ESE 531: Digital Signal Processing

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Discrete Time Signals and Systems

Discrete Time Signals

Signals

Signals carry information
Examples:
- Speech signals transmit language via acoustic waves
- Radar signals transmit the position and velocity of targets via electromagnetic waves
- Electrophysiology signals transmit information about processes inside the body
- Financial signals transmit information about events in the economy

Signal processing systems manipulate the information carried by signals

Signals are Functions

A signal is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of $n$ produces the value $x[n]$
- In this course we will focus on discrete-time signals:
  - Independent variable is an integer: $n \in \mathbb{Z}$ (will refer to it as time $n$)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$

A Menagerie of Signals

- Google Share daily share price for 5 months
- Temperature at Houston International Airport in 2013
- Excerpt from a reading of Shakespeare’s Hamlet
Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable $n$ is discrete.
- To plot a discrete-time signal in a program like Matlab, you should use the `stem` or similar command and not the `plot` command.

Correct:

Incorrect:

Unit Sample

- The delta function $\delta[n]$ peaks up at $n = m$; here $m = 9$.

Unit Step

- The unit step $u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0 
\end{cases}$

- The shifted unit step $u[n-m]$ ramps from 0 to 1 at $n = m$; here $m = 9$.

Unit Pulse

- The unit pulse $p[n] = \begin{cases} 
1 & n = N_2 \\
0 & N_1 \leq n \leq N_2 \\
1 & n > N_2 
\end{cases}$

- Ex: $p[0]$ for $N_1 = 5$ and $N_2 = 3$.

- One of many different formulas for the unit pulse:
  
  $p[n] = u(n=N_1) - u(n=N_2)$

Real Exponential

- The real exponential $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$.

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$.

- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$.

Sinusoids

- There are two natural real-value sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$.

- Frequency: $\omega$ (units: radians/sample).

- Phase: $\phi$ (units: radians).
**Sinusoid Examples**

- $\cos(0n)$
- $\sin(0n)$
- $\sin\left(\frac{\pi}{4}n\right)$
- $\cos(\pi n)$

**Sinusoid in Matlab**

- It’s easy to play around in Matlab to get comfortable with the properties of sinusoids

```matlab
% Sinusoid in Matlab
n = 0:10; % Use a finer step for visualization
omega = pi/4; % Frequency
phi = pi/4; % Phase
x = cos(omega*n + phi); % Sinusoid
plot(n, x)
```

**Complex Sinusoid**

- The complex-value sinusoid combines both the cos and sin terms using Euler’s identity:

\[ e^{j(\omega t + \phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi) \]

**Complex Sinusoid as Helix**

- A complex sinusoid is a helix in 3D space $(\Re[\{\}], \Im[\{\}], \phi)$
- Real part (cos term) is the projection onto the $\Re[\{\}]$ axis
- Imaginary part (sin term) is the projection onto the $\Im[\{\}]$ axis
- Frequency determines rotation speed and direction of helix
- $\omega \in \Re$ = anti-clockwise rotation
- $\omega < 0$ = clockwise rotation

**Negative Frequency?**

- Negative frequency is nothing to be afraid of!

**Negative Frequency**

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency:

\[ e^{-j\omega t} = e^{j(-\omega t)} = \cos(-\omega t) + j\sin(-\omega t) = \cos(\omega t) - j\sin(\omega t) \]

- Also note: $e^{-j\phi} = e^{j(-\phi)} = (e^{j\phi})^*$

- Takeaway: negating the frequency is equivalent to complex conjugating a complex sinusoid—flips the sign of the imaginary sin term.
Phase of a Sinusoid

- $\phi$ is a (frequency independent) shift that is referenced to one period of oscillation

\[
\begin{align*}
\cos (\phi) & = \cos(\pi) \\
\cos (\phi - \pi) & = \cos(0) = 1 \\
\cos (\phi - 2\pi) & = \cos(\pi) = -1 \\
\end{align*}
\]

