqrt{h}}, \qquad X(h) = X(0)e^{-\sigma\sqrt{h}}$$

- Possible returns are therefore

$$r_1 = \frac{X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 = e^{\sigma\sqrt{h}} - 1, \quad r_2 = \frac{X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\sigma\sqrt{h}} - 1$$

Present value of returns

- One dollar at time h is not the same as 1 dollar at time 0
- Interest rate of a risk-free investment is α continuously compounded
- In practice, α is the money market rate
- Prices have to be compared at their present value

- The present value of $X(h)$ at time 0 is $X(h)e^{-\alpha h}$
- Then, return on investment is $e^{-\alpha h}X(h)/X(0) - 1$
- Present value of possible returns (whether $Y_0 = 1$ or $Y_0 = -1$) are

$$r_1 = \frac{e^{-\alpha h}X(0)e^{\sigma\sqrt{h}}}{X(0)} - 1 \quad = e^{-\alpha h}e^{\sigma\sqrt{h}} - 1,$$

$$r_2 = \frac{e^{-\alpha h}X(0)e^{-\sigma\sqrt{h}}}{X(0)} - 1 = e^{-\alpha h}e^{-\sigma\sqrt{h}} - 1$$

No arbitrage condition

- Arbitrage not possible if and only if there exists $0 \leq q \leq 1$ such that

$$qr_1 + (1-q)r_2 = 0$$

- Arbitrage theorem in 1 dimension (only one bet, buy stock)
- Substituting r_1 and r_2 for their respective values

$$q\left(e^{-\alpha h}e^{\sigma\sqrt{h}} - 1\right) + (1-q)\left(e^{-\alpha h}e^{-\sigma\sqrt{h}} - 1\right) = 0$$

- Can be easily solved for q. Expanding product and reordering terms

$$qe^{-\alpha h}e^{\sigma\sqrt{h}} + (1-q)e^{-\alpha h}e^{-\sigma\sqrt{h}} = 1$$

- Multiplying by $e^{\alpha h}$ and grouping terms with a q factor

$$q\left(e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right) = e^{\alpha h} - e^{-\sigma\sqrt{h}}$$

No arbitrage condition (continued)

- Solving for q finally yields $\Rightarrow q = \dfrac{e^{\alpha h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$

- For small h we have $e^{\alpha h} \approx 1 + \alpha h$ and $e^{\pm\sigma\sqrt{h}} \approx 1 \pm \sigma\sqrt{h} + \sigma^2 h/2$
- Thus, the value of q as $h \to 0$ may be approximated as

$$q \approx \frac{1 + \alpha h - \left(1 - \sigma\sqrt{h} + \sigma^2 h/2\right)}{1 + \sigma\sqrt{h} - \left(1 - \sigma\sqrt{h}\right)} = \frac{\sigma\sqrt{h} + \left(\alpha - \sigma^2/2\right) h}{2\sigma\sqrt{h}}$$

$$= \frac{1}{2}\left(1 + \frac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$$

- Approximation proves that at least for small h $0 < q < 1$

  $\Rightarrow$ Arbitrage not possible

- Also, suspiciously similar to probabilities of geometric random walk

  $\Rightarrow$ Fundamental observation as we'll see next

Risk neutral measure

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

No arbitrage condition on geometric random walk

- Stock prices $X(t)$ follow geometric random walk (drift μ, variance σ^2)
- Risk free investment has return α (cost of money, money market)
- Arbitrage is not possible in stock flips if there is $0 \leq q \leq 1$ such that

$$q = \frac{e^{\alpha h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$$

- Notice that q satisfies the equation (which we'll use later on)

$$qe^{\sigma\sqrt{h}} + (1-q)e^{-\sigma\sqrt{h}} = e^{\alpha h}$$

- Can we have arbitrage using a more complex set of possible bets?

