Lecture Outline

- Discrete Time Signals
- Signal Properties
- Discrete Time Systems
Discrete Time Signals
Signals

- Signals carry information
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy
- Signal processing systems manipulate the information carried by signals
Signals are Functions

A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of $n$ produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an integer: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$
A Menagerie of Signals

- Google Share daily share price for 5 months

- Temperature at Houston Intercontinental Airport in 2013 (Celcius)

- Excerpt from Shakespeare’s *Hamlet*
Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable $n$ is discrete (integer).

- To plot a discrete-time signal in a program like Matlab, you should use the `stem` or similar command and not the `plot` command.

- Correct:

```
1

0

-1

-15 -10 -5 0 5 10 15

n

x[n]
```

- Incorrect:

```
1

0

-1

-15 -10 -5 0 5 10 15

n

x[n]
```
The **delta function** (aka unit impulse) \( \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \)

- The shifted delta function \( \delta[n - m] \) peaks up at \( n = m \); here \( m = 9 \)
The unit step $u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases}$

- The shifted unit step $u[n - m]$ jumps from 0 to 1 at $n = m$; here $m = 5$
Unit Pulse

The unit pulse (aka boxcar)

\[ p[n] = \begin{cases} 
0 & n < N_1 \\
1 & N_1 \leq n \leq N_2 \\
0 & n > N_2 
\end{cases} \]

Example: \( p[n] \) for \( N_1 = -5 \) and \( N_2 = 3 \)

One of many different formulas for the unit pulse

\[ p[n] = u[n - N_1] - u[n - (N_2 + 1)] \]
The real exponential \( r[n] = a^n, \ a \in \mathbb{R}, \ a \geq 0 \)

- For \( a > 1 \), \( r[n] \) shrinks to the left and grows to the right; here \( a = 1.1 \)

- For \( 0 < a < 1 \), \( r[n] \) grows to the left and shrinks to the right; here \( a = 0.9 \)
Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency**: $\omega$ (units: radians/sample)
- **Phase**: $\phi$ (units: radians)

- $\cos(\omega n)$

- $\sin(\omega n)$
Sinusoid Examples

- $\cos(0n)$
- $\sin(0n)$
- $\sin\left(\frac{\pi}{4}n + \frac{2\pi}{6}\right)$
- $\cos(\pi n)$
Sinusoid in Matlab

It's easy to play around in Matlab to get comfortable with the properties of sinusoids

N=36;
n=0:N-1;
omega=pi/6;
phi=pi/4;
x=cos(omega*n+phi);
stem(n,x)
The complex-valued sinusoid combines both the \( \cos \) and \( \sin \) terms (via Euler's identity):

\[
e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)
\]
Complex Sinusoid as Helix

\[ e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi) \]

A complex sinusoid is a **helix** in 3D space \((\text{Re}\{\}, \text{Im}\{\}, n)\)
- **Real part** (cos term) is the projection onto the \(\text{Re}\{\}\) axis
- **Imaginary part** (sin term) is the projection onto the \(\text{Im}\{\}\) axis

- **Frequency** \(\omega\) determines rotation speed and **direction** of helix
  - \(\omega > 0\) \(\Rightarrow\) anticlockwise rotation
  - \(\omega < 0\) \(\Rightarrow\) clockwise rotation

Animation: [https://upload.wikimedia.org/wikipedia/commons/4/41/Rising_circular.gif](https://upload.wikimedia.org/wikipedia/commons/4/41/Rising_circular.gif)
Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

  $$e^{j(-\omega)n} = e^{-j\omega n} = \cos(-\omega n) + j \sin(-\omega n) = \cos(\omega n) - j \sin(\omega n)$$

- Also note: $e^{j(-\omega)n} = e^{-j\omega n} = (e^{j\omega n})^*$

- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, sin term

  $$\text{Re}(e^{j\omega n}) = \cos(\omega n)$$

  $$\text{Im}(e^{j\omega n}) = \sin(\omega n)$$

  $$\text{Re}(e^{-j\omega n}) = \cos(\omega n)$$

  $$\text{Im}(e^{-j\omega n}) = -\sin(\omega n)$$
Phase of a Sinusoid

- $\phi$ is a (frequency independent) shift that is referenced to one period of oscillation

