Laplace Transform Review

- Table of Laplace transforms
- Pole – zero patterns
- Circuit examples
- Circuit/system stability

Laplace Transforms

Laplace transform of a time function:

$$L\{f(t)\}=F(s)=\int_0^{\infty} f(t)e^{-st}\,dt \qquad s=\alpha+j\omega$$

One necessary condition for convergence:

$$f(t)e^{-\alpha t}\rightarrow 0 \quad as \quad t\rightarrow\infty$$

There are others that we will not discuss here.

The inverse Laplace transform of a frequency function:

$$f(t)=\frac{1}{j2\pi}\oint F(s)e^{st}\,ds$$

We will not need to evaluate this complex plane *contour integral,* since we can use a table of time function transforms instead.

Transform Table Construction

Without repeating the niceties of mathematical rigor, we will construct a table of Laplace transforms sufficient for our purposes. For successively more detailed and precise explanations of the theory consult the automatic control literature, or math texts on the theory of complex variables.

Transform of a constant $f(t) = K\,u(t)$, where $u(t) = 1$ for $t \geq 0$ and 0 for $t < 0$:

$$F(s) = \int_0^\infty K e^{-st}\,dt = K \int_0^\infty \frac{1}{-s} e^{-st}(-s\,dt) = \frac{K}{s}\left(e^0 - e^{-\infty}\right) = \frac{K}{s}$$

$$\boxed{K\,u(t) \Leftrightarrow \frac{K}{s}}$$

Transform Table Continued

A useful theorem:

$$L\{e^{-at}f(t)\}=\int_0^\infty f(t)e^{-(s+a)}dt=F(s+a)$$

Since $$K\,u(t)\Leftrightarrow\frac{K}{s}$$

It follows that:

$$L\{e^{-at}u(t)\}=\frac{1}{s+a}$$

$$\boxed{e^{-at}\Leftrightarrow\frac{1}{s+a}}$$

Transform Table Continued

For sinusoids:

$$L\{\cos(\omega_1 t)u(t)\}=L\{\frac{e^{j\omega_1 t}u(t)+e^{-j\omega_1 t}u(t)}{2}\}$$

$$L\{\cos(\omega_1 t)u(t)\}=\frac{1}{2}\left(L\{e^{j\omega_1 t}u(t)\}+L\{e^{-j\omega_1 t}u(t)\}\right)$$

$$L\{\cos(\omega_1 t)u(t)\}=\frac{1}{2}\left(\frac{1}{s-j\omega_1}+\frac{1}{s+j\omega_1}\right)$$

$$L\{\cos(\omega_1 t)u(t)\}=\frac{s}{s^2+\omega_1^2}$$

$$\boxed{\cos(\omega_1 t)u(t)\Leftrightarrow\frac{s}{s^2+\omega_1^2}}$$

Transform Table Continued

Sinusoids continued:

Use exponential definition of the sin function:

$$\sin(\omega_1 t)=\frac{e^{j\omega_1 t}-e^{-j\omega_1 t}}{2j}$$

And using algebraic manipulations as for the cosine function:

