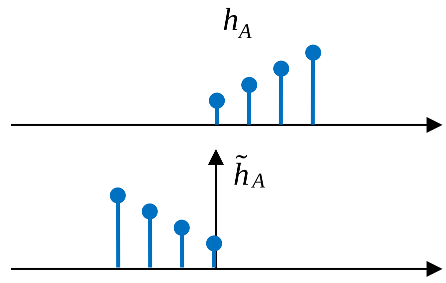
Dec. 11, 2015	ESE 531	Final Exam Solution

Problem 1

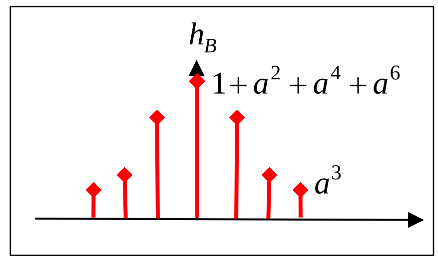


(a) $H_B(z) = H_A(z)H_A(1/z)$

For system A, $h_A = \{1, a, a^2, a^3\}$ for $L = 4$

$$\tilde{H}_A(z) = H_A(1/z) = \sum_{n=0}^{3} a^n z^n = a^3z^3 + a^2z^2 + az + 1$$

Corresponding unit-sample response $\tilde{h}_A$ is $\{a^3, a^2, a, 1\}$
$\uparrow n = 0$



h_B is convolution of h_A and $\tilde{h}_A$

Note $H_B(e^{j\omega}) = H_A(e^{j\omega})H_A(e^{-j\omega}) = \left|H_A(e^{j\omega})\right|^2$ real

- *Therefore it is a linear phase system, with zero phase.*

(b) $H_C(z) = H_A(-z) - H_A(-