- $\cos\left(\frac{\pi}{6}n - 0\right)$

- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$

- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$

- $\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$
Complex Exponentials

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{j\phi}$

- Generalize to $e^{j\phi}$

- Consider the general complex number $z = |z|e^{j\omega}$, $z \in \mathbb{C}$
  - $|z| =$ magnitude of $z$
  - $\omega = \angle(z)$, phase angle of $z$
  - Can visualize $z \in \mathbb{C}$ as a point in the complex plane
Complex Exponentials

- Complex sinusoid \( e^{j(\omega n + \phi)} \) is of the form \( e^{\text{Purely Imaginary Numbers}} \)

- Generalize to \( e^{	ext{General Complex Numbers}} \)

- Consider the general complex number \( z = |z|e^{j\omega}, z \in \mathbb{C} \)
  - \(|z| = \text{magnitude of } z\)
  - \(\omega = \angle(z), \text{phase angle of } z\)
  - Can visualize \( z \in \mathbb{C} \) as a point in the complex plane

- Now we have
  \[ z^n = (|z|e^{j\omega})^n = |z|^n(e^{j\omega})^n = |z|^n e^{j\omega n} \]
  - \(|z|^n \) is a real exponential \((a^n \text{ with } a = |z|)\)
  - \(e^{j\omega n} \) is a complex sinusoid
Complex Exponentials

\[ z^n = (|z| e^{i\omega n})^n = |z|^n e^{i\omega n} \]

- \( |z|^n \) is a real exponential envelope \((a^n \text{ with } a = |z|)\)
- \( e^{i\omega n} \) is a complex sinusoid

\[ |z| < 1 \quad |z| > 1 \]

**Bounded**

**Unbounded**
Digital signals are a special sub-class of discrete-time signals

- Independent variable is still an integer: \( n \in \mathbb{Z} \)

- Dependent variable is from a finite set of integers: \( x[n] \in \{0, 1, \ldots, D - 1\} \)

- Typically, choose \( D = 2^q \) and represent each possible level of \( x[n] \) as a digital code with \( q \) bits

- Ex: Digital signal with \( q = 2 \) bits \( \Rightarrow D = 2^2 = 4 \) levels

- Ex: Compact discs use \( q = 16 \) bits \( \Rightarrow D = 2^{16} = 65536 \) levels
Signal Properties
Finite/Infinite Length Sequences

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$

- A **finite-length** discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$

- Important: a finite-length signal is **undefined** for $n < N_1$ and $n > N_2$
Windowing

- Converts a longer signal into a shorter one

\[ y[n] = \begin{cases} 
  x[n] & N_1 \leq n \leq N_2 \\
  0 & \text{otherwise}
\end{cases} \]
Zero Padding

- Converts a shorter signal into a longer one
- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$
- Given $N_0 \leq N_1 \leq N_2 \leq N_3$, $y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$
Periodic Signals

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$

**Notes:**
- The period $N$ must be an integer
- A periodic signal is infinite in length

A discrete-time signal is **aperiodic** if it is not periodic
Periodization

- Converts a finite-length signal into an infinite-length, periodic signal
- Given finite-length $x[n]$, replicate $x[n]$ periodically with period $N$

$$y[n] = \sum_{m=-\infty}^{\infty} x[n - mN], \quad n \in \mathbb{Z}$$

$$= \ldots + x[n + 2N] + x[n + N] + x[n] + x[n - N] + x[n - 2N] + \ldots$$
Causal Signals

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.

A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$.

A signal $x[n]$ is **acausal** if it is not causal.
Even Signals

**Definition**

A real signal \(x[n]\) is **even** if \(x[-n] = x[n]\)

- Even signals are symmetrical around the point \(n = 0\)
Odd Signals

**Definition**

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$.