General investment strategy

- Consider the following general investment strategy:

  Observe: Observe the stock price at times $h, 2h, \ldots, nh$

  Compare: Is $X(h) = x_1, X(2h) = x_2, \ldots, X(nh) = x_n$?

  Buy: If above answer is yes, buy stock at price $X(nh)$

  Sell: Sell stock at time mh for price $X(mh)$

- Possible bets are the observed values of the stock $x_1, x_2, \ldots, x_l$

  $\Rightarrow$ There are 2^n possible bets

- Possible outcomes are value at time mh and observed values

  $\Rightarrow$ There are 2^m possible outcomes

Explanation of general investment strategy

- Bet 1 $=$ n price increases, bet 2 $=$ price increases in $1, \ldots, n-1$ and price decrease in n ...
- For each bet we have 2^{m-n} possible outcomes: $m-n$ price increases, price increases in $n+1, \ldots, m-1$ and price decrease in m ...

	$X(h)$	$X(2h)$	$X(3h)$	...	$X(nh)$
bet 1	$e^{\sigma\sqrt{h}}$	$e^{2\sigma\sqrt{h}}$	$e^{3\sigma\sqrt{h}}$	...	$e^{n\sigma\sqrt{h}}$
bet 2	$e^{\sigma\sqrt{h}}$	$e^{2\sigma\sqrt{h}}$	$e^{3\sigma\sqrt{h}}$	...	$e^{(n-2)\sigma\sqrt{h}}$
⋮	⋮	⋮	⋮		⋮
bet 2^n	$e^{-\sigma\sqrt{h}}$	$e^{-2\sigma\sqrt{h}}$	$e^{-3\sigma\sqrt{h}}$	...	$e^{-n\sigma\sqrt{h}}$

$X((n+1)h)$	$X((n+2)h)$	...	$X(mh)$
$X(nh)e^{\sigma\sqrt{h}}$	$X(nh)e^{2\sigma\sqrt{h}}$	...	$X(nh)e^{m\sigma\sqrt{h}}$
$X(nh)e^{\sigma\sqrt{h}}$	$X(nh)e^{2\sigma\sqrt{h}}$	...	$X(nh)e^{(m-2)\sigma\sqrt{h}}$
⋮	⋮		⋮
$X(nh)e^{-\sigma\sqrt{h}}$	$X(nh)e^{-2\sigma\sqrt{h}}$	...	$X(nh)e^{-m\sigma\sqrt{h}}$

outcomes per each bet

- Figure assumes $X(0)=1$ for simplicity

Explanation of general investment strategy

- Define the prob. distribution **q** over possible outcomes as follows
- Start with a sequence of independent identically distributed Y_n
- Each element Y_n is a binary random variable with probabilities

$$\mathrm{P}\left[Y_n = 1\right] = q, \quad \mathrm{P}\left[Y_n = -1\right] = 1 - q$$

- With $q = \left(e^{\alpha h} - e^{-\sigma\sqrt{h}}\right) / \left(e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}\right)$ as in slide 16
- Joint prob. distribution **q** on $X(h), X(2h), \ldots, X\big((n+m)h\big)$ outcomes obtained through transformation

$$X\big((n+1)h\big) = X(nh)e^{\sigma\sqrt{h}Y_n}$$

- Notice once more that this is **NOT** the prob. distribution of $X(nh)$
- Will show that expected value of earnings with respect to **q** is null
  - $\Rightarrow$ Thus, arbitrages are not possible

Return for given outcome

- Consider a time 0 unit investment in given arbitrary outcome
- Stock units purchased depend on the price $X(nh)$ at buying time

$$\text{Units bought } = \frac{1}{X(nh)e^{-\alpha nh}}$$

- Have corrected $X(nh)$ to express it in time 0 values

- Cash after selling stock given by price $X(mh)$ at sell time $m+n$
- Expressed in time 0 values

$$\text{Cash after sell } = \frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}}$$