Complex Exponentials

- Complex sinusoid $e^{j(\omega t + \phi)}$ is of the form $e^{j\omega t}$, $\phi$ is frequency independent
- Generalize to $e^{j\alpha}$ where $\alpha$ is a constant

Consider the general complex number $z = |z| e^{j\phi}$, $z \in \mathbb{C}$

- $|z|$ = magnitude of $z$
- $\phi$ = phase angle of $z$
- Can visualize $z \in \mathbb{C}$ as a point in the complex plane

Now we have

$z^n = (|z|e^{j\phi})^n = |z|^n e^{j\alpha n}$

- $|z|$ is a real exponential of $|z|$ with $n = |z|$ (where $n$ is an integer)
- $e^{j\alpha n}$ is a complex sinusoid

Digital Signals

- Digital signals are a special subclass of discrete-time signals
  - Independent variable is still an integer: $n \in \mathbb{Z}$
  - Dependent variable is from a finite set of integers: $x[n] \in \{0, 1, \ldots, D - 1\}$
  - Typically, choose $D = 2^d$ and represent each possible level of $x[n]$ as a digital code with $d$ bits
  - Ex. Digital signal with $d = 2$ bits: $D = 2^2 = 4$ levels
  - Ex. Compact discs use $d = 16$ bits: $D = 2^{16} = 65536$ levels
Signal Properties

Finite/Infinite Length Sequences

- An infinite-length discrete-time signal $x[n]$ is defined for $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$.

- A finite-length discrete-time signal $x[n]$ is defined only for a finite range of $N_L \leq n \leq N_R$.

- Important: a finite-length signal is undefined for $n < N_L$ and $n > N_R$.

Windowing

- Converts a longer signal into a shorter one.

- $y[n] = \begin{cases} x[n] & N_L \leq n \leq N_R \\ 0 & \text{otherwise} \end{cases}$

Zero Padding

- Converts a shorter signal into a longer one.

- $x[n]$ is defined for $N_L \leq n \leq N_R$.

- Given $N_L \leq N_L' \leq N_R' \leq N_R$,

  $y[n] = \begin{cases} x[n] & N_L \leq n \leq N_R \\ 0 & \text{otherwise} \end{cases}$

Periodic Signals

- A discrete-time signal is periodic if it repeats with period $N \in \mathbb{Z}$:

  $x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$

Notes:
- The period $N$ must be an integer.
- A periodic signal is finite in length.
- A discrete-time signal is aperiodic if it is not periodic.

Periodization

- Converts a finite-length signal into an infinite-length, periodic signal.

- Given finite-length $x[n]$, replicate $x[n]$ periodically with period $N$:

  $y[n] = \sum_{m=-\infty}^{\infty} x[n - mN] \quad \forall n \in \mathbb{Z}$

  $= \ldots + x[n + 2N] + x[n + N] + x[n] + x[n - N] + x[n - 2N] + \ldots$
Causal Signals

- A signal $x[n]$ is causal if $x[n] = 0$ for all $n < 0$.
- A signal $x[n]$ is anti-causal if $x[n] = 0$ for all $n \geq 0$.
- A signal $x[n]$ is non-causal if it is not causal.

Even Signals

- A real signal $x[n]$ is even if $x[-n] = x[n]$.
- Even signals are symmetrical around the point $n = 0$.

Odd Signals

- A real signal $x[n]$ is odd if $x[-n] = -x[n]$.
- Odd signals are anti-symmetrical around the point $n = 0$.

Signal Decomposition

- Useful fact: Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part.
- Even part: $x_e[n] = \frac{1}{2} [x[n] + x[-n]]$ (easy to verify that $x_e[n]$ is even).
- Odd part: $x_o[n] = \frac{1}{2} [x[n] - x[-n]]$ (easy to verify that $x_o[n]$ is odd).
- Decomposition: $x[n] = x_e[n] + x_o[n]$. 
- Verify the decomposition: 
  $x[n] + x[n] = \frac{1}{2} [x[n] + x[-n] + x[n] - x[-n]] = \frac{1}{2} [x[n] + x[-n] + x[n] - x[-n]]$ where $x[n] + x[n] = x[n]$. 

