ESE370: Circuit-Level Modeling, Design, and Optimization for Digital Systems

Lec 23: November 1, 2021

Driving Large Capacitive Loads

Today

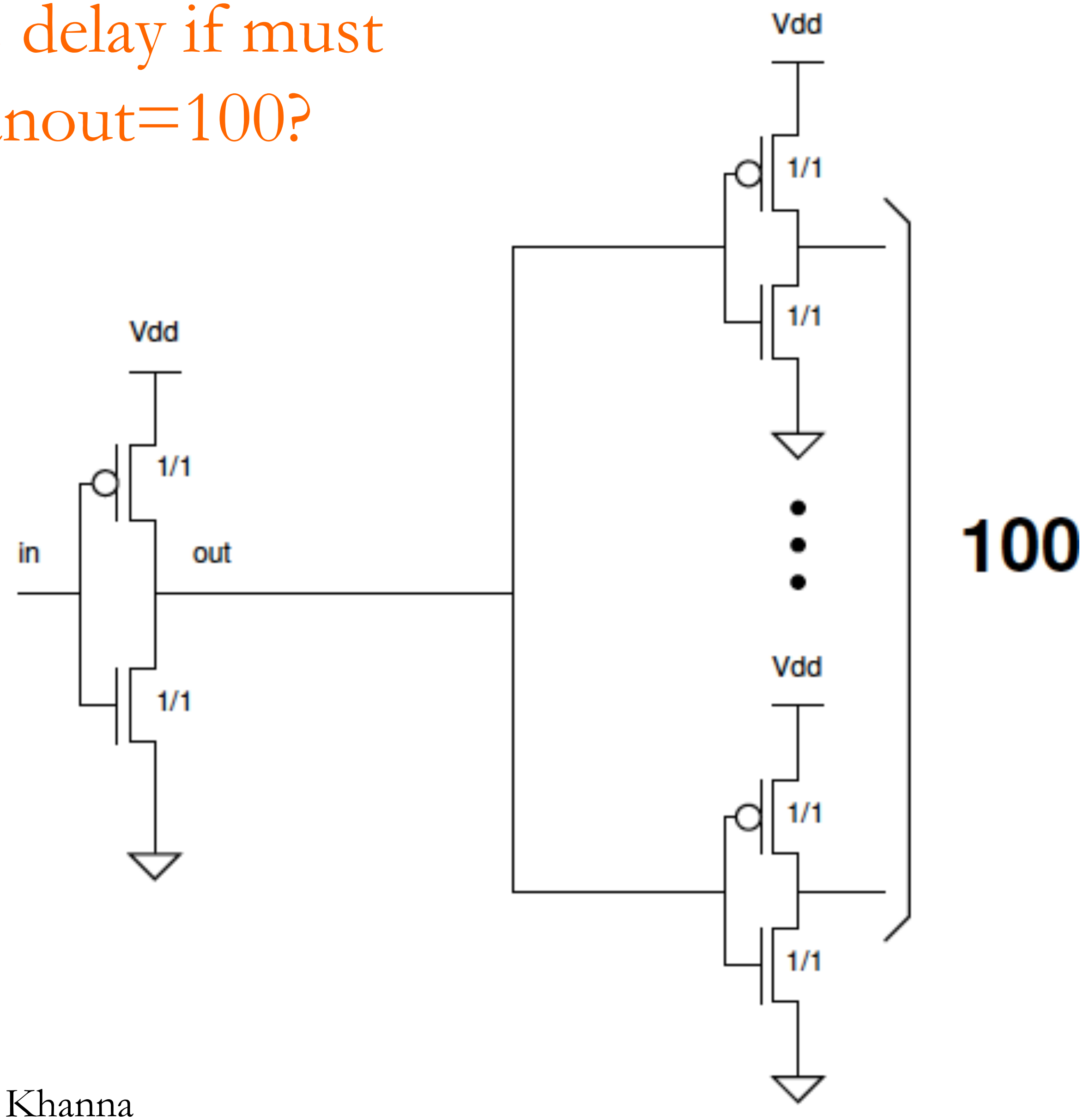
- Back to CMOS today
- How do we drive a large capacitive load?
  - Stages and buffer sizing
  - Minimum delay

Message

- To drive large loads
  - Scale buffers geometrically
  - Exponential scale up in buffer size
- Scale factor: 3—4 typically
  - One origin of FO4 target
- Drains contribute capacitance too (C_{diff})
- Can formulate sizing to optimize

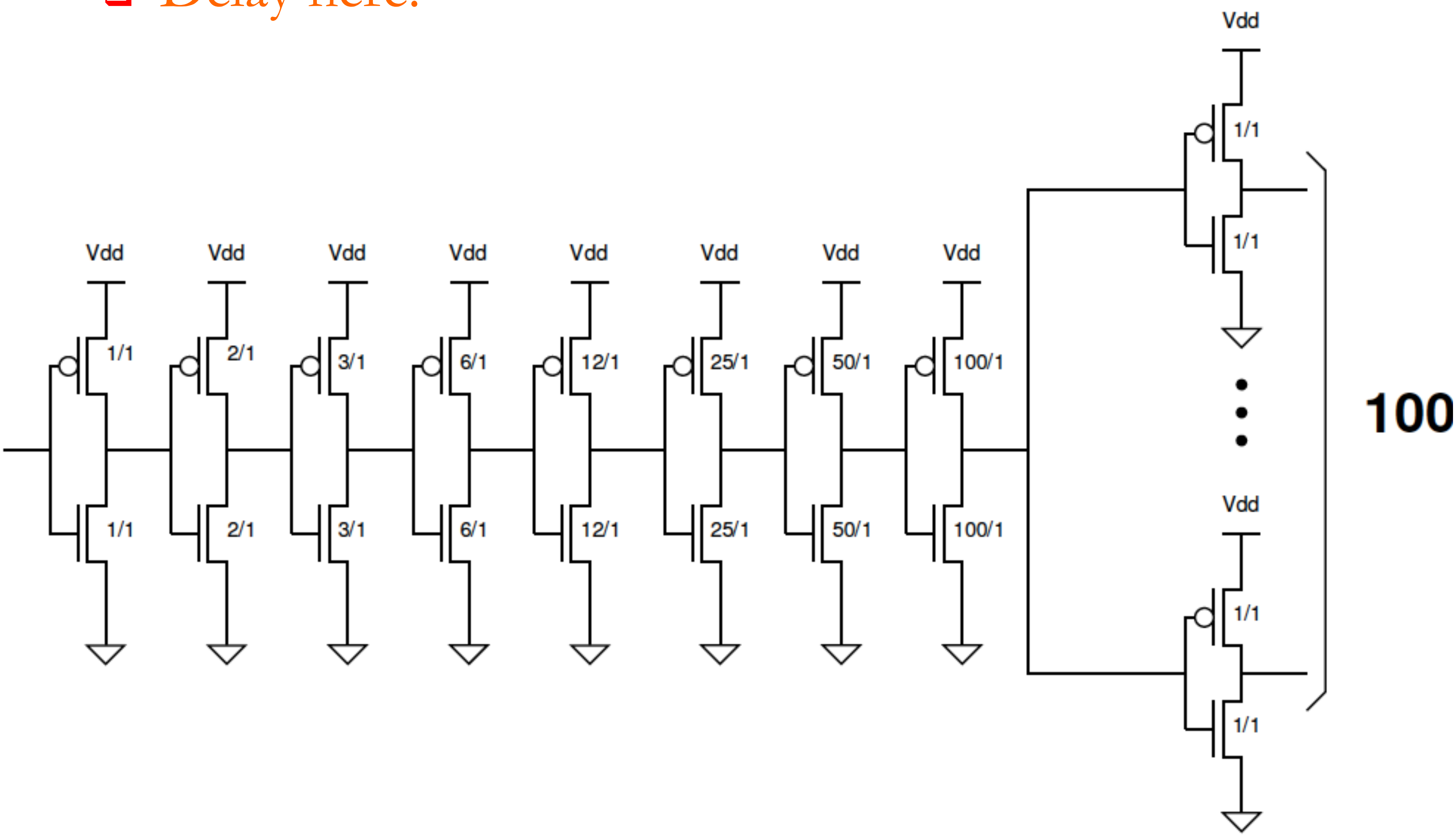
Call back: Large Fanout Delay

- What is delay if must drive fanout=100?



Call back: …and Again

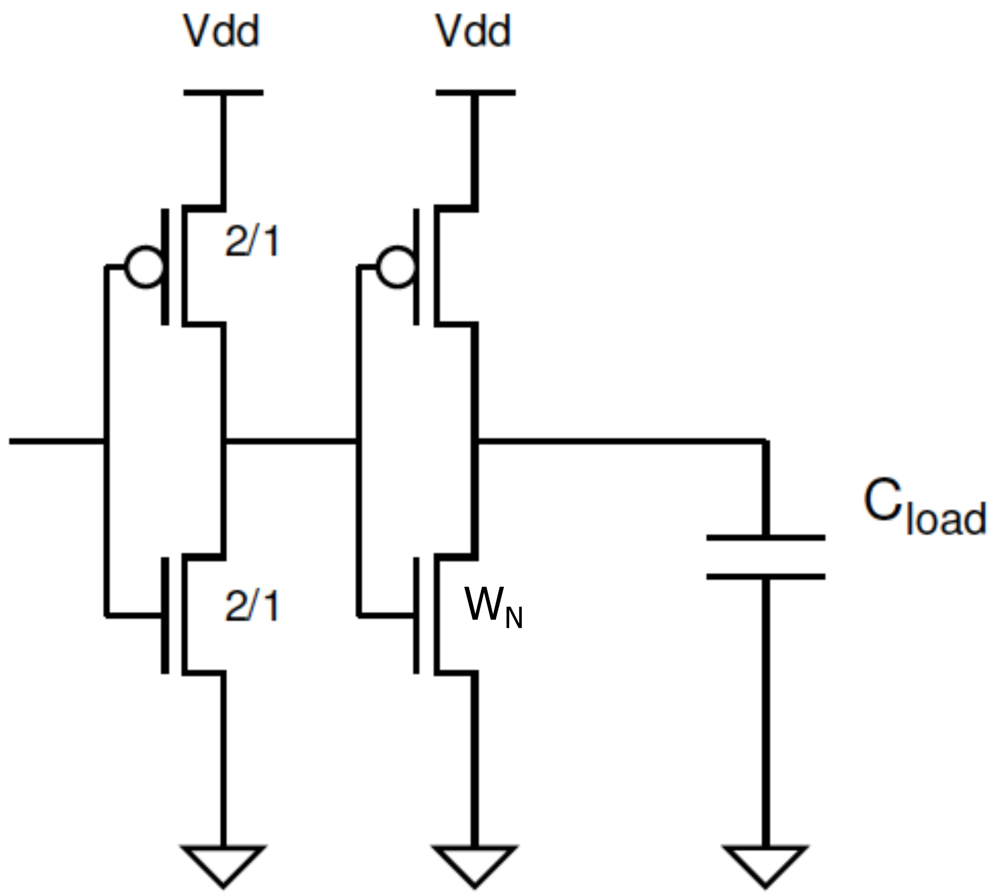
- Delay here?