- Return is then $\Rightarrow r\big(X(h), \ldots, X(mh)\big) = \dfrac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1$
- Depends on $X(mh)$ and $X(nh)$ only

Expected return with respect to measure $\mathbf{q}$

- Consider expected value of all possible returns with respect to $\mathbf{q}$

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\ldots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}}\left[\frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

- Condition on observed values $X(h),\ldots,X(nh)$

$$\begin{aligned}\mathbb{E}_{\mathbf{q}}&\left[r\big(X(h),\ldots,X(mh)\big)\right] \\ &= \mathbb{E}_{\mathbf{q}(1:n)}\left[\mathbb{E}_{\mathbf{q}(n+1:m)}\left[\frac{X(mh)e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1 \,\Big|\, X(h),\ldots,X(nh)\right]\right]\end{aligned}$$

- In innermost expectation $X(nh)$ is given. Furthermore, process $X(t)$ is Markov, thus conditioning on $X(h),\ldots,X((n-1)h)$ is irrelevant. Thus

$$\mathbb{E}_{\mathbf{q}}\left[r\big(X(h),\ldots,X(mh)\big)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[\frac{\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\big|\,X(nh)\right]e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

Expected value of future values (measure q)

- Need to find expectation of future value $\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh) \,\middle|\, X(nh)\right]$
- From recursive relation for $X(nh)$ in terms of Y_n sequence

$$\begin{aligned} X(mh) &= X\big((m-1)h\big)e^{\sigma\sqrt{h}Y_{m-1}} \\ &= X\big((m-2)h\big)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}} \\ &\vdots \\ &= X\big(nh\big)e^{\sigma\sqrt{h}Y_{m-1}}e^{\sigma\sqrt{h}Y_{m-2}}\dots e^{\sigma\sqrt{h}Y_{n+1}} \end{aligned}$$

- All the Y_n are independent. Then, upon taking expected value

$$\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh) \,\middle|\, X(nh)\right] = X\big(nh\big)\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{m-1}}\right]\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{m-2}}\right]\dots\mathbb{E}\left[e^{\sigma\sqrt{h}Y_{n+1}}\right]$$

- Need to determine expectation of relative price increase $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right]$

Expectation of relative price increase (measure **q**)

- The expected value of the relative price increase $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right]$ is

$$\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\sigma\sqrt{h}}\Pr\left[Y_n = 1\right] + e^{-\sigma\sqrt{h}}\Pr\left[Y_n = -1\right]$$

- According to definition of measure **q**, it holds

$$\Pr\left[Y_n = 1\right] = q, \qquad \Pr\left[Y_n = -1\right] = 1 - q$$

- Substituting in expression for $\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right]$

$$\mathbb{E}\left[e^{\sigma\sqrt{h}Y_n}\right] = e^{\sigma\sqrt{h}}\,q + e^{-\sigma\sqrt{h}}(1-q) = e^{\alpha h}$$

- where last equality follows from definition of probability q
- Reweave the quilt $\Rightarrow$ use expected relative price increase to compute expected future value to find expected return

Reweave the quilt

- Substitute expected relative price increase into expression for expected future value

$$\mathbb{E}_{\mathbf{q}(n+1:m)}\left[X(mh)\,\middle|\,X(nh)\right] = X\left(nh\right)e^{\alpha h}e^{\alpha h}\ldots e^{\alpha h} = X\left(nh\right)e^{\alpha(m-n)h}$$

- Substitute result into expression for expected return

$$\mathbb{E}_{\mathbf{q}}\left[r\left(X(h),\ldots,X(mh)\right)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[\frac{X\left(nh\right)e^{\alpha(m-n)h}e^{-\alpha mh}}{X(nh)e^{-\alpha nh}} - 1\right]$$

- Exponentials cancel each other, finally yielding

$$\mathbb{E}_{\mathbf{q}}\left[r\left(X(h),\ldots,X(mh)\right)\right] = \mathbb{E}_{\mathbf{q}(1:n)}\left[1-1\right] = 0$$

