Colpitts Oscillator

- Basic positive feedback oscillator
- The Colpitts LC Oscillator circuit
- Open-loop analysis
- Closed-loop analysis
- Root locus
- Stability limit
- Colpitts design
Basic Positive Feedback Oscillator

closed-loop oscillator

\[ V_i = 0 \quad + \quad A(s) \quad + \quad V_o \]

pos. fdbk

\[ A_{cl}(s) = \frac{A(s)}{1 - A(s)} \]

\[ V_o = A(s)(V_i + V_f) = A(s)(0 + V_o) \Rightarrow \]

\[ V_o(1 - A(s)) = 0 \]

Since: \( V_o \neq 0 \Rightarrow 1 - A(s) = 0 \Rightarrow A(s) = 1 \Rightarrow D(s) - K N(s) = 0 \]

Condition for oscillation at \( s = j\omega_0 \):

\[ A(s) = 1 e^{j\pm2k\pi} \quad \text{Barkhausen criterion} \]

for \( k = 0, 1, 2 \ldots \)
Colpitts Oscillator Basic Schematic

Assumptions:
1. $r_\pi$ large (compared to $1/\omega C_2$).
2. $C_\mu$ negligible (compared to $C_1$, $C_2$)
3. $C_\pi$ part of $C_2$ (in closed loop)
4. $R$ represents total resistance in collector circuit, i.e. $R||r_o \approx R$
Loop-Gain Analysis

![Circuit Diagram]

Node equation at $V_c$:

$$(sC_1 + \frac{1}{sL} + \frac{1}{R})V_c - \frac{1}{sL} V_f + (g_m - sC_\mu)V_\pi = 0$$

$$\left| j \omega C_\mu \right| \ll g_m$$

Note that

$V_f = V_f(s)$

$V_c = V_c(s)$

$V_\pi = V_\pi(s)$

at $V_f$:

$$-\frac{1}{sL} V_c + \left( \frac{1}{sL} + sC_2 \right) V_f = 0$$
Open Loop Analysis - cont.

Rearranging (1) and (2):

\[
\left( sC_1 + \frac{1}{sL} + \frac{1}{R} \right) V_c - \frac{1}{sL} V_f = -g_m V_\pi \quad (3)
\]

\[-\frac{1}{sL} V_c + \left( sC_2 + \frac{1}{sL} \right) V_f = 0 \quad (4)
\]

Further rearrangement of (3) and (4):

\[
\left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) V_c - \frac{1}{sL} V_f = -g_m V_\pi \quad (5)
\]

\[-\frac{1}{sL} V_c + \left( \frac{s^2 LC_2 + 1}{sL} \right) V_f = 0 \quad (6)
\]

from previous slide

\[
\left( sC_1 + \frac{1}{sL} + \frac{1}{R} \right) V_c - \frac{1}{sL} V_f + g_m V_\pi = 0 \quad (1)
\]

\[-\frac{1}{sL} V_c + \left( \frac{1}{sL} + sC_2 \right) V_f = 0 \quad (2)
\]
Open Loop Analysis - cont.

Prepare to add the two equations:

\[
\frac{1}{sL} \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) V_c - \left( \frac{1}{sL} \right)^2 V_f = -\frac{g_m}{sL} V_\pi
\]

\[
-\frac{1}{sL} \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) V_c + \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 L C_2 + 1}{sL} \right) V_f = 0
\]

Adding (to eliminate \( V_c \) terms):

\[
\left( -\left( \frac{1}{sL} \right)^2 + \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 L C_2 + 1}{sL} \right) \right) V_f = -\frac{g_m}{sL} V_\pi
\]

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\[
\left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) V_c - \frac{1}{sL} V_f = -g_m V_\pi \quad (5)
\]

\[
-\frac{1}{sL} V_c + \left( \frac{s^2 L C_2 + 1}{sL} \right) V_f = 0 \quad (6)
\]

\[
Eq(5) \ast \frac{1}{sL}
\]

\[
Eq(6) \ast \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right)
\]
Open Loop Analysis – cont.