- Odd signals are anti-symmetrical around the point $n = 0$. 
Signal Decomposition

- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
  - Even part: $e[n] = \frac{1}{2} (x[n] + x[-n])$  \hspace{1cm} (easy to verify that $e[n]$ is even)
  - Odd part: $o[n] = \frac{1}{2} (x[n] - x[-n])$  \hspace{1cm} (easy to verify that $o[n]$ is odd)

- **Decomposition** $x[n] = e[n] + o[n]$

- Verify the decomposition:
  
  $$
  e[n] + o[n] = \frac{1}{2} (x[n] + x[-n]) + \frac{1}{2} (x[n] - x[-n]) \\
  = \frac{1}{2} (x[n] + x[-n] + x[n] - x[-n]) \\
  = \frac{1}{2} (2x[n]) = x[n] \; \checkmark
  $$
Decomposition Example
Decomposition Example

\[
\frac{1}{2} \begin{pmatrix} x[n] \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x[-n] \\ 0 \end{pmatrix} = e[-n]
\]

\[
\frac{1}{2} \begin{pmatrix} x[n] \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x[-n] \\ 0 \end{pmatrix} = o[-n]
\]
Decomposition Example

\[ X[n] + X[-n] = e[-n] \]
\[ = o[-n] \]
Decomposition Example

\[\frac{1}{2} X[n] + \frac{1}{2} X[-n] = e[-n] \]

\[\frac{1}{2} X[n] - \frac{1}{2} X[-n] = o[-n] \]
Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(ωn+φ)}$ have two counterintuitive properties.

- Both involve the frequency $ω$.

- Weird property #1: Aliasing.

- Weird property #2: Aperiodicity.
Property #1: Aliasing of Sinusoids

Consider two sinusoids with two different frequencies

- $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
- $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$
Property #1: Aliasing of Sinusoids

Consider two sinusoids with two different frequencies

- $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
- $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi) n + \phi)}$

But note that

$$x_2[n] = e^{j((\omega + 2\pi) n + \phi)} = e^{j(\omega n + \phi) + j2\pi n} = e^{j(\omega n + \phi)} e^{j2\pi n} = e^{j(\omega n + \phi)} = x_1[n]$$

The signals $x_1$ and $x_2$ have different frequencies but are identical!

We say that $x_1$ and $x_2$ are aliases; this phenomenon is called aliasing

Note: Any integer multiple of $2\pi$ will do; try with $x_3[n] = e^{j((\omega + 2\pi m) n + \phi)}$, $m \in \mathbb{Z}$
Aliasing Example

\[ x_1[n] = \cos \left( \frac{\pi}{6} n \right) \]

\[ x_2[n] = \cos \left( \frac{13\pi}{6} n \right) = \cos \left( \left( \frac{\pi}{6} + 2\pi \right) n \right) \]
**Aliasing Example**

\[ x_1[n] = \cos \left( \frac{\pi}{6} n \right) \]
Alias-Free Frequencies

Since

\[ x_3[n] = e^{i(\omega + 2\pi m) n + \phi)} = e^{i(\omega n + \phi)} = x_1[n] \quad \forall m \in \mathbb{Z} \]

the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length \( 2\pi \)

Two intervals are typically used in the signal processing literature (and in this course)

- \( 0 \leq \omega < 2\pi \)
- \( -\pi < \omega \leq \pi \)
Which is higher in frequency?

- $\cos(\pi n)$ or $\cos\left(\frac{3\pi}{2n}\right)$?
Low and High Frequencies

- **Low frequencies:** $\omega$ close to 0 or $2\pi$ rad
  - Ex: $\cos\left(\frac{\pi}{10} n\right)$

- **High frequencies:** $\omega$ close to $\pi$ or $-\pi$ rad
  - Ex: $\cos\left(\frac{9\pi}{10} n\right)$
Increasing Frequency

\[ \omega_0 = 0 \]

\[ \omega_0 = \frac{\pi}{8} \]

\[ \omega_0 = \frac{\pi}{4} \]

\[ \omega_0 = \pi \]
Decreasing Frequency

\[ \omega_0 = 2\pi \]

\[ \omega_0 = \frac{15}{8}\pi \]

\[ \omega_0 = \frac{7}{4}\pi \]

\[ \omega_0 = \pi \]
Property #2: Periodicity of Sinusoids

Consider \( x_1[n] = e^{j(\omega_n + \phi)} \) with frequency \( \omega = \frac{2\pi k}{N} \), \( k, N \in \mathbb{Z} \) (harmonic frequency).