- Arbitrage not possible in any trading strategy if $0 \leq q \leq 1$ exists

If prices follow geometric Brownian motion

- Stock prices follow a geometric Brownian motion, i.e.,

$$X(t) = X(0)e^{Y(t)}$$

- with $Y(t)$ Brownian motion with drift μ and variance σ^2

- What is the no arbitrage condition?
- Approximate geometric Brownian motion by geometric random walk
- No arbitrage measure $\mathbf{q}$ exists for geometric random walk
  - This requires h sufficiently small
  - Notice that prob. distribution $\mathbf{q} = \mathbf{q}(h)$ is a function of h
- Approximation arbitrarily accurate by letting $h \to 0$
- Existence of the prob. distribution $\mathbf{q} := \lim_{h\to 0} \mathbf{q}(h)$ proves that arbitrages are not possible in stock trading

No arbitrage probability distribution

- Recall that as $h \to 0 \;\Rightarrow q \approx \dfrac{1}{2}\left(1 + \dfrac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$
- And consequently $\Rightarrow (1-q) \approx \dfrac{1}{2}\left(1 - \dfrac{\alpha - \sigma^2/2}{\sigma}\sqrt{h}\right)$
- Thus, measure $\mathbf{q} := \lim_{h\to 0} \mathbf{q}(h)$ is geometric Brownian motion
  $\Rightarrow$ Variance $\Rightarrow \sigma^2$ (same as stock price)
  $\Rightarrow$ Drift $\Rightarrow \alpha - \sigma^2/2$
- Measure showing arbitrage not possible is a geometric random walk
- Which is also the way stock prices evolve
- Furthermore, the variance is the same as that of stock prices
- The drifts are different $\Rightarrow \mu$ for stocks and $\alpha - \sigma^2/2$ for no arbitrage

Expected investment growth

- Compute expected return on an investment on stock $X(t)$
- Buy 1 share of stock at time 0. Cash invested $\Rightarrow X(0)$
- Sell stock at time t. Cash after sell $\Rightarrow X(t)$
- Expected value of cash after sell given $X(0)$ is

$$\mathbb{E}\left[X(t) \,\middle|\, X(0)\right] = X(0)e^{(\mu+\sigma^2/2)t}$$

- Alternatively, invest $X(0)$ risk free in the money market
- Guaranteed cash at time t is $X(0)e^{\alpha t}$
- Invest in stock only if $\mu + \sigma^2/2 > \alpha$ $\Rightarrow$ risk premium

Risk neutral measure

- Compute expected return as if **q** were the actual distribution
  - And recall that **q** is NOT the actual distribution
- As before, cash invested is $X(0)$ and cash after sale is $X(t)$
- Expected cash value is different because prob. distribution is different

$$\mathbb{E}_{\mathbf{q}}\left[X(t) \,\middle|\, X(0)\right] = X(0)e^{(\alpha-\sigma^2/2+\sigma^2/2)t} = X(0)e^{\alpha t}$$

- Same return as risk free investment regardless of parameters' values

- Measure **q** is called risk neutral measure
- Risky stock investments yield same return as risk free investments
- "Alternate universe" in which investors do not demand risk premiums

- Pricing of derivatives, e.g., options, is always based on expected returns with respect to risk neutral valuation (pricing in alternate universe)
- Basis for Black-Scholes. More later

Arbitrages

Risk neutral measure

Black-Scholes formula for option pricing

Options

- An option is a contract to buy shares of a stock at a future time
- Strike time t = Convened time for stock purchase
- Strike price K = Price at which stock is purchased at strike time
- At time t, option holder may decide to
  - $\Rightarrow$ Buy a stock at strike price K = exercise the option
  - $\Rightarrow$ Do not exercise the option

- May buy option at time 0 for price c
- How do we determine the option's worth, i.e., price c, at time 0 ?
- Answer given by Black-Scholes formula for option pricing