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\[
\left( -\left( \frac{1}{sL} \right)^2 + \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 LC_2 + 1}{sL} \right) \right) V_f = \frac{-g_m}{sL} V_\pi
\]

Multiply (7) by \((sL)^2\):

\[
\left( -1 + \left( s^2 LC_1 + 1 + \frac{sL}{R} \right) \left( s^2 LC_2 + 1 \right) \right) V_f = -g_m sL V_\pi
\]

Expand and collect terms according to \(s^n\):

\[
\left( -1 + s^4 C_1 C_2 L^2 + s^2 \left( LC_1 + LC_2 \right) + s^3 \frac{L^2 C_2}{R} + s \frac{L}{R} + 1 \right) V_f = -g_m sL V_\pi
\]

(8)
Open Loop Analysis - cont.

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\[
\begin{align*}
\left(-1 + s^4 C_1 C_2 L^2 + s^2 (L C_1 + L C_2) + \frac{s^3 L^2 C_2}{R} + \frac{s L}{R} + 1\right) V_f &= -g_m s L V_\pi \quad (8)
\end{align*}
\]

Canceling (-1 by 1) in (8) and dividing by \(sL\):

\[
\begin{align*}
\left(s^3 C_1 C_2 L + s (C_1 + C_2) + \frac{s^2 L C_2}{R} + \frac{1}{R}\right) V_f &= -g_m V_\pi \\
\end{align*}
\]

Multiply by \(R\):

\[
\begin{align*}
\left(s^3 R C_1 C_2 L + s R (C_1 + C_2) + s^2 L C_2 + 1\right) V_f &= -g_m R V_\pi \quad (9)
\end{align*}
\]
Open Loop Analysis - cont.

From previous slide

\[
\left( s^3 R C_1 C_2 L + s R \left( C_1 + C_2 \right) + s^2 L C_2 + 1 \right) V_f = -g_m R V_{\pi} \tag{9}
\]

Divide (9) by \( R C_1 C_2 L \):

\[
\left( s^3 + s \frac{C_1 + C_2}{C_1 C_2 L} + s^2 \frac{1}{R C_1} + \frac{1}{R C_1 C_2 L} \right) V_f = \frac{-g_m R}{R C_1 C_2 L} V_{\pi}
\]

The loop-gain transfer function:

\[
\frac{V_f}{V_{\pi}} = A(s) = \left( -\frac{g_m}{C_1 C_2 L} s^3 + s^2 \frac{1}{R C_1} + s \frac{C_1 + C_2}{C_1 C_2 L} + \frac{1}{R C_1 C_2 L} \right) = \frac{K N \left( s \right)}{D \left( s \right)} \tag{10}
\]
Closed Loop Analysis - cont.

The closed loop equation:

\[ A_{cl}(s) = \frac{A(s)}{1 - A(s)} = \frac{K N(s)}{D(s) - K N(s)} = \frac{K N(s)}{D'(s)} \]

where:

\[ A(s) = \frac{K N(s)}{D(s)} \quad K = \frac{-g_m}{C_1 C_2 L} \quad N(s) = 1 \]

\[ D(s) = s^3 + s^2 \frac{1}{RC_1} + s \frac{C_1 + C_2}{C_1 C_2 L} + \frac{1}{RC_1 C_2 L} \] (11)

then:

\[ D'(s) = D(s) - \frac{-g_m}{C_1 C_2 L} = s^3 + s^2 \frac{1}{RC_1} + s \frac{C_1 + C_2}{C_1 C_2 L} + \frac{1 + g_m R}{RC_1 C_2 L} \] (12)
Closed Loop Analysis - cont.