Decomposition Example

- $X[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$ with $x[k] = \delta[k]$.
Decomposition Example

\[ X[n] \]

\[ X[-n] \]

Discrete-Time Sinusoids

- Discrete-time sinusoids \( e^{j(n+\omega)} \) have two counterintuitive properties
  - Both involve the frequency \( \omega \)
  - Weird property #1: Aliasing
  - Weird property #2: Aperiodicity

Property #1: Aliasing of Sinusoids

Consider two sinusoids with two different frequencies

\[ \omega \quad \Rightarrow \quad x_1[n] = e^{j(n+\omega)} \]
\[ \omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j(n+2\pi)} \]

But note that

\[ x_1[n] = e^{j(n+\omega+2\pi)} = e^{j(n+\omega)} = x_2[n] \]

The signals \( x_1 \) and \( x_2 \) have different frequencies but are identical!

We say that \( x_1 \) and \( x_2 \) are aliased; this phenomenon is called aliasing

Note: Any integer multiple of \( 2\pi \) will do; try with \( x_3[n] = e^{j(3\pi(n+2\pi))} = x_1[n] \), \( m \in \mathbb{Z} \)
**Aliasing Example**

- $x_1[n] = \cos \left( \frac{\pi}{2} n \right)$

- $x_2[n] = \cos \left( \frac{3\pi}{2} n \right) = \cos \left( \frac{\pi}{2} + 2\pi n \right)$

**Alias-Free Frequencies**

- Since $x_2[n] = e^{i\omega_0 n} e^{i\omega n} = e^{i\omega n}$, $\omega_0 \in \mathbb{R}$

  the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length $2\pi$

- Two intervals are typically used in the signal processing literature (and in this course)
  - $0 \leq \omega < 2\pi$
  - $-\pi < \omega \leq \pi$

**Which is higher in frequency?**

- $\cos(\pi n)$ or $\cos\left(\frac{3\pi}{2n}\right)$

**Low and High Frequencies**

- **Low frequencies**: $\omega$ close to 0 or $2\pi$ rad
  - Ex: $\cos \left( \frac{\pi}{2} n \right)$

- **High frequencies**: $\omega$ close to $\pi$ or $-\pi$ rad
  - Ex: $\cos \left( \frac{3\pi}{2} n \right)$

**Increasing Frequency**

- $\omega_0 = 0$

- $\omega_0 = \frac{\pi}{8}$
Property #2: Periodicity of Sinusoids

- Consider $x[n] = e^{j\omega n}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that $x[n]$ is periodic with period $N$, since
  
  $x[n+N] = e^{j\omega(n+N)} = e^{j\omega n}e^{j\omega N} = e^{j2\pi}x[n] = x[n]$.

- Ex: $x[n] = \cos(\frac{\pi}{2}n)$, $N = 16$

- Note: $x[n]$ is periodic with the (smaller) period of $\frac{N}{d}$ when $d$ is an integer

Aperiodicity of Sinusoids

- Consider $y[n] = e^{j\omega n}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is $y[n]$ periodic?
  
  $y[n+N] = e^{j\omega(n+N)} = e^{j\omega n}e^{j\omega N} = e^{j\omega n}e^{j\omega N} \neq e^{j\omega n} \neq y[n]$. NO!

- Ex: $y[n] = \cos(1.38n)$
Harmonic Sinusoids

- Semi-analyzing fact: The only periodic discrete-time sinusoids are those with harmonic frequencies $\omega = \frac{2\pi k}{N}, k, N \in \mathbb{Z}$.
- Which means that:
  - Most discrete-time sinusoids are not periodic.
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT).

Periodic or not?

- $\cos\left(\frac{5}{7} \pi n\right)$
  - $N=14, k=5$
  - $\cos\left(\frac{5\pi}{14}\pi n\right)$
  - Repeats every $N=14$ samples
- $\cos\left(\frac{\pi}{5} n\right)$
  - $N=10, k=1$
  - $\cos\left(\frac{\pi}{10}\pi n\right)$
  - Repeats every $N=10$ samples

What are $N$ and $k$? (i.e. How many samples is one period?)