Start C_{diff}=0

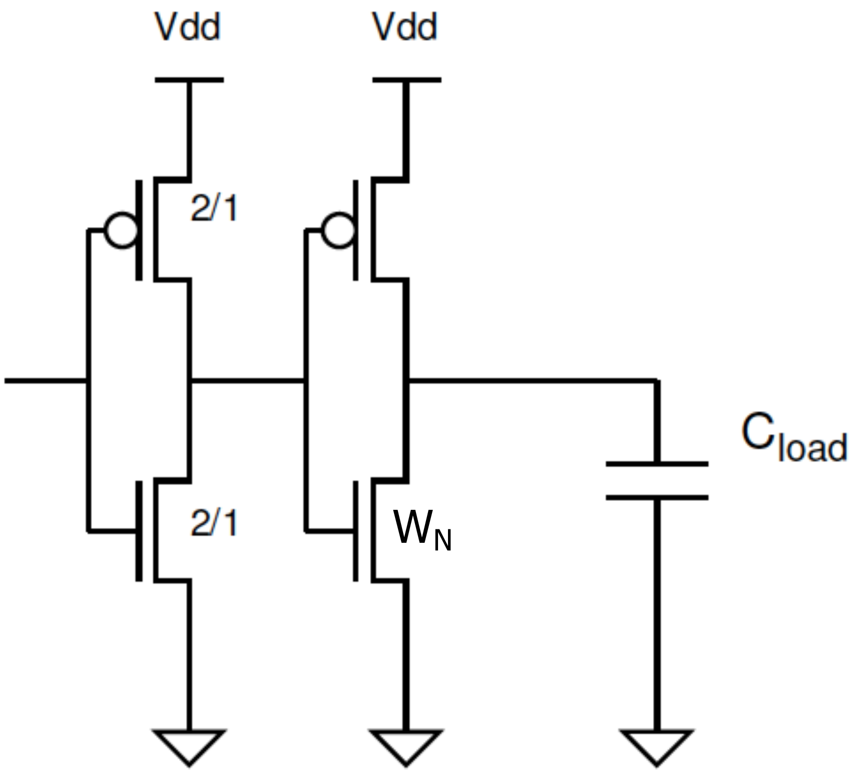
One Stage (Preclass 1)

- How do we size to minimize delay?



One Stage (Preclass 1)

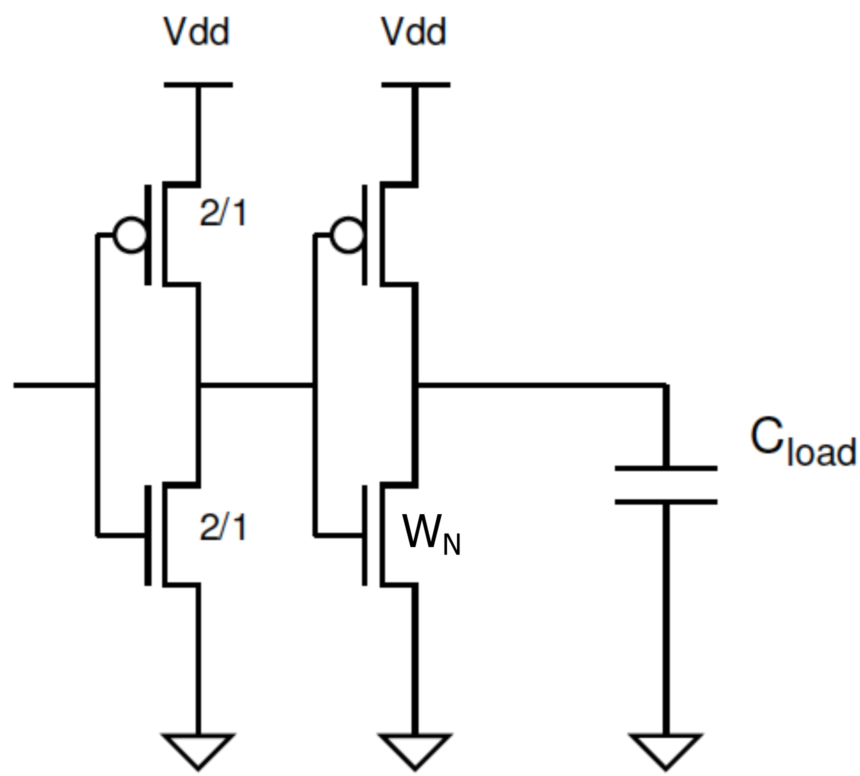
- Delay equation?



One Stage (Preclass 1)

- Delay equation?

$$delay = \frac{R_0}{2} 2W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{load}$$



Minimize (Preclass 1)

$$delay = R_0 W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{load}$$

- Differentiate and set to zero

$$R_0 C_0 - \frac{R_0}{W_N^2} \cdot C_{load} = 0$$

- What's W_N?

Minimize

$$delay = R_0 W_N \cdot C_0 + \frac{R_0}{W_N} \cdot C_{load}$$

- Differentiate and set to zero

$$R_0 C_0 - \frac{R_0}{W_N^2} \cdot C_{load} = 0$$

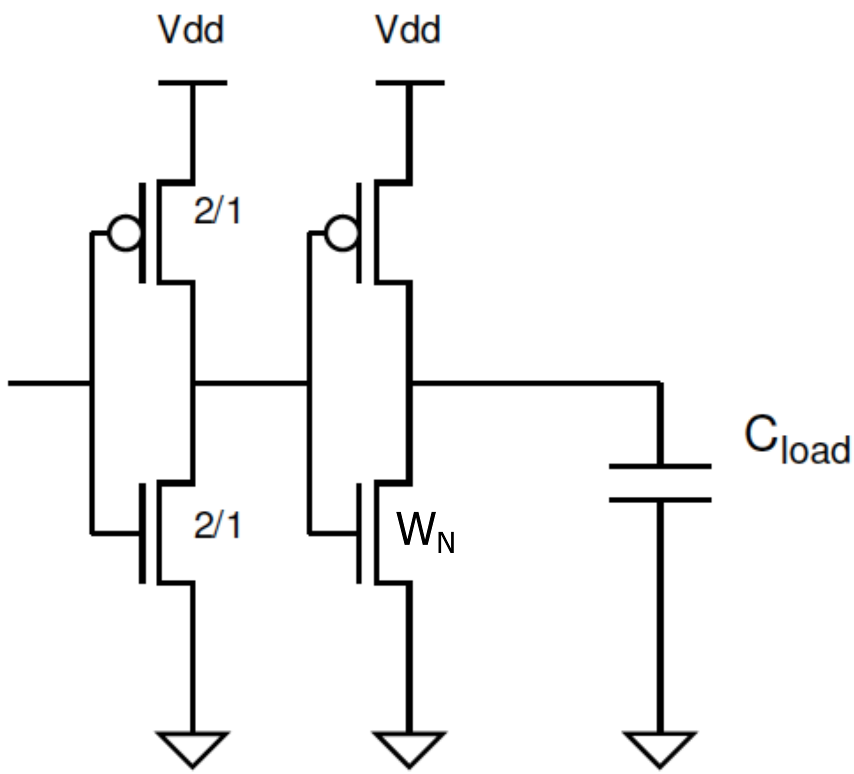
- What's W_N?