$$L\{\sin(\omega_1 t)u(t)\}=\frac{\omega_1}{s^2+\omega_1^2}$$

$$\boxed{\sin(\omega_1 t)u(t)\Leftrightarrow\frac{\omega_1}{s^2+\omega_1^2}}$$

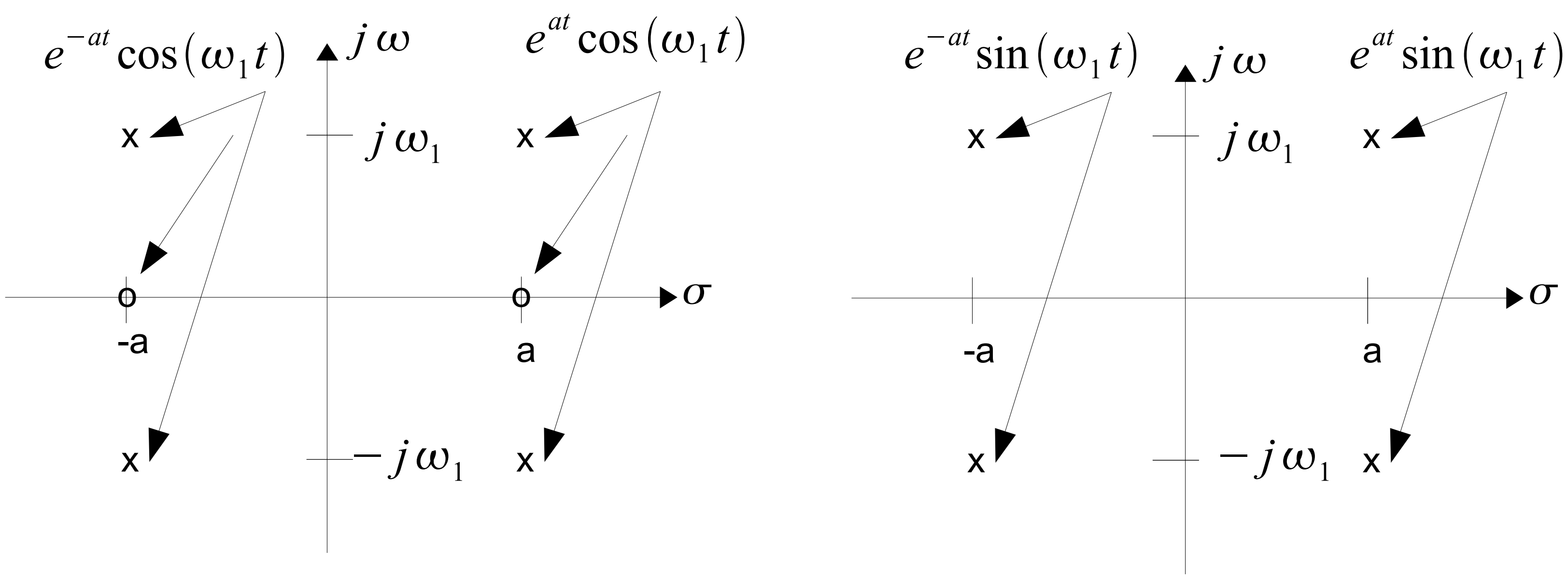
Transform Table Continued

Use the “exponential” theorem to write:

$$e^{-at}\cos(\omega_1 t)u(t) \Leftrightarrow \frac{s+a}{(s+a)^2+\omega_1^2} \qquad e^{-at}\sin(\omega_1 t)u(t) \Leftrightarrow \frac{\omega_1}{(s+a)^2+\omega_1^2}$$

Finally, we make the observation, that our working with the sinusoidal frequency variable *“jω”* transformed the differential equations of circuit theory to complex number algebraic equations. If we generalize by replacing *“ jω”* with *“s ”* we can transform the circuit equations with an exponential class of “forcing functions” (so far here with zero initial conditions) to algebraic equations that we can solve by matching transform pairs.

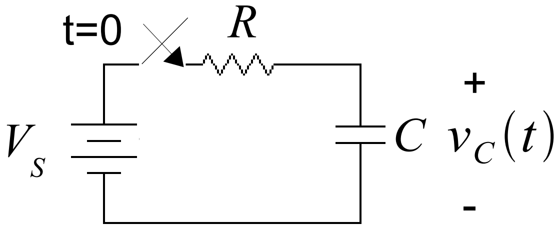
Pole – Zero Patterns



Roots of denominator polynomial are called “poles.” Those of numerator polynomial are called “zeros.” Poles with RH plane real parts correspond to exponentially growing signals – sinusoids in this case.

Example 1

Write the phasor/Laplace equations directly:



$$v_C(0)=0$$

$$V_S u(t) \Leftrightarrow \frac{V_S}{s}$$

$$\frac{V_C(s)-\frac{V_S}{s}}{R}+sCV_C(s)=0$$

$$sCV_C(s)+\frac{V_C(s)}{R}=\frac{V_S}{Rs}$$

$$V_C(s)=\frac{\frac{1}{R}}{sC+\frac{1}{R}}\frac{V_S}{s}$$

$$V_C(s)=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\frac{V_S}{s}$$

Example 1 Continued

$$V_C(s)=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\frac{V_S}{s}$$

We need to "undo" the common denominator to obtain terms that lie in our transform table. A partial fraction expansion is the tool to use. Write:

$$V_C(s)=\frac{K_1}{s+\frac{1}{RC}}+\frac{K_2}{s}$$

Example 1 Continued

a. $$V_C(s)=\boxed{\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\frac{V_S}{s}}$$

b. $$V_C(s)=\boxed{\frac{K_1}{s+\frac{1}{RC}}+\frac{K_2}{s}}$$

1. Recombine the terms in b; 2. Equate numerators in a and b.

$$\boxed{\frac{\frac{V_S}{RC}}{s\left(s+\frac{1}{RC}\right)}}=\boxed{\frac{K_1 s+K_2\left(s+\frac{1}{RC}\right)}{s\left(s+\frac{1}{RC}\right)}}$$

$$\frac{1}{RC}V_S=K_2\left(s+\frac{1}{RC}\right)+K_1 s$$

$$K_1+K_2=0,\ \frac{K_2}{RC}=\frac{V_S}{RC}$$

$$V_C=\frac{-V_S}{s+\frac{1}{RC}}+\frac{V_S}{s}$$

Table match to obtain:

$$v_c(t)=V_S\left(1-e^{-\frac{t}{RC}}\right)u(t)$$

Example 1 Continued

A " simpler" way is to equate the two expressions, i.e.