- $\cos\left(\frac{5}{7} \pi n\right) + \cos\left(\frac{\pi}{5} n\right)$
  - $N=\text{SCM}(10,14)=70$
  - $\cos\left(\frac{5\pi}{7}\pi n\right) + \cos\left(\frac{\pi}{5}\pi n\right)$
  - $n=N=70 \Rightarrow \cos\left(\frac{5\pi}{7}\pi n\right) + \cos\left(\frac{\pi}{5}\pi n\right) = \cos\left(\frac{25\pi}{7}\pi n\right) + \cos\left(\frac{7\pi}{2}\pi n\right)$

Discrete-Time Systems
Discrete Time Systems

A discrete-time system $Y$ is a transformation (a rule or formula) that maps a discrete-time input signal $x$ into a discrete-time output signal $y$.

- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A scale system transforms the received scale value to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of brain activity into voxel-by-voxel estimates of brain activity
  - A 30-day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems

- Recall that there are two kinds of signals: infinite-length and finite-length
- Accordingly, we will consider two kinds of systems:
  - Systems that transform an infinite-length signal $x$ into an infinite-length signal $y$
  - Systems that transform a length-$N$ signal $x$ into a length-$N$ signal $y$
    - Such systems can also be used to process periodic signals with period $N$.
- For generality, we will assume that the input and output signals are complex valued.

System Examples

- **Identity**
  \[ y[n] = x[n] \quad \forall n \]
- **Scaling**
  \[ y[n] = 2x[n] \quad \forall n \]
- **Offset**
  \[ y[n] = x[n] + 2 \quad \forall n \]
- **Square signal**
  \[ y[n] = (x[n])^2 \quad \forall n \]
- **Shift**
  \[ y[n] = x[n+2] \quad \forall n \]
- **Decimate**
  \[ y[n] = x[2n] \quad \forall n \]
- **Square time**
  \[ y[n] = x[n^2] \quad \forall n \]

System Properties

- **Memoryless**
- **Linearity**
- **Time Invariance**
- **Causality**
- **BIBO Stability**

Memoryless

- $y[n]$ depends only on $x[n]$
- Examples:
  - Ideal delay system (or shift system):
    - $y[n]=x[n-m]$ **memoryless**?
  - Square system:
    - $y[n]=(x[n])^2$ **memoryless**?
Linear Systems

A system \( H \) is (zero-state) linear if it satisfies the following two properties:

- **Scalability**: If \( y(n) = H(x(n)) \), then \( y(\alpha n) = \alpha y(n) \) for all \( \alpha \in \mathbb{R} \).

- **Additivity**: If \( y_1(n) = H(x_1(n)) \) and \( y_2(n) = H(x_2(n)) \), then \( y_1(n) + y_2(n) = H(x_1(n) + x_2(n)) \).

Proving Linearity

- A system that is not linear is called nonlinear.
- To prove that a system is linear, you must prove rigorously that it has both the scaling and additivity properties for arbitrary input signals.
- To prove that a system is nonlinear, it is sufficient to exhibit a counterexample.

Linearity Example: Moving Average

\[ x[n] \rightarrow R \rightarrow y[n] = (x[n] + x[n-1]) \]

- **Scaling**: (Strategy to prove: Scale input \( x \) by \( \alpha \in \mathbb{C} \), compute output \( y \) via the formula at top, and verify that it is scaled as well)
  - Let \( x' = \alpha x \), \( \alpha \in \mathbb{C} \)
  - Let \( y' \) denote the output when \( x' \) is input (that is, \( y' = H(x') \))
  - Then:
    \[ y'(n) = \frac{1}{2}(x'_n + x'_n-1) = \alpha y(n) \]