$$W_N^2 = \frac{C_{load}}{C_0} \qquad W_N = \sqrt{\frac{C_{load}}{C_0}}$$

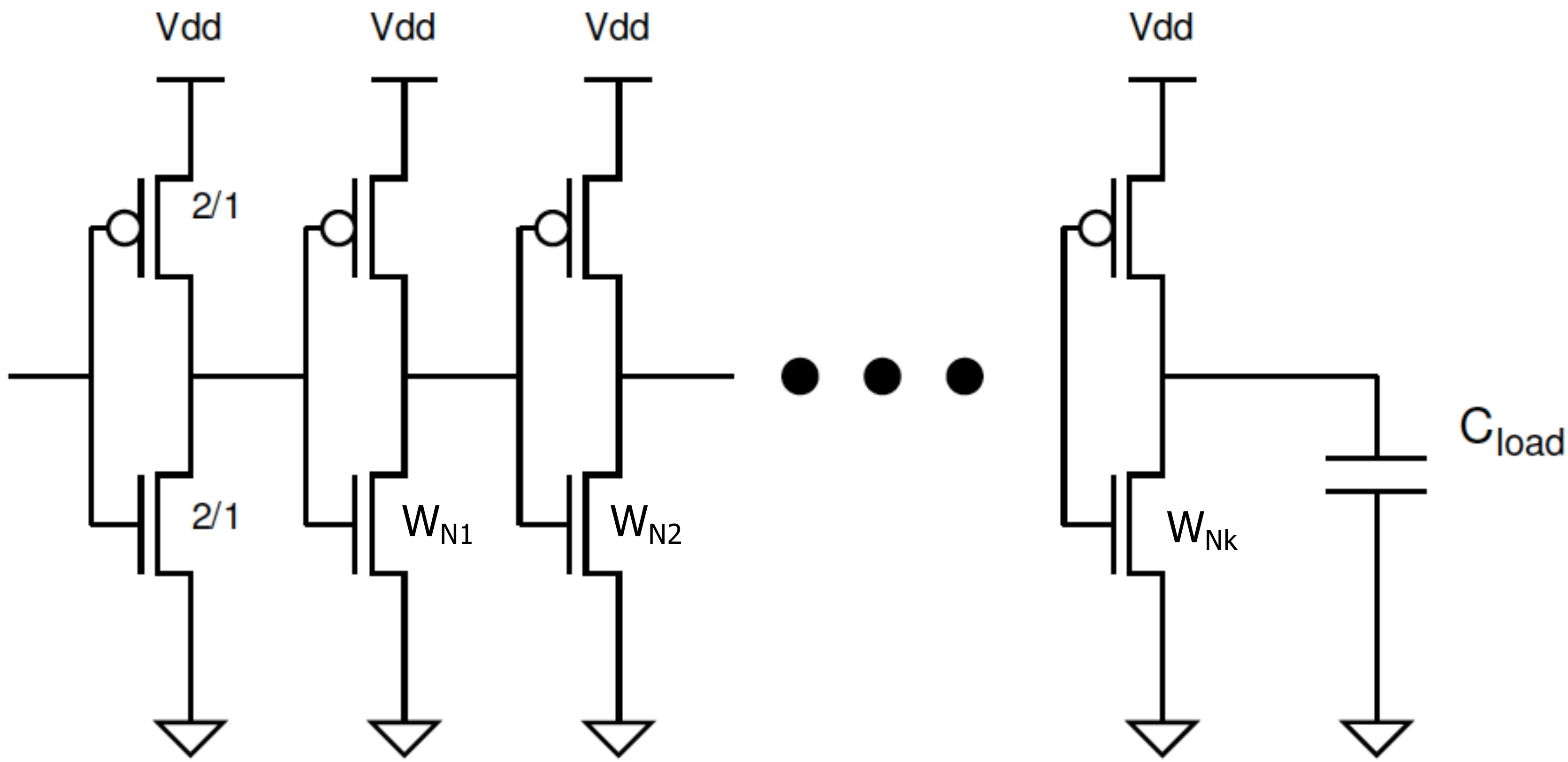
Concrete?

- What is W_N for $C_{load}=4x10^4C_0$?

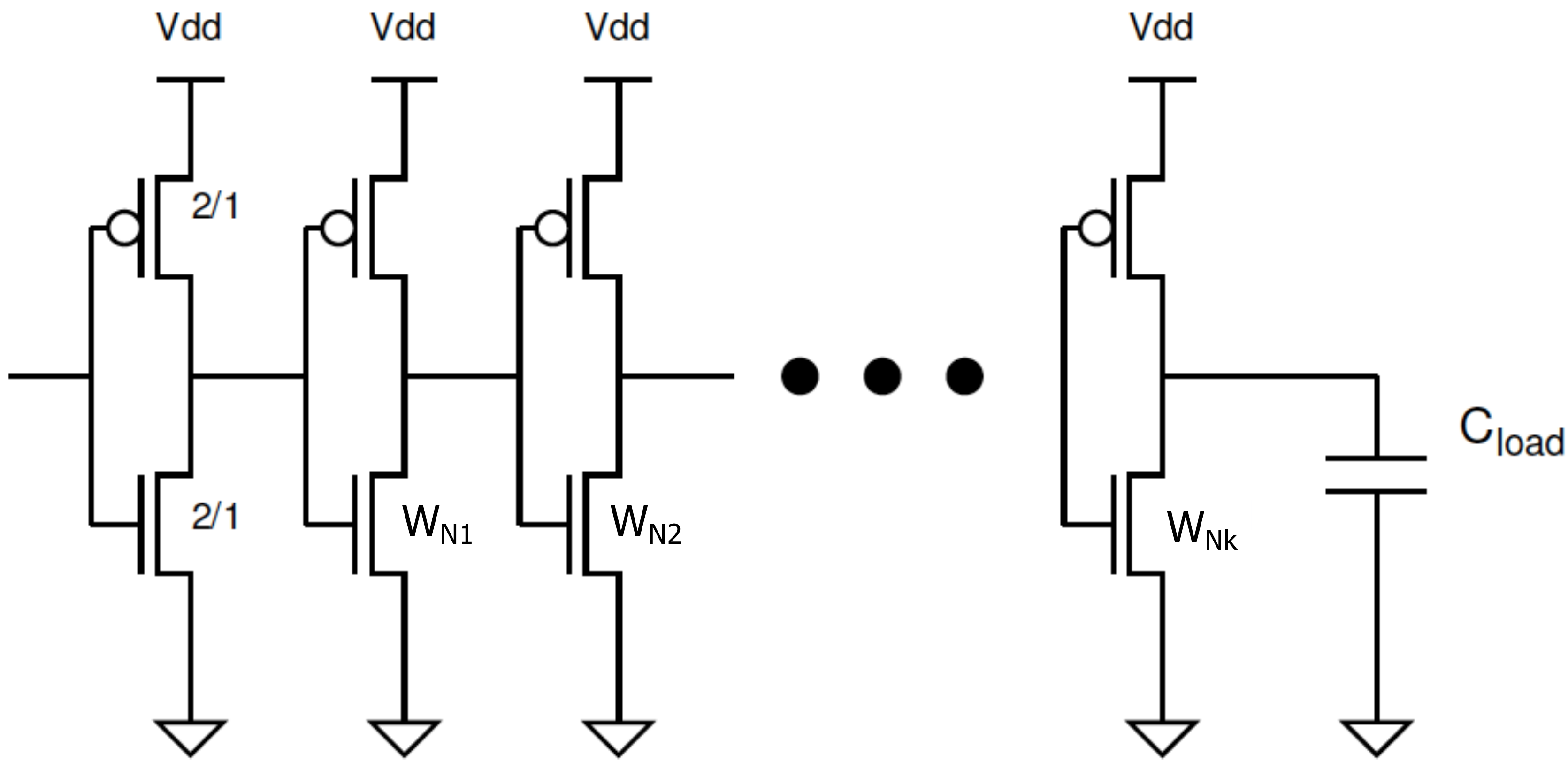
$$W_N = \sqrt{\frac{C_{load}}{C_0}}$$



k-stage (Preclass 2)



k-stage Delay (Preclass 2)



$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

Size W_{Ni} to minimize delay (Preclass 2)

- How do we minimize?

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

Size W_{Ni} to minimize delay (Preclass 2)

- Take partial derivative with respect to W_{Ni} and set = 0

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

$$2\tau\left(0+0+0+\ldots+\frac{1}{W_{N(i-1)}}-\frac{W_{N(i+1)}}{\left(W_{Ni}\right)^2}+\ldots+0\right)+0=0$$

Size W_{Ni} to minimize delay (Preclass 2)

- Take partial derivative with respect to W_{Ni} and set = 0

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

$$2\tau\left(0+0+0+\ldots+\frac{1}{W_{N(i-1)}}-\frac{W_{N(i+1)}}{\left(W_{Ni}\right)^2}+\ldots+0\right)+0=0$$

$$\frac{1}{W_{N(i-1)}}=\frac{W_{N(i+1)}}{W_{Ni}^2}\rightarrow\frac{W_{Ni}}{W_{N(i-1)}}=\frac{W_{N(i+1)}}{W_{Ni}}$$

Delay

- **Conclude:** at optimal sizing, ratio of stages is same:

$$\frac{1}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}^2} \rightarrow \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$

Delay

- Call that ratio ***ρ***

$$\rho = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}}$$

Stage Delay

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

Stage Delay

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau\cdot W_{Nk}}\times C_{load}\right)$$

Stage Delay

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau\cdot W_{Nk}}\times C_{load}\right)$$

$$\rho=\frac{W_{N1}}{2}=\frac{W_{Ni}}{W_{N(i-1)}}=\frac{W_{N(i+1)}}{W_{Ni}}=\frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

Stage Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

Two math simplifications: 1) to relate ρ and k, 2) total delay in terms of ρ and k

Stage Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

$$\left(\frac{W_{N1}}{2}\right)\left(\frac{W_{N2}}{W_{N1}}\right)\left(\frac{W_{N3}}{W_{N2}}\right)\cdots\left(\frac{W_{Ni}}{W_{N(i-1)}}\right)\left(\frac{W_{N(i+1)}}{W_{Ni}}\right)\cdots\left(\frac{W_{Nk}}{W_{N(k-1)}}\right)\frac{C_{load}}{W_{Nk}\left(2C_0\right)} = \rho^{k+1}$$

Stage Delay

$$\rho^{k+1} = \frac{C_{load}}{4C_0} \rightarrow \rho = \sqrt[k+1]{\frac{C_{load}}{4C_0}}$$

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

Total Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

$$2\tau\left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \ldots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \ldots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load}\right)$$