We know that the open loop system $A(s)$ is stable. It has poles in the left half plane, since it is a passive RLC circuit. We also know that it has 3 poles. One is negative-real, the other 2 can be negative-real or LHP complex conjugates.

$$D(s) = s^3 + s^2 \frac{1}{RC_1} + s \frac{C_1 + C_2}{C_1 C_2 L} + \frac{1}{RC_1 C_2 L} \quad (11)$$

So, let's do a rough sketch of the root locus for a feedback system with a 3 pole $A(s)$.
Root Locus Characteristic

- The loop will become unstable for any value of $\Gamma_F > \Gamma_{Fx}$.

- Rather than sketch the root locus in more exacting detail – it has served its purpose by verifying that oscillation is possible.

- Let's solve for the required $\Gamma_{Fx}$.

$$D(s) = s^3 + s^2 \frac{1}{RC_1} + s \left( \frac{C_1 + C_2}{C_1 C_2 L} \right) + \frac{1}{RC_1 C_2 L} \quad (11)$$

$$D'(s) = D(s) - KN(s) = s^3 + s^2 \frac{1}{RC_1} + s \left( \frac{C_1 + C_2}{C_1 C_2 L} \right) + \frac{1 + g_m R}{RC_1 C_2 L} \quad (12)$$
Stability Limit Calculation

If the closed loop system is at the stability limit point:

\[ D'(s) = D(s) + KN(s) = (s + a)(s^2 + \omega_x^2) \]

Multiplying terms:

\[ D'(s) = s^3 + as^2 + \omega_x^2 s + a \omega_x^2 \Rightarrow D'(j\omega) = (a \omega_x^2 - a \omega^2) + j(\omega^2 \omega - \omega^3) \]

Match term by term with:

\[ D'(s) = s^3 + s^2 \frac{1}{RC_1} + s \frac{(C_1 + C_2)}{C_1 C_2 L} + 1 + \frac{g_m R}{RC_1 C_2 L} \]

\[ a = \frac{1}{RC_1} \quad \omega_x^2 = \frac{(C_1 + C_2)}{C_1 C_2 L} \quad a \omega_x^2 = \frac{1 + g_m R}{RC_1 C_2 L} \quad \text{s.t.} \quad a \omega_x = a \omega_x^2 \]

to oscillate at \( \omega = \omega_x \)

= 0

= 0
Stability Limit

\[ a = \frac{1}{RC_1} \quad \omega^2_x = \frac{C_1 + C_2}{C_1 C_2 L} \quad a \omega^2_x = \frac{1 + g_m R}{RC_1 C_2 L} \quad a \omega^2_x = a \omega^2_x \]

To find the “gain” requirement for oscillation, equate

\[ a \omega^2_x = a \omega^2_x \]

\[
\begin{bmatrix}
\frac{1}{RC_1} & \frac{C_1 + C_2}{C_1 C_2 L} & \frac{C_1 + C_2}{RC_1^2 C_2 L} & \frac{1 + g_m R}{RC_1 C_2 L} \\
\end{bmatrix}
\]

\[ g_m R = \frac{C_2}{C_1} \quad \omega_x = \sqrt{\frac{C_1 + C_2}{C_1 C_2 L}} \]
Oscillator Design Summary

The oscillation frequency:

\[ \omega_x = \sqrt{\frac{C_1 + C_2}{C_1 C_2 L}} \]

The required feedback gain \( \Gamma_F = g_m R \):

\[ \Gamma_{Fx} = (g_m)R = \frac{C_2}{C_1} \] poles at \( \pm j\omega_x \)

To insure the start-up of oscillation:

\[ (g_m)R > \frac{C_2}{C_1} \]

Comments:

1. Unfortunately we don't control “R”.

2. Set \( C_2 \) & \( C_1 \) to convenient values, say \( C_2 = C_1 = C \), and adjust \( g_m \) using \( I_C \) s.t. \( g_m R > C_2 / C_1 \); then adjust “L” to maintain \( \omega_x \) at the desired oscillation frequency.
Practical Colpitts Oscillator Circuit

RFC = RF choke -> large reactance at $\omega_x$

low resistance at dc