It is easy to show that \( x_1 \) is periodic with period \( N \), since

\[
x_1[n + N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega N + \phi)} e^{j(\omega n)} = e^{j(\omega + \phi)} e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark
\]
Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that $x_1$ is periodic with period $N$, since

$$x_1[n + N] = e^{j(\omega (n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} = e^{j(\omega n + \phi)} e^{j\left(\frac{2\pi k}{N} N\right)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos\left(\frac{2\pi^3}{16} n\right)$, $N = 16$

- Note: $x_1$ is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer
Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}, \ k, N \in \mathbb{Z}$ (not harmonic frequency)
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is $x_2$ periodic?

$$x_2[n + N] = e^{j(\omega(n + N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$
Aperiodicity of Sinusoids

- Consider \( x_2[n] = e^{j(\omega n + \phi)} \) with frequency \( \omega \neq \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z} \) (not harmonic frequency)

- Is \( x_2 \) periodic?

\[
x_2[n + N] = e^{j(\omega (n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}
\]

- Ex: \( x_2[n] = \cos(1.16 \ n) \)

- If its frequency \( \omega \) is not harmonic, then a sinusoid oscillates but is not periodic!
Harmonic Sinusoids

$$e^{j(\omega n + \phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with *harmonic frequencies*

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that

  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)
Periodic or not?

- $\cos(\frac{5}{7}\pi n)$

- $\cos(\frac{\pi}{5n})$

- What are $N$ and $k$? (I.e. How many samples is one period?)
Periodic or not?

- \( \cos\left(\frac{5}{7} \pi n\right) \)
  - \( N=14, k=5 \)
  - \( \cos(5/14*2 \pi n) \)
  - Repeats every \( N=14 \) samples

- \( \cos\left(\frac{\pi}{5}n\right) \)
  - \( N=10, k=1 \)
  - \( \cos(1/10*2 \pi n) \)
  - Repeats every \( N=10 \) samples
Periodic or not?

- \( \cos\left(\frac{5}{7}\pi n\right) \)
  - \( N=14, \ k=5 \)
  - \( \cos\left(\frac{5}{14}\pi \cdot 2\pi n\right) \)
  - Repeats every \( N=14 \) samples

- \( \cos\left(\frac{\pi}{5}n\right) \)
  - \( N=10, \ k=1 \)
  - \( \cos\left(\frac{1}{10}\pi \cdot 2\pi n\right) \)
  - Repeats every \( N=10 \) samples

- \( \cos\left(\frac{5}{7}\pi n\right)+\cos\left(\frac{\pi}{5}n\right) \) ?
Periodic or not?

- \( \cos\left(\frac{5}{7} \pi n\right) + \cos\left(\frac{\pi}{5n}\right) \) ?
  - \( N = \text{SCM}\{10, 14\} = 70 \)
  - \( \cos\left(\frac{5}{7} \pi n\right) + \cos\left(\frac{\pi}{5n}\right) \)
    - \( n = N = 70 \rightarrow \cos\left(\frac{5}{7} \times 70 \pi\right) + \cos\left(\frac{\pi}{5} \times 70\right) = \cos\left(25 \times 2 \pi\right) + \cos\left(7 \times 2 \pi\right) \)
Discrete-Time Systems
Discrete Time Systems

A discrete-time system $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal $x$ into a discrete-time output signal $y$

\[ y = \mathcal{H}\{x\} \]

- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price
Recall that there are two kinds of signals: infinite-length and finite-length.