Stock price model

- Let $e^{\alpha t}$ be the compounding of a risk free investment
- Let $X(t)$ be the stock's price at time t
- Price modeled as geometric Brownian motion, drift μ, variance σ^2
- Risk neutral measure $\mathbf{q}$ is also a geometric Brownian motion
  $\Rightarrow$ Variance σ^2 and drift $\alpha - \sigma^2/2$

Return of option investment

- At time t, the option's worth depends on the stock's price $X(t)$
- If stock's price smaller or equal than strike price $\Rightarrow X(t) \leq K$
  $\Rightarrow$ Option is worthless (better to buy stock at current price)
- Since had paid c for the option at time 0, lost c on this investment
  $\Rightarrow$ return on investment is $r = -c$
- If stock's price larger than strike price $\Rightarrow X(t) > K$
  $\Rightarrow$ Exercise option and realize a gain of $X(t) - K$
- To obtain return express as time 0 values and subtract c

$$r = e^{-\alpha t}\big(X(t) - K\big) - c$$

- May combine both in single equation $\Rightarrow r = e^{-\alpha t}\big(X(t) - K\big)^{+} - c$
- $(\cdot)^{+}$ denotes projection on positive reals

Expected return under risk neutral measure

- Consider mixed positions on stocks and options
- Is there a position guaranteeing positive return, i.e., an arbitrage?
- Assume expected return under risk neutral measure is nonzero

$$\mathbb{E}_{\mathbf{q}}\left[r\right] = \mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big(X(t) - K\big)^{+} - c\right] \neq 0$$

- Then, an arbitrage is possible according to arbitrage theorem
- If expected return under risk neutral measure is zero

$$\mathbb{E}_{\mathbf{q}}\left[r\right] = \mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big(X(t) - K\big)^{+} - c\right] = 0$$

- Then, no arbitrage is possible according to arbitrage theorem
- Select options price c to prevent arbitrage opportunities

Options pricing

- To have no arbitrage, must select option's price c so that

$$\mathbb{E}_{\mathbf{q}}\left[e^{-\alpha t}\big(X(t)-K\big)^{+}-c\right]=0$$

- where expectation is with respect to risk neutral measure
- From above condition, the no-arbitrage price of the option is

$$c=e^{-\alpha t}\mathbb{E}_{\mathbf{q}}\left[\big(X(t)-K\big)^{+}\right]$$

- Source of Black-Scholes formula for option valuation
- Rest of derivation is just evaluation of expected value
- Same argument used to price any derivative of the stock's price

Use fact that prices are a geometric random walk

- Let us evaluate expectation to compute option's price c

- Prices follow a geometric random walk $\Rightarrow X(t) = X_0 e^{Y(t)}$
- $X_0 =$ price at time 0,
- $Y(t)$ random walk with drift parameter μ and variance parameter σ^2
- Can rewrite no arbitrage condition as

$$c = e^{-\alpha t} \mathbb{E}_{\mathbf{q}} \left[\left(X_0 e^{Y(t)} - K \right)^+ \right]$$

- $Y(t)$ random walk. Then, in particular, $Y(t) \sim \mathcal{N}(\mu t, t\sigma^2)$

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t \sigma^2}} \int_{-\infty}^{\infty} (X_0 e^y - K)^+ \, e^{-(y-\mu t)^2/(2t\sigma^2)} \, dy$$

Evaluation of the integral

- Note that $\left(X_0 e^{Y(t)} - K\right)^+ = 0$ for all values $Y(t) \leq \log(K/X_0)$
- Because integrand is null for $Y(t) \leq \log(K/X_0)$ can write

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi t\sigma^2}} \int_{\log(K/X_0)}^{\infty} \left(X_0 e^{y} - K\right) e^{-(y-\mu t)^2/(2t\sigma^2)}\, dy$$

- Change of variables $z = (y - \mu t)/\sqrt{t\sigma^2}$. Associated replacements