Linearity Example: Moving Average

- **Additivity**: (Strategy to prove: Input two signals \( x_1, x_2 \) into the system and verify that the output equals the sum of the respective outputs)
  - Let \( x_1[n] \) and \( x_2[n] \) denote the output when \( x_1[n] \) and \( x_2[n] \) are input into the system.
  - Then:
    \[ y[n] = \frac{1}{2}(x_1[n] + x_2[n] + x_1[n-1] + x_2[n-1]) = \frac{1}{2}(x_1[n] + x_2[n] + x_1[n-1] + x_2[n-1]) \]
Example: Squaring is Nonlinear

\[ z[n] = x[n] \rightarrow y[n] = (x[n])^2 \]

- **Additivity**: Input two signals into the system and see what happens
  - Let \( x_1[n] = (n)(n)_1 \), \( x_2[n] = (n)(n)_2 \)
  - Set \( x[n] = x_1[n] + x_2[n] \)

Time-Invariant Systems

A system \( \mathcal{H} \) processing infinite-length signals is time-invariant (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal:

\[ \begin{align*}
  x[n] &\rightarrow y[n] \\
  x[n-q] &\rightarrow y[n-q]
\end{align*} \]

- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called time-varying

Example: Moving Average

\[ z[n] = x[n] \rightarrow y[n] = \frac{1}{2}(x[n] + x[n-1]) \]

- Let \( x'[n] = x[n] \), \( y' \in \mathbb{Z} \)
- Let \( y'[n] \) denote the output when \( x' \) is input (that is, \( y' = \mathcal{H}(x') \))
- Then \( y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) \)
Example: Decimation

\[ x[n] \rightarrow \mathcal{H} \rightarrow y[n] = x[2n] \]

- This system is time-varying; demonstrate with a counter-example
- Let \( y'[n] = x[n-1] \)
- Let \( y \) denote the output when \( x' \) is input (that is, \( y = y'(x') \))
- Then \( y[n] = x'[2n] = x[2(n-1)] \neq x[2(n-1)] = y[n-1] \)

Causal Systems

A system \( \mathcal{H} \) is causal if the output \( y[n] \) at time \( n \) depends only the input \( x[n] \) for times \( n \leq n \). In words, causal systems do not look into the future.

- Forward difference system:
  \( y[n] = x[n+1] - x[n] \) causal

- Backward difference system:
  \( y[n] = x[n] - x[n-1] \) causal

Stability

- BIBO Stability
  - Bounded-input bounded-output Stability

An LTI system is bounded-input bounded-output (BIBO) stable if a bounded input \( x \) always produces a bounded output \( y \).

System Properties - Summary

- Causality
  - \( y[n] \) only depends on \( x[m] \) for \( m \leq n \)
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    \[ A_1 x_1[n] + A_2 x_2[n] \rightarrow A_1 y_1[n] + A_2 y_2[n] \]
- Memoryless
  - \( y[n] \) depends only on \( x[n] \)
- Time Invariance
  - Shifted input results in shifted output
    \[ x[n-q] \rightarrow y[n-q] \]
- BIBO Stability
  - A bounded input results in a bounded output (i.e. max signal value exists for output if max)

Examples

  - BIBO Stable?
- Time Shift:
  \[ y[n] = x[n-m] \]
- Accumulator:
  \[ y[n] = \sum_{k=-\infty}^{n} x[k] \]
- Compressor (M>1):
  \[ y[n] = x[Mn] \]

Big Ideas

- Discrete Time Signals
  - Unit impulse, unit step, exponential, sinusoids, complex sinusoids
  - Can be finite length, infinite length
  - Properties
    - Even, odd, causal
    - Periodicity and aliasing
    - Discrete frequency bounded!
- Discrete Time Systems
  - Transform one signal to another
  - Properties
    - Linear, Time-invariance, memoryless, causality, BIBO stability
Admin

- Behind the scenes programming note:
  - Additional grader: Zhefu Peng
- Enroll in Piazza site:
  - piazza.com/upenn/spring2019/ese531
- HW 0: Brush up on background and Matlab tutorial