Total Delay (Preclass 2)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

$$2\tau\left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \ldots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \ldots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load}\right)$$

$$TotalDelay = 2\tau(k+1)\rho$$

Total Delay

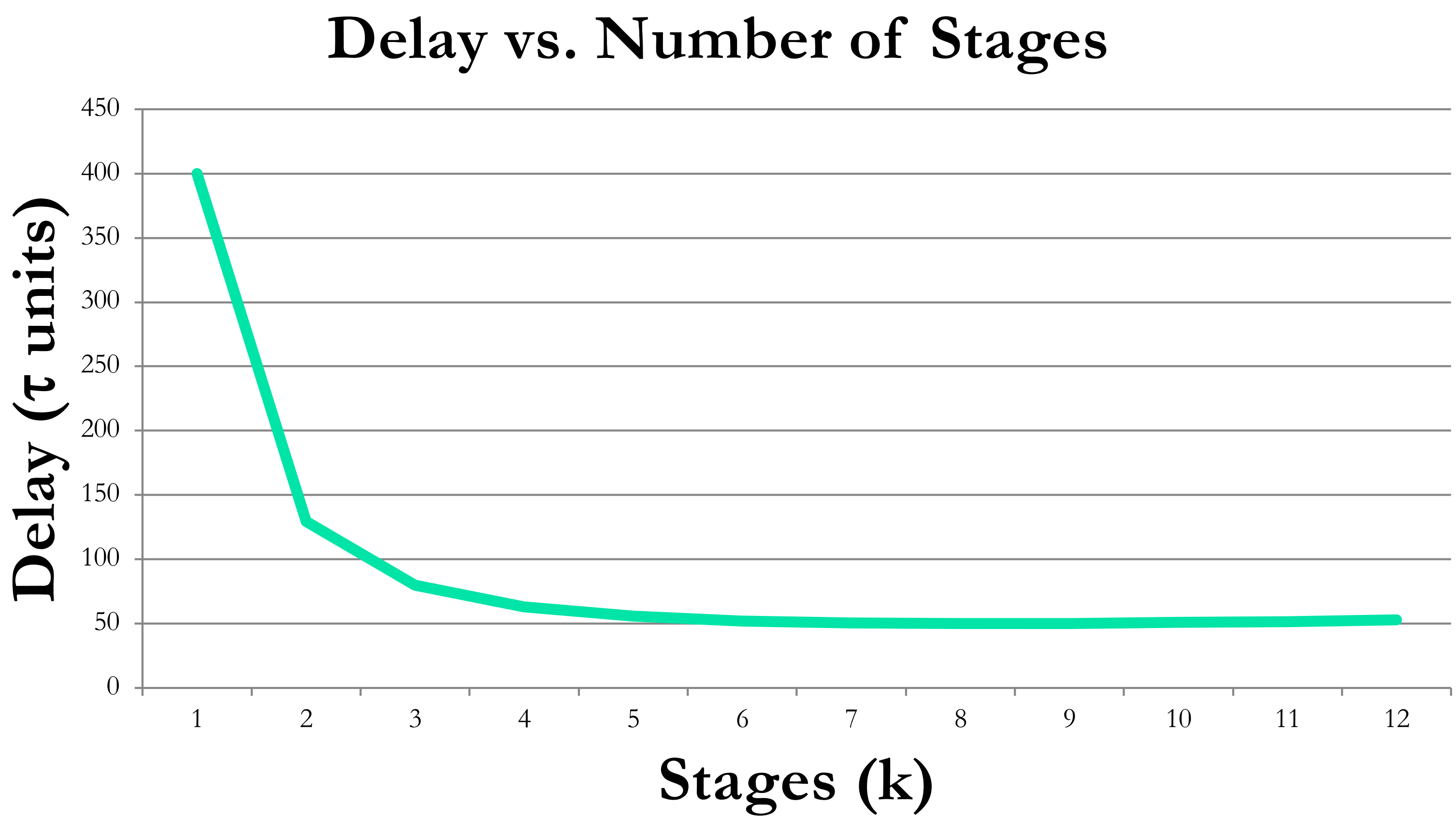
$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$TotalDelay = 2\tau(k+1)\rho$$

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

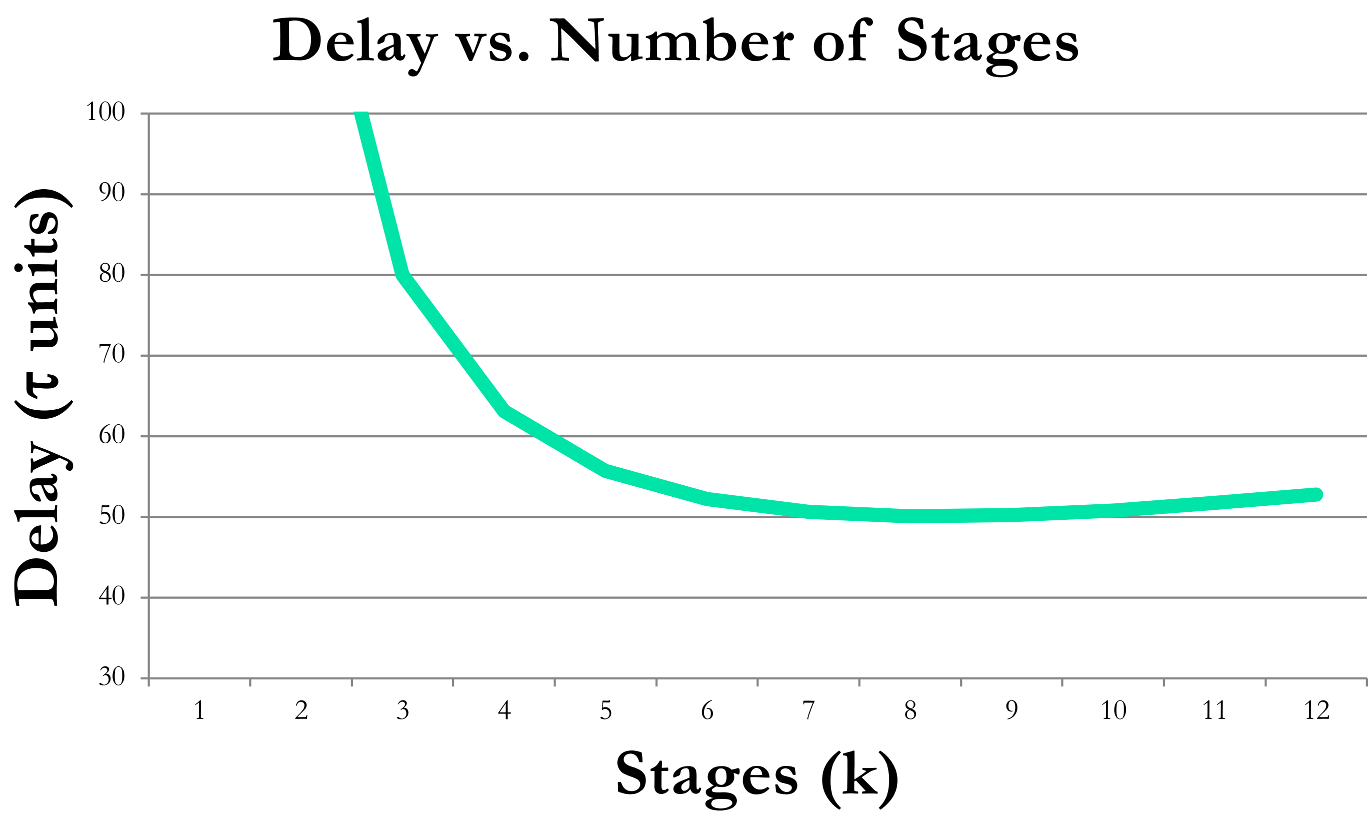
Plot Delay vs. k ($C_{load}=4x10^4C_0$)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$



Zoom: Plot Delay vs. k (C_{load}=4x10^4C_0)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$



Minimize (Preclass 3)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$0 = 2\tau\left[\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1)\cdot\ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}\left(\frac{1}{k+1}\right)^2\right]$$

$$\frac{d(b^x)}{dx} = \ln(b)\cdot b^x$$

$$\frac{d(1/x)}{dx} = -\frac{1}{x^2}$$

Minimize (Preclass 3)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$0 = 2\tau\left[\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1)\cdot\ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}\left(\frac{1}{k+1}\right)^2\right]$$