Variable:	y	$\Rightarrow \sqrt{t\sigma^2}z + \mu t$
Differential:	dy	$\Rightarrow \sqrt{t\sigma^2}\, dz$
Integration limit:	$\log(K/X_0)$	$\Rightarrow a := \dfrac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}}$

- Option price then given by

$$c = e^{-\alpha t} \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} \left(X_0 e^{\sqrt{t\sigma^2}z + \mu t} - K\right) e^{-z^2/2}\, dz$$

Evaluation of the integral (continued)

- Separate in two integrals $c = e^{-\alpha t}(I_1 - I_2)$ where

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{t\sigma^2}z + \mu t} e^{-z^2/2}\, dz$$
$$I_2 := \frac{K}{\sqrt{2\pi}} \int_a^\infty e^{-z^2/2}\, dz$$

- Gaussian Q function (ccdf of normal RV with mean 0 and variance 1)

$$Q(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2}\, dz$$

- Comparing last two equations we have $I_2 = KQ(a)$

- I_1 requires some more work

Evaluation of the integral (continued)

- Reorder terms in integral I_2

$$I_1 := \frac{1}{\sqrt{2\pi}} \int_a^\infty X_0 e^{\sqrt{t\sigma^2}z+\mu t} e^{-z^2/2}\, dz = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{\sqrt{t\sigma^2}z - z^2/2}\, dz$$

- The exponent can be written as a square minus a "constant" (no z)

$$-\left(z-\sqrt{t\sigma^2}\right)^2/2 + t\sigma^2/2 = -z^2/2 + \sqrt{t\sigma^2}z - t\sigma^2/2 + t\sigma^2/2$$

- Substituting the latter into I_1 yields

$$I_1 = \frac{X_0 e^{\mu t}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z-\sqrt{t\sigma^2}\right)^2/2 + t\sigma^2/2}\, dz = \frac{X_0 e^{\mu t + t\sigma^2/2}}{\sqrt{2\pi}} \int_a^\infty e^{-\left(z-\sqrt{t\sigma^2}\right)^2/2}\, dz$$

Evaluation of the integral (continued)

- Change of variables $u = z - \sqrt{t\sigma^2} \;\Rightarrow\; du = dz$ and integration limit

$$a \;\Rightarrow\; b := a - \sqrt{t\sigma^2} = \frac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}} - \sqrt{t\sigma^2}$$

- Implementing change of variables in I_1

$$I_1 = \frac{X_0 e^{\mu t + t\sigma^2/2}}{\sqrt{2\pi}} \int_b^{\infty} e^{u^2/2}\, du = X_0 e^{\mu t + t\sigma^2/2} Q(b)$$

- Putting together results for I_1 and I_2

$$c = e^{-\alpha t}(I_1 - I_2) = e^{-\alpha t} X_0 e^{\mu t + t\sigma^2/2} Q(b) - e^{-\alpha t} K Q(a)$$

- For non-arbitrage stock prices $\Rightarrow \alpha = \mu + \sigma^2/2$
- Substitute to obtain Black-Scholes formula

Black-Scholes

- Black-Scholes formula for option pricing

$$c = X_0 Q(b) - e^{-\alpha t} K Q(a)$$

- Where $\Rightarrow a := \dfrac{\log(K/X_0) - \mu t}{\sqrt{t\sigma^2}}$ and $b := a - \sqrt{t\sigma^2}$
- Note further that $\mu = \alpha - \sigma^2/2$. Can then write a as

$$a = \frac{\log(K/X_0) - \left(\alpha - \sigma^2/2\right) t}{\sqrt{t\sigma^2}}$$

- X_0 = stock price at time 0, c = option cost at time 0,
- K = option's strike price, t = option's strike time
- α = benchmark risk-free rate of return (cost of money)
- σ^2 = volatility of stock
- Black-Scholes formula independent of stock's mean tendency μ