Accordingly, we will consider two kinds of systems:

1. Systems that transform an infinite-length-signal \( x \) into an infinite-length signal \( y \)
2. Systems that transform a length-\( N \) signal \( x \) into a length-\( N \) signal \( y \)
   (Such systems can also be used to process periodic signals with period \( N \))

For generality, we will assume that the input and output signals are complex valued.
System Examples

- Identity
  \[ y[n] = x[n] \quad \forall n \]

- Scaling
  \[ y[n] = 2x[n] \quad \forall n \]

- Offset
  \[ y[n] = x[n] + 2 \quad \forall n \]

- Square signal
  \[ y[n] = (x[n])^2 \quad \forall n \]

- Shift
  \[ y[n] = x[n + 2] \quad \forall n \]

- Decimate
  \[ y[n] = x[2n] \quad \forall n \]

- Square time
  \[ y[n] = x[n^2] \quad \forall n \]
System Examples

- **Shift system** ($m \in \mathbb{Z}$ fixed)
  \[ y[n] = x[n - m] \quad \forall n \]

- **Moving average** (combines shift, sum, scale)
  \[ y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n \]

- **Recursive average**
  \[ y[n] = x[n] + \alpha y[n - 1] \quad \forall n \]
System Properties

- Memoryless
- Linearity
- Time Invariance
- Causality
- BIBO Stability
Memoryless

- $y[n]$ depends only on $x[n]$

- Examples:
  - Ideal delay system (or shift system):
    - $y[n] = x[n-m]$ memoryless?
  - Square system:
    - $y[n] = (x[n])^2$ memoryless?
A system $\mathcal{H}$ is (zero-state) **linear** if it satisfies the following two properties:

1. **Scaling**
   \[ \mathcal{H}\{\alpha x\} = \alpha \mathcal{H}\{x\} \quad \forall \alpha \in \mathbb{C} \]

   ![Diagram](image)

2. **Additivity**
   If $y_1 = \mathcal{H}\{x_1\}$ and $y_2 = \mathcal{H}\{x_2\}$ then
   \[ \mathcal{H}\{x_1 + x_2\} = y_1 + y_2 \]

   ![Diagram](image)
Proving Linearity

- A system that is not linear is called **nonlinear**

- To prove that a system is linear, you must prove rigorously that it has both the scaling and additivity properties for arbitrary input signals

- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**
Linearity Example: Moving Average

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2} (x[n] + x[n-1]) \]

- **Scaling**: (Strategy to prove – Scale input \( x \) by \( \alpha \in \mathbb{C} \), compute output \( y \) via the formula at top, and verify that it is scaled as well)
  - Let
    \[ x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C} \]
  - Let \( y' \) denote the output when \( x' \) is input (that is, \( y' = \mathcal{H}(x') \))
  - Then
    \[ y'[n] = \frac{1}{2} (x'[n] + x'[n-1]) \]
Linearity Example: Moving Average

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2} (x[n] + x[n - 1]) \]

- **Scaling:** (Strategy to prove – Scale input \( x \) by \( \alpha \in \mathbb{C} \), compute output \( y \) via the formula at top, and verify that it is scaled as well)
  
  - Let
    \[ x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C} \]
  
  - Let \( y' \) denote the output when \( x' \) is input (that is, \( y' = \mathcal{H}\{x'\} \))
  
  - Then
    \[
    y'[n] = \frac{1}{2} (x'[n] + x'[n - 1]) = \frac{1}{2} (\alpha x[n] + \alpha x[n - 1]) = \alpha \left( \frac{1}{2} (x[n] + x[n - 1]) \right) = \alpha y[n] \]
Linearity Example: Moving Average

\[
x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1])
\]

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
    \[
x'[n] = x_1[n] + x_2[n]
    \]
  - Let \(y'/y_1/y_2\) denote the output when \(x'/x_1/x_2\) is input
Linearity Example: Moving Average

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1]) \]

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
    \[ x'[n] = x_1[n] + x_2[n] \]
  - Let \( y'/y_1/y_2 \) denote the output when \( x'/x_1/x_2 \) is input
  - Then
    \[ y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}((x_1[n] + x_2[n]) + (x_1[n-1] + x_2[n-1])) \]
Linearity Example: Moving Average

\[ x[n] \rightarrow \mathcal{H} \rightarrow y[n] = \frac{1}{2}(x[n] + x[n - 1]) \]