$$0 = 1 - \left(\frac{1}{k+1}\right)\ln\left(\frac{C_{load}}{4C_0}\right)$$

$$\frac{d(b^x)}{dx} = \ln(b)\cdot b^x$$

$$\frac{d(1/x)}{dx} = -\frac{1}{x^2}$$

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$

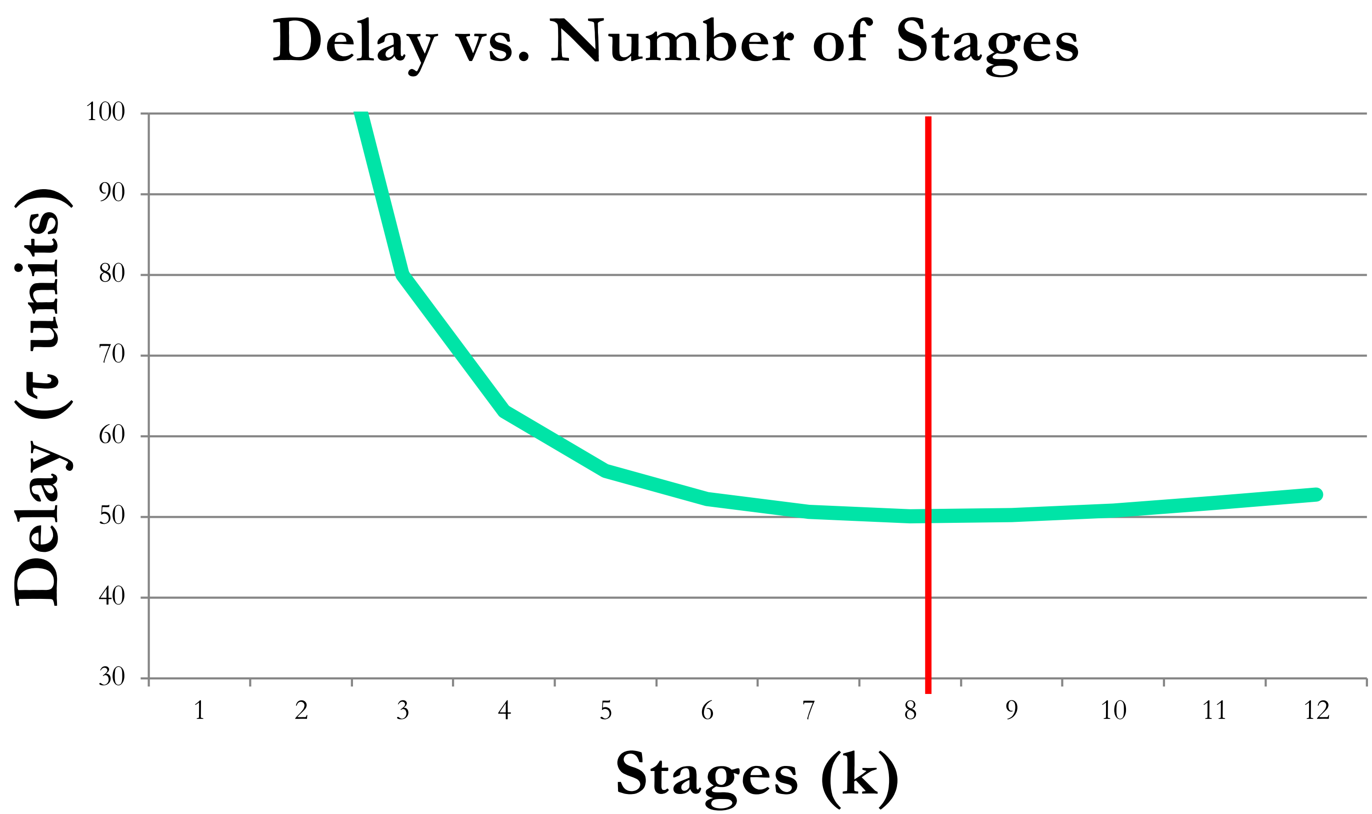
Concrete (Preclass 3)

- What is optimal k for $C_{load}=4x10^4C_0$?

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$

Zoom: Plot Delay vs. k ($C_{load}=4x10^4C_0$)

$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$



Optimum Scale Up

- For optimum delay

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1$$

- What is ρ?

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

Optimum Scale Up

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{\ln\left(\frac{C_{load}}{4C_0}\right)}\right)} = (Y)^{\left(\frac{1}{\ln(Y)}\right)}$$

Optimum Scale Up

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{\ln\left(\frac{C_{load}}{4C_0}\right)}\right)} = (Y)^{\left(\frac{1}{\ln(Y)}\right)}$$

$$\ln(\rho) = \frac{1}{\ln(Y)}\ln(Y) = 1$$

$$\rho = e$$

Call Back: Total Delay

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

$$2\tau\left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \ldots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \ldots + \frac{W_{Nk}}{W_{N(k-1)}}\right) + \frac{R_0}{W_{Nk}} \times C_{load}$$

$$2\tau\left(\frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \ldots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \ldots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \times C_{load}\right)$$

$$TotalDelay = 2\tau(k+1)\rho$$

Delay at Optimum

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1 \qquad \rho = e$$

$$TotalDelay = 2\tau(k+1)\rho$$

Delay at Optimum (Preclass 3)

$$k = \ln\left(\frac{C_{load}}{4C_0}\right) - 1 \qquad \rho = e$$

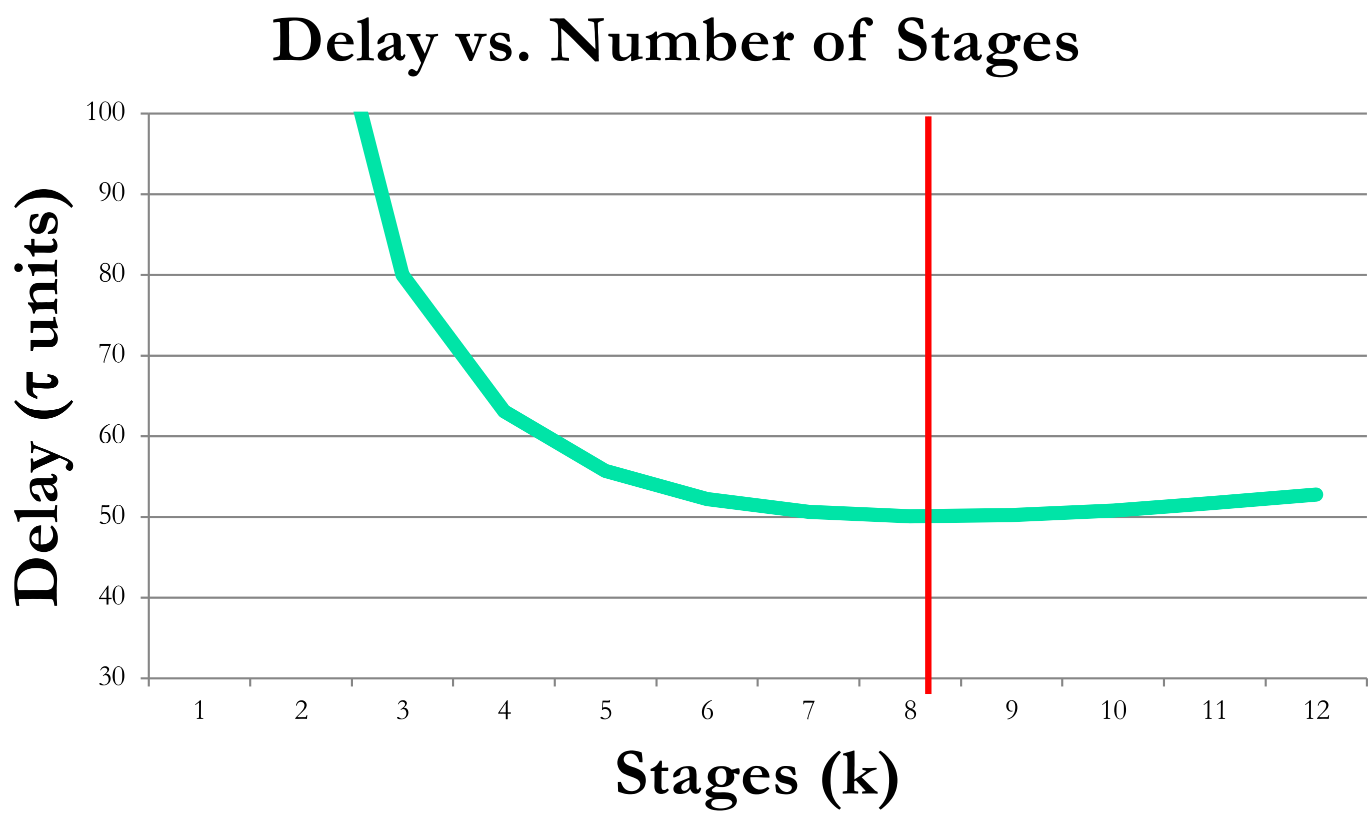
$$TotalDelay = 2\tau(k+1)\rho$$

$$TotalDelay = 2\tau \cdot \ln\left(\frac{C_{load}}{4C_0}\right) \cdot e$$

- What is optimal delay for $C_{load}=4\text{x}10^4 C_0$ in tau units?

Zoom: Plot Delay vs. k (C_{load}=4x$10^4 C_0$)

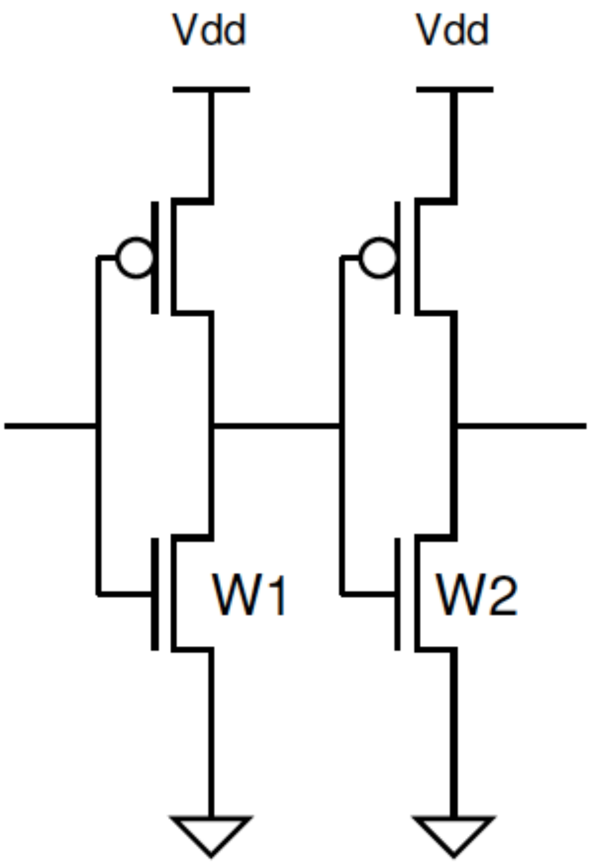
$$TotalDelay = 2\tau(k+1)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$



$C_{diff}=\gamma C_{gate}$

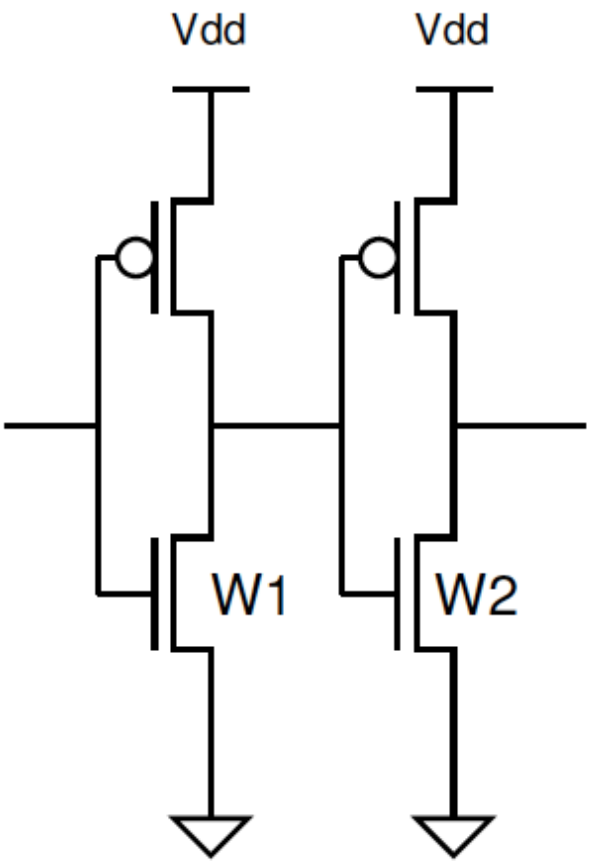
Diffusion Capacitance (Preclass 4)

- What does this do to τ model?
  - Delay of middle stage cascade?