$$\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\frac{V_S}{s}=\frac{K_1}{s+\frac{1}{RC}}+\frac{K_2}{s}$$

Multiply both sides by one denominator factor, say s and let it approach zero. We then have (for the s factor):

$$\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\frac{V_S}{\cancel{s}}\cancel{s}=K_1\frac{s}{s+\frac{1}{RC}}+\frac{K_2}{\cancel{s}}\cancel{s}$$

$$s=0\Rightarrow\frac{\frac{1}{RC}}{\frac{1}{RC}}V_S=K_1\cdot 0+K_2\Rightarrow K_2=V_S$$

Example 1 Continued

For the second factor $s+\frac{1}{RC}$:

$$\frac{1}{RC}\frac{\cancel{s+\frac{1}{RC}}}{\cancel{s+\frac{1}{RC}}}\frac{V_S}{s}=K_1\frac{\cancel{s+\frac{1}{RC}}}{\cancel{s+\frac{1}{RC}}}+K_2\frac{s+\frac{1}{RC}}{s}$$

Let $s = -1/RC$:

$$s=\frac{-1}{RC}\Rightarrow\frac{1}{RC}\frac{V_S}{\frac{-1}{RC}}=K_1+K_2\cdot 0\Rightarrow K_1=-V_S$$

Comments on the Laplace Transform

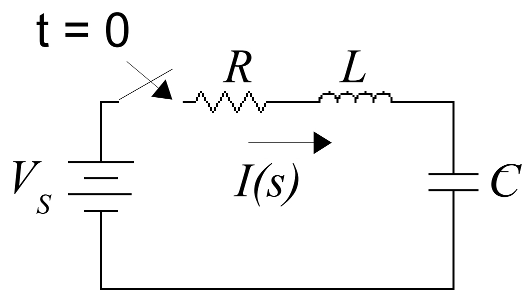
As in Bode plotting, we need to factor the denominator polynomial to obtain open loop or closed loop solutions.

If the time-base solution of the equation contains a sinusoid, the denominator polynomial will have a pair of complex conjugate $(a+j\omega, a-j\omega)$ roots.

These roots can represent decaying sinusoids ($a < 0$), sinusoids ($a = 0$), or growing sinusoids ($a > 0$). It is convenient to keep these conjugate root-pairs in quadratic form, i.e.

$$D(s)=(s+a)^2+\omega^2$$

Example 2



$$I(s)=\frac{C}{s^2LC+sCR+1}V_s$$

$$I(s)=\frac{\frac{1}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}V_s$$

Loop equation:

$$\left(R+sL+\frac{1}{sC}\right)I(s)=\frac{V_s}{s}$$

Multiplying both sides by "sC"

$$(s^2LC+sCR+1)I(s)=C\,V_s$$

Example 2

Assume R/L is "small" and complete the square in the denominator:

$$I(s)=\frac{\frac{1}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}V_s$$

$$I(s)=\frac{\frac{1}{L}}{s^2+\frac{R}{L}s+\left(\frac{1}{2}\frac{R}{L}\right)^2+\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2}V_s=\frac{\frac{1}{L}}{\left(s+\frac{1}{2}\frac{R}{L}\right)^2+\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)}V_s$$

Example 2

$$I=\frac{\frac{1}{L}}{\left(s+\frac{1}{2}\frac{R}{L}\right)^2+\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)}V_s$$

If: $\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)>0$

We have a solution of the form of a damped sinusoid:

$$e^{-at}\sin(\omega_1 t)\Leftrightarrow\frac{\omega_1}{(s+a)^2+\omega_1^2}$$

where:

$$a=\frac{1}{2}\frac{R}{L} \qquad \omega_1=\sqrt{\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)}$$

Example 2 cont.

If R could be made zero: $$\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)=0$$

The response would be a sine wave.

And if: $$\left(\frac{1}{LC}-\left(\frac{1}{2}\frac{R}{L}\right)^2\right)<0$$

We won't have sinusoidal response. The quadratic will factor into two real terms and we will obtain a response of the form:

$$i(t)=K_{11}e^{-bt}+K_2 e^{-ct}$$

Circuit/System Stability

1. Simply put, a linear system such as an electronic circuit is “stable” if it has no poles (denominator roots or zeros) in the right-half of the complex s-plane.

2. Right-half plane poles correspond to growing exponential terms in the solution of the system/circuit differential equation.

3. To determine stability, we need to check for right-half plane poles. This requires either factoring the denominator polynomial or employing a number of other techniques developed to assist in the pencil and paper design of linear systems.