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
    \[ x'[n] = x_1[n] + x_2[n] \]
  - Let \( y'/y_1/y_2 \) denote the output when \( x'/x_1/x_2 \) is input
  - Then
    \[
    y'[n] = \frac{1}{2}(x'[n] + x'[n - 1]) = \frac{1}{2}((x_1[n] + x_2[n]) + (x_1[n - 1] + x_2[n - 1]))
    \]
    \[
    = \frac{1}{2}(x_1[n] + x_1[n - 1]) + \frac{1}{2}(x_2[n] + x_2[n - 1]) = y_1[n] + y_2[n] \quad \checkmark
    \]
Example: Squaring is Nonlinear

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = (x[n])^2 \]

- **Additivity**: Input two signals into the system and see what happens
  
  - Let
    \[ y_1[n] = (x_1[n])^2, \quad y_2[n] = (x_2[n])^2 \]
  
  - Set
    \[ x'[n] = x_1[n] + x_2[n] \]
  
  - Then
    \[ y'[n] = (x'[n])^2 = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \neq y_1[n] + y_2[n] \]
  
  - Nonlinear!
Time-Invariant Systems

A system \( \mathcal{H} \) processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal.

\[
\begin{align*}
x[n] &\rightarrow \mathcal{H} \rightarrow y[n] \\
x[n - q] &\rightarrow \mathcal{H} \rightarrow y[n - q]
\end{align*}
\]

- Intuition: A time-invariant system behaves the same no matter when the input is applied.
- A system that is not time-invariant is called **time-varying**.
Example: Moving Average

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2} (x[n] + x[n-1]) \]

- Let
  \[ x'[n] = x[n-q], \quad q \in \mathbb{Z} \]

- Let \( y' \) denote the output when \( x' \) is input (that is, \( y' = \mathcal{H}\{x'\} \))

- Then
  \[ y'[n] = \frac{1}{2} (x'[n] + x'[n-1]) = \frac{1}{2} (x[n-q] + x[n-q-1]) = y[n-q] \]
Example: Decimation

\[ x[n] \xrightarrow{\mathcal{H}} y[n] = x[2n] \]

- This system is time-varying; demonstrate with a counter-example

- Let

\[ x'[n] = x[n - 1] \]

- Let \( y' \) denote the output when \( x' \) is input (that is, \( y' = \mathcal{H}\{x'\} \))

- Then

\[ y'[n] = x'[2n] = x[2n - 1] \neq x[2(n - 1)] = y[n - 1] \]
Causal Systems

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time $n$ depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future.

- **Forward difference system:**
  - $y[n] = x[n+1] - x[n]$ causal?

- **Backward difference system:**
  - $y[n] = x[n] - x[n-1]$ causal?
Stability

- BIBO Stability
  - Bounded-input bounded-output Stability

**Definition**

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input $x$ always produces a bounded output $y$.

Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all $n$.
System Properties - Summary

- Causality
  - $y[n]$ only depends on $x[m]$ for $m \leq n$

- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $A x_1[n] + B x_2[n] \rightarrow A y_1[n] + B y_2[n]$

- Memoryless
  - $y[n]$ depends only on $x[n]$

- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$

- BIBO Stability
  - A bounded input results in a bounded output (i.e., max signal value exists for output if max)
Examples

- Time Shift:
  - \( y[n] = x[n - m] \)
- Accumulator:
  - \( y[n] = \sum_{k=\infty}^{n} x[k] \)
- Compressor (\( M > 1 \)):  
  - \( y[n] = x[Mn] \)
Big Ideas

■ Discrete Time Signals
  ■ Unit impulse, unit step, exponential, sinusoids, complex sinusoids
  ■ Can be finite length, infinite length
  ■ Properties
    ■ Even, odd, causal
    ■ Periodicity and aliasing
      ■ Discrete frequency bounded!

■ Discrete Time Systems
  ■ Transform one signal to another
  ■ Properties
    ■ Linear, Time-invariance, memoryless, causality, BIBO stability
Admin

- Enroll in Piazza site:
  - piazza.com/upenn/spring2018/ese531
- HW 1 out after class