Diffusion Capacitance (Preclass 4)

- What does this do to τ model?
  - Delay of middle stage cascade?



$$delay_{W1\rightarrow W2} = \left(\frac{R_0}{W_1}\right)\left(2W_1 \cdot C_{diff0} + 2W_2 \cdot C_0\right)$$

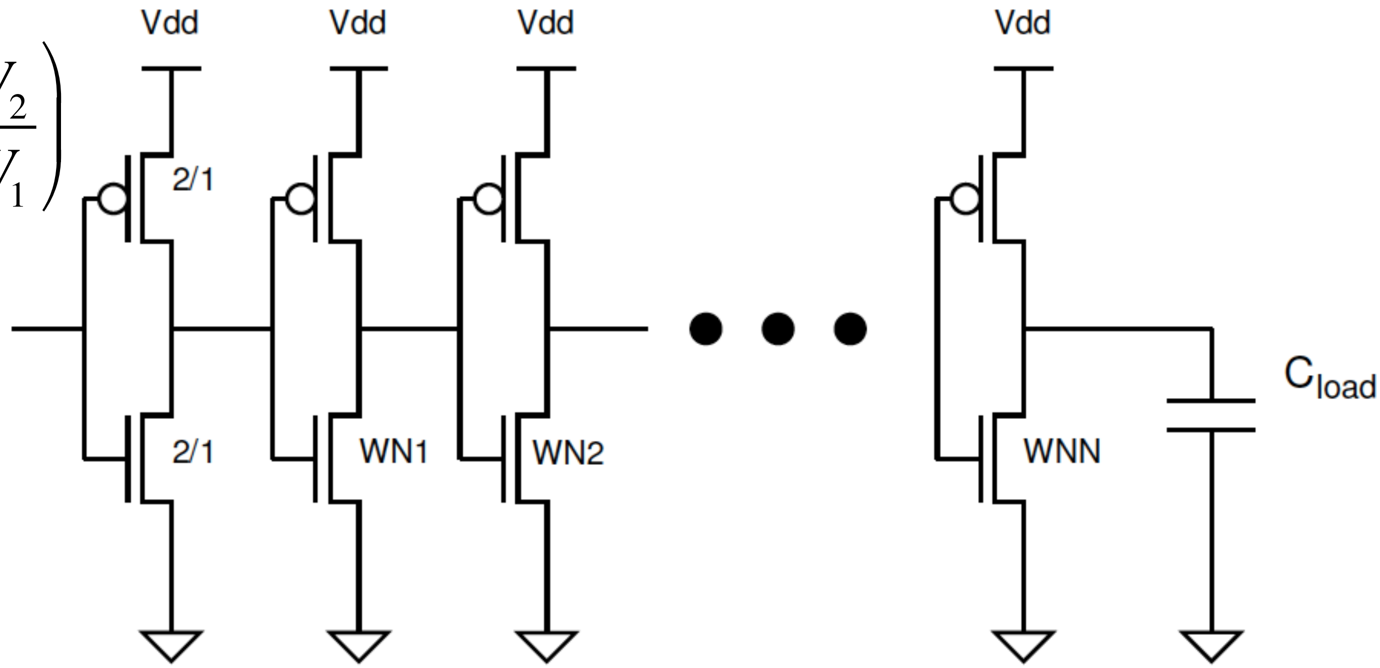
$$delay_{W1\rightarrow W2} = 2\left(\frac{R_0}{W_1}\right)\left(W_1 \cdot \gamma C_0 + W_2 \cdot C_0\right)$$

$$delay_{W1\rightarrow W2} = 2\tau\left(\gamma + \frac{W_2}{W_1}\right)$$

k-stage Delay

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

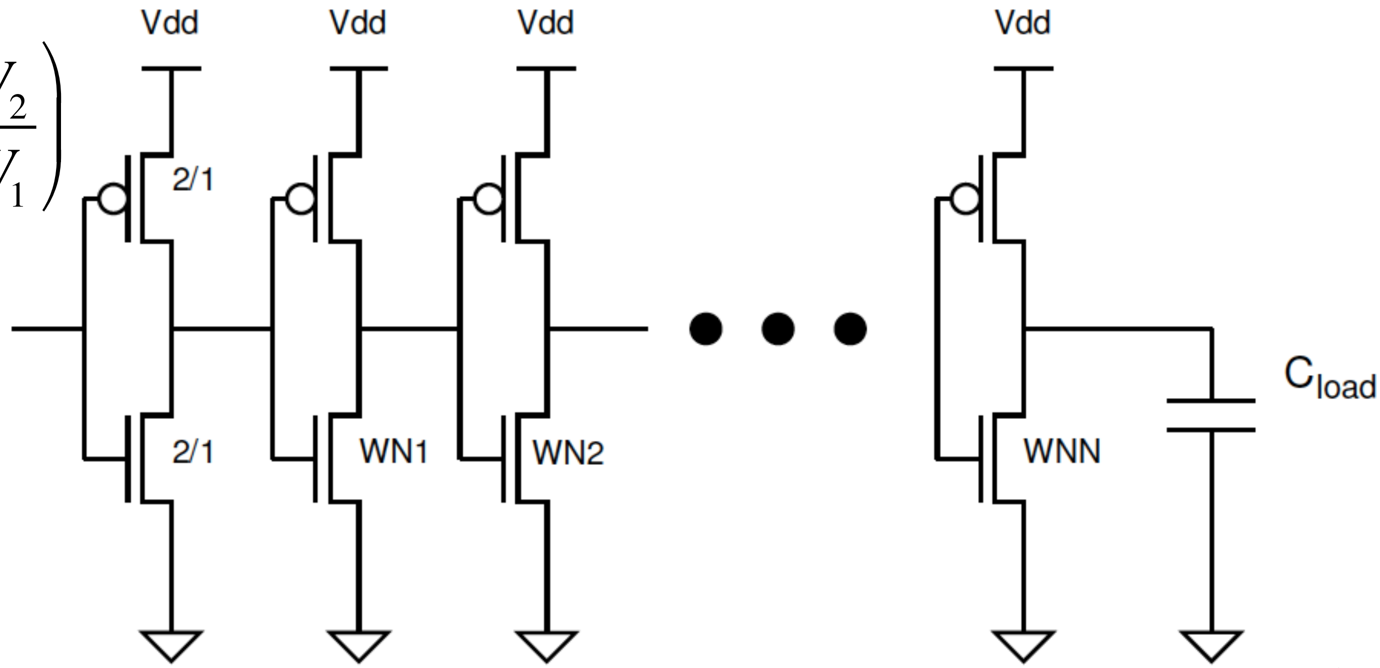
$$delay_{W1\rightarrow W2}=2\tau\left(\gamma+\frac{W_2}{W_1}\right)$$



k-stage Delay (Preclass 4)

$$2\tau\left(\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}\times C_{load}$$

$$delay_{W1\rightarrow W2}=2\tau\left(\gamma+\frac{W_2}{W_1}\right)$$



$$2\tau\left(\gamma+\frac{W_{N1}}{2}+\gamma+\frac{W_{N2}}{W_{N1}}+\gamma+\frac{W_{N3}}{W_{N2}}+\ldots+\gamma+\frac{W_{Ni}}{W_{N(i-1)}}+\gamma+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\gamma+\frac{W_{Nk}}{W_{N(k-1)}}\right)$$

$$+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

k-stage Delay

$$2\tau\left(\gamma+\frac{W_{N1}}{2}+\gamma+\frac{W_{N2}}{W_{N1}}+\gamma+\frac{W_{N3}}{W_{N2}}+\ldots+\gamma+\frac{W_{Ni}}{W_{N(i-1)}}+\gamma+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\gamma+\frac{W_{Nk}}{W_{N(k-1)}}\right)$$

$$+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

k-stage Delay

$$2\tau\left(\gamma+\frac{W_{N1}}{2}+\gamma+\frac{W_{N2}}{W_{N1}}+\gamma+\frac{W_{N3}}{W_{N2}}+\ldots+\gamma+\frac{W_{Ni}}{W_{N(i-1)}}+\gamma+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\gamma+\frac{W_{Nk}}{W_{N(k-1)}}\right)$$

$$+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}C_{load}+2\tau\gamma$$

k-stage Delay

$$2\tau\left(\gamma+\frac{W_{N1}}{2}+\gamma+\frac{W_{N2}}{W_{N1}}+\gamma+\frac{W_{N3}}{W_{N2}}+\ldots+\gamma+\frac{W_{Ni}}{W_{N(i-1)}}+\gamma+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\gamma+\frac{W_{Nk}}{W_{N(k-1)}}\right)$$

$$+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}C_{load}+2\tau\gamma$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau W_{Nk}}C_{load}+\gamma\right)$$

k-stage Delay

$$2\tau\left(\gamma+\frac{W_{N1}}{2}+\gamma+\frac{W_{N2}}{W_{N1}}+\gamma+\frac{W_{N3}}{W_{N2}}+\ldots+\gamma+\frac{W_{Ni}}{W_{N(i-1)}}+\gamma+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\gamma+\frac{W_{Nk}}{W_{N(k-1)}}\right)$$

$$+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}(C_{load}+2\gamma W_{Nk}C_0)$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}\right)+\frac{R_0}{W_{Nk}}C_{load}+2\tau\gamma$$

$$2\tau\left(\gamma k+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau W_{Nk}}C_{load}+\gamma\right)$$

$$2\tau\left(\gamma(k+1)+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau\cdot W_{Nk}}\cdot C_{load}\right)$$

Size W_{Ni} to minimize delay

- Take partial derivative with respect to $W_{Ni} = 0$

$$2\tau\left(\gamma(k+1)+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau\cdot W_{Nk}}\cdot C_{load}\right)$$

$$2\tau\left(0+0+0+\ldots+\frac{1}{W_{N(i-1)}}-\frac{W_{N(i+1)}}{\left(W_{Ni}\right)^2}+\ldots+0\right)+0=0$$

$$\frac{1}{W_{N(i-1)}}=\frac{W_{N(i+1)}}{W_{Ni}^2}\rightarrow\frac{W_{Ni}}{W_{N(i-1)}}=\frac{W_{N(i+1)}}{W_{Ni}}$$

Impact on Minimum W_{Ni} ?

- Partial derivative unchanged

$$2\tau\left(\gamma(k+1)+\frac{W_{N1}}{2}+\frac{W_{N2}}{W_{N1}}+\frac{W_{N3}}{W_{N2}}+\ldots+\frac{W_{Ni}}{W_{N(i-1)}}+\frac{W_{N(i+1)}}{W_{Ni}}+\ldots+\frac{W_{Nk}}{W_{N(k-1)}}+\frac{R_0}{2\tau\cdot W_{Nk}}\cdot C_{load}\right)$$

What does this say about ρ?

Stage Delay: ρ unchanged (for fixed k)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

Stage Delay: ρ unchanged (for fixed k)

$$\rho = \frac{W_{N1}}{2} = \frac{W_{Ni}}{W_{N(i-1)}} = \frac{W_{N(i+1)}}{W_{Ni}} = \frac{C_{load}}{W_{Nk}\left(2C_0\right)}$$

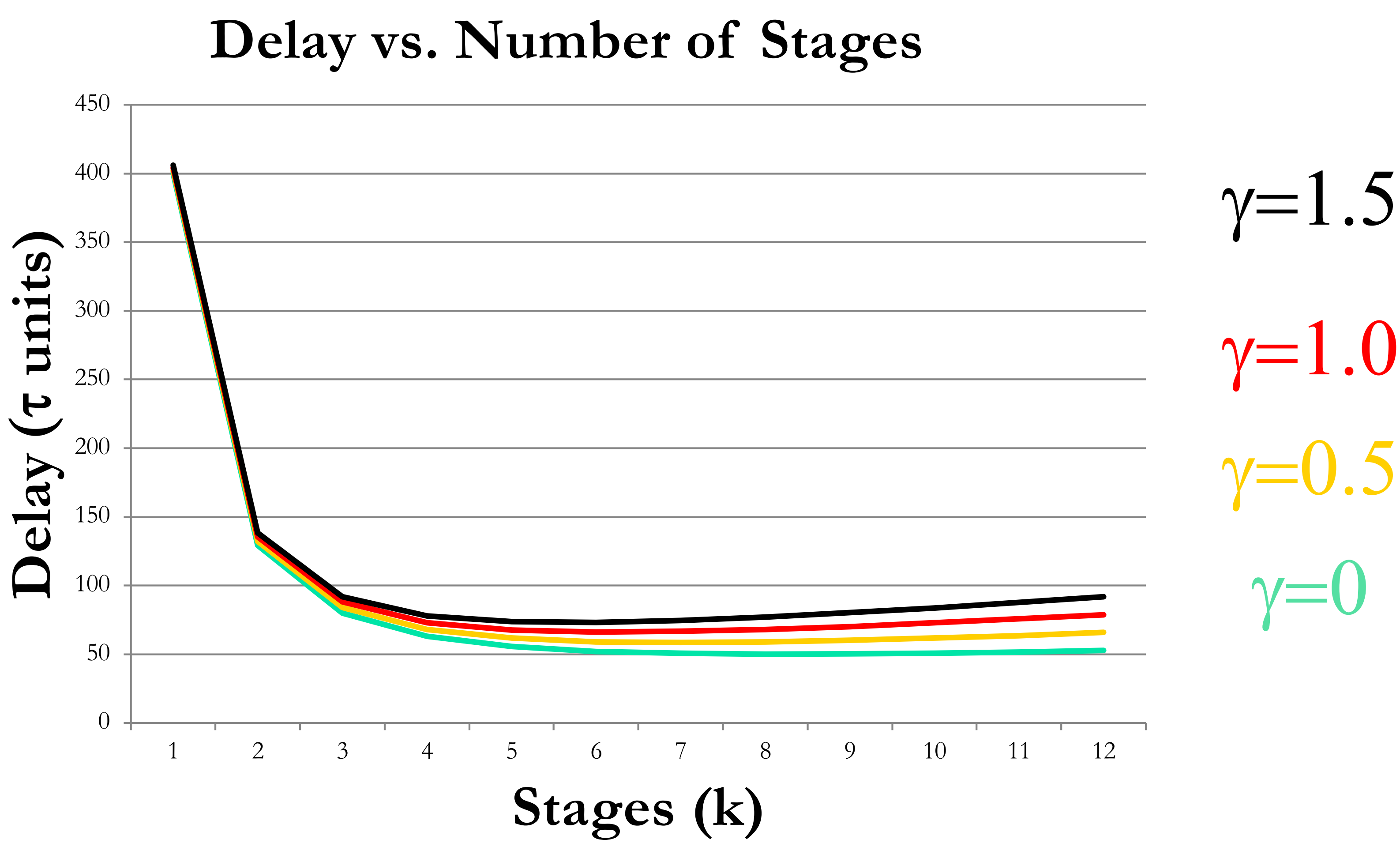
$$\rho = \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}$$

$$2\tau\left(\gamma(k+1) + \frac{W_{N1}}{2} + \frac{W_{N2}}{W_{N1}} + \frac{W_{N3}}{W_{N2}} + \ldots + \frac{W_{Ni}}{W_{N(i-1)}} + \frac{W_{N(i+1)}}{W_{Ni}} + \ldots + \frac{W_{Nk}}{W_{N(k-1)}} + \frac{R_0}{2\tau \cdot W_{Nk}} \cdot C_{load}\right)$$

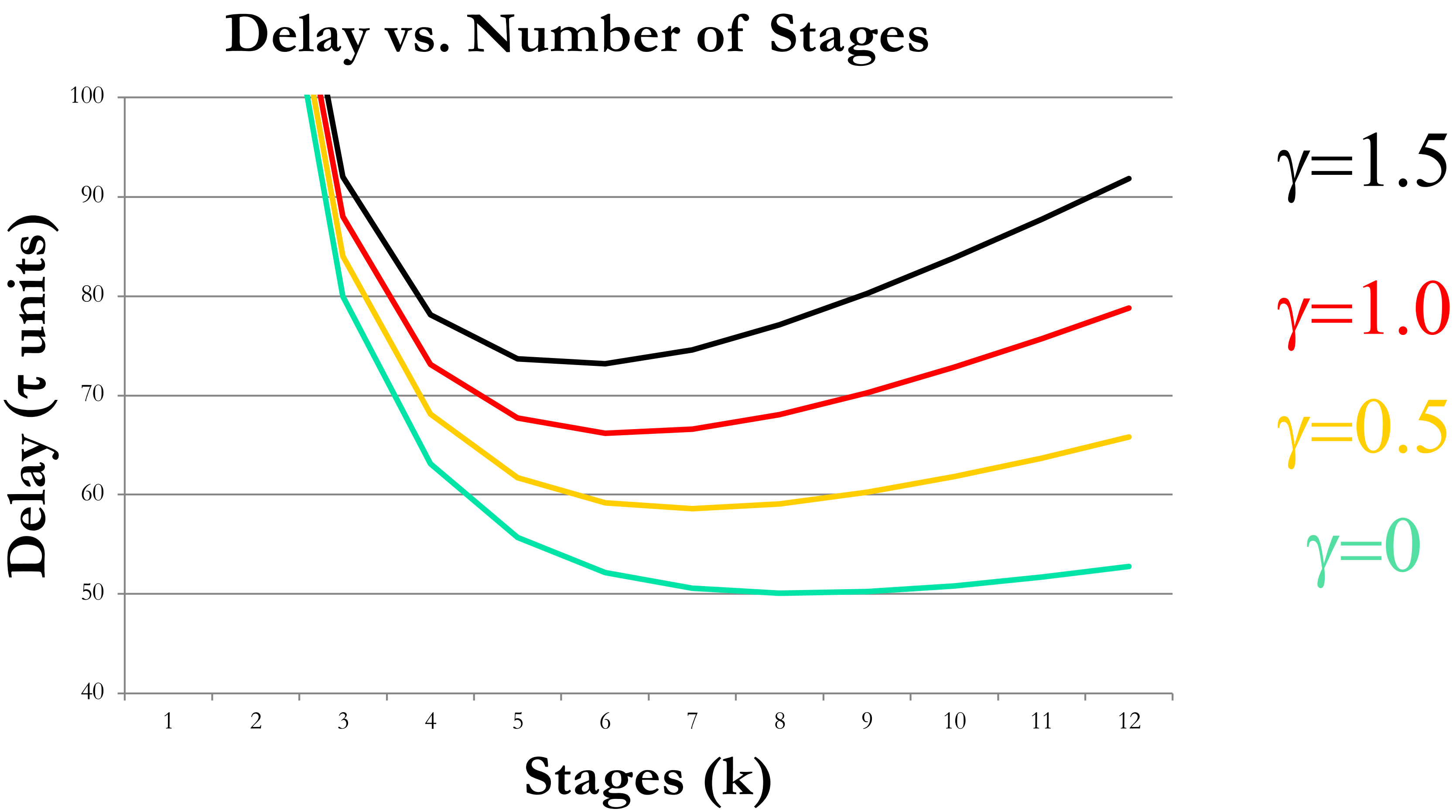
$$TotalDelay = 2\tau(k+1)(\rho+\gamma)$$

$$TotalDelay = 2\tau(k+1)\left(\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} + \gamma\right)$$

Impact of Gamma



Impact of Gamma



Minimize

$$TotalDelay = 2\tau(k+1)(\rho+\gamma)$$

$$TotalDelay = 2\tau(k+1)\left(\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} + \gamma\right)$$

$$0 = 2\tau\left[\gamma + \left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)} - (k+1)\cdot\ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{C_{load}}{4C_0}\right)^{\left(\frac{1}{k+1}\right)}\left(\frac{1}{k+1}\right)^2\right]$$

$$0 = \gamma + \rho - (k+1)\cdot\ln\left(\frac{C_{load}}{4C_0}\right)\rho\left(\frac{1}{k+1}\right)^2$$

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$

Solve

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)$$

Solve

$$\gamma + \rho = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)\rho$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\frac{C_{load}}{4C_0}\right)\left(\frac{1}{k+1}\right)$$

$$\frac{\gamma}{\rho} + 1 = \ln\left(\left(\frac{C_{load}}{4C_0}\right)^{\frac{1}{k+1}}\right)$$

$$\frac{\gamma}{\rho} + 1 = \ln(\rho)$$

Optimal Staging Any γ

$$\rho = e^{\left(\frac{\gamma}{\rho}+1\right)}$$

ρ and γ? (Preclass 4)

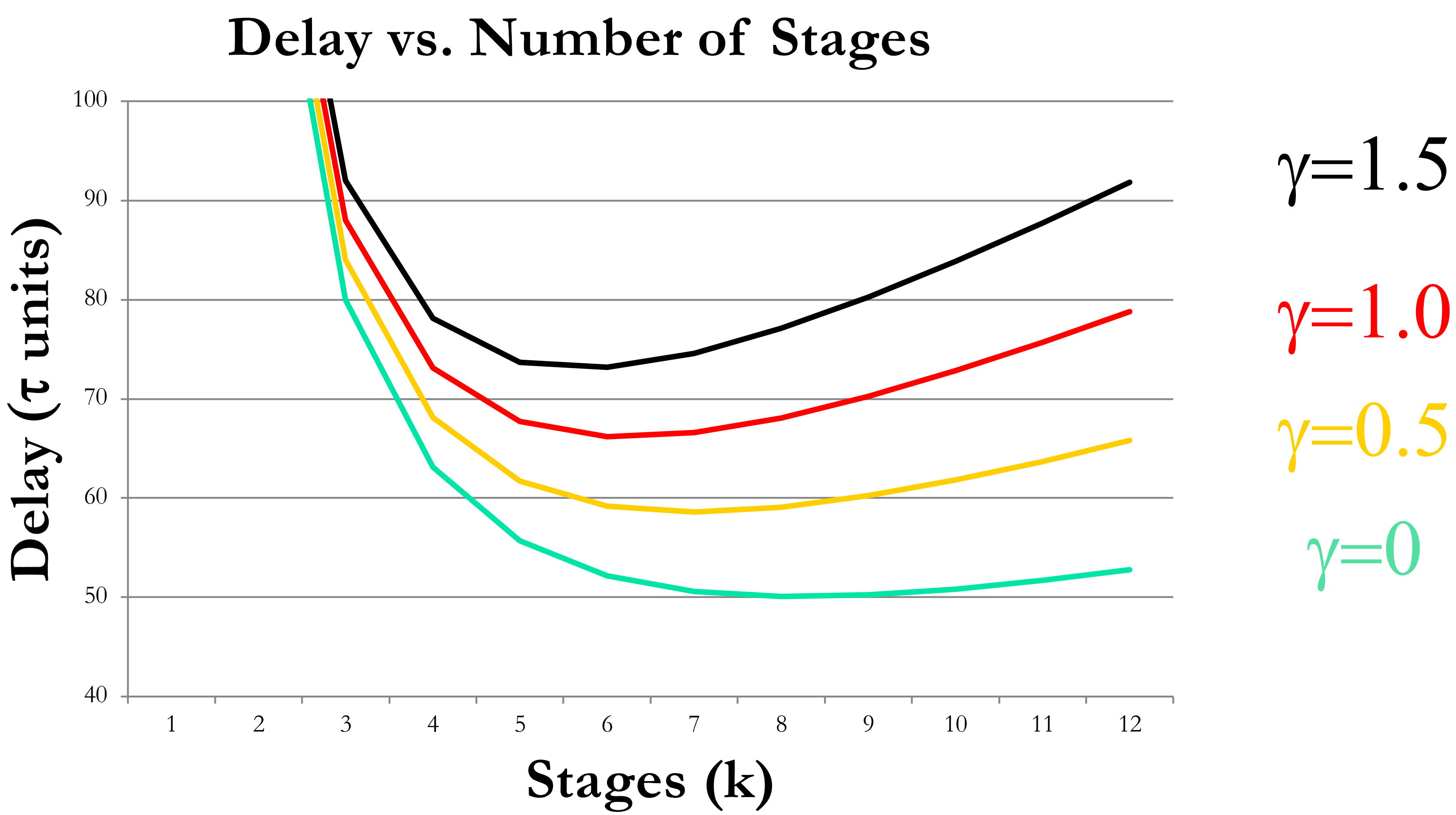
- ρ=4 is optimal for what γ?
- ρ=3 is optimal for what γ?

$$\rho = e^{\left(\frac{\gamma}{\rho}+1\right)}$$

$$\ln \rho = \frac{\gamma}{\rho} + 1$$

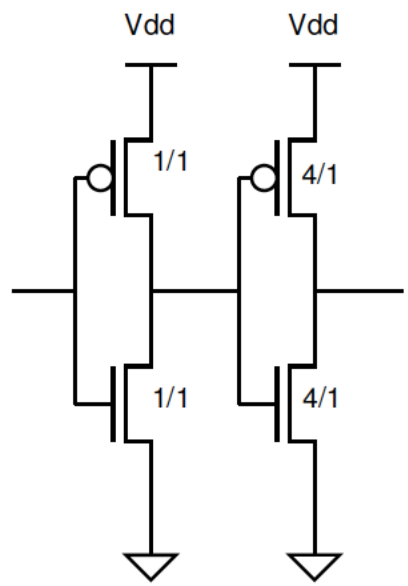
$$\rho \ln \rho - \rho = \gamma$$

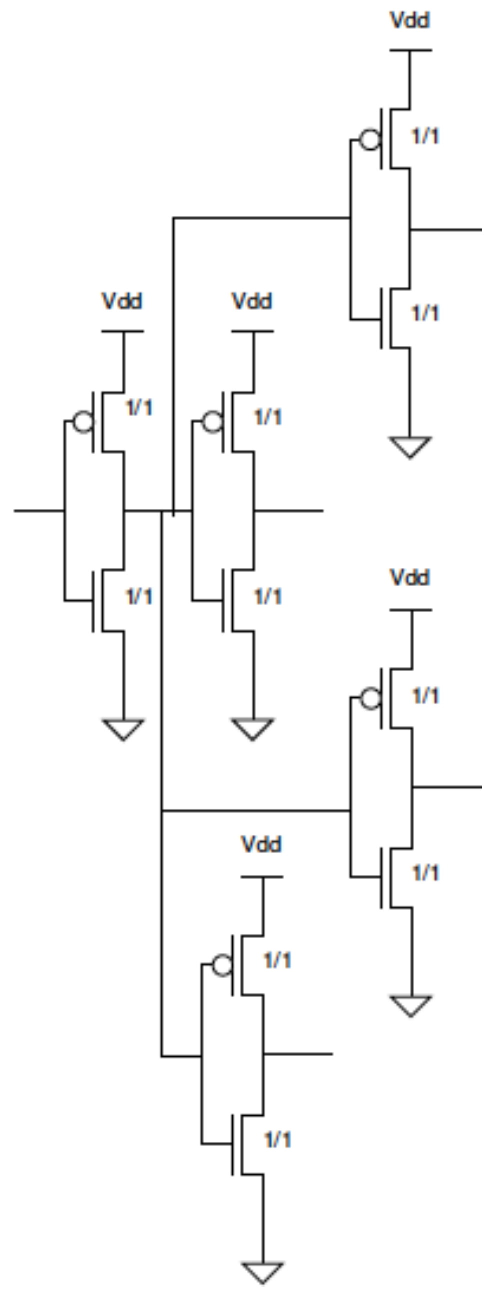
Impact of Gamma



Optimal Fanout

- Clearer why we use ρ=4 as our benchmark?





Idea

- To drive large loads
  - Scale buffers geometrically
  - Exponential scale up in buffer size ($\rho = e$)
- Scale factor: 3—4 typically
  - One origin of fanout 4 target
- Drains contribute capacitance too (C_{diff})
- Can formulate sizing to optimize

Admin

- Wednesday 11/3 Midterm 2 (next week)
  - 7-9pm DRLB 3C2
  - Lectures 1-22
  - Closed note, calculator allowed
  - All old exams online
    - focus on 2015-2019 ← taught in person by me
    - Study them!
  - Felicity review session
    - Monday 11/1 @ 3:30, link in Piazza
- HW 6 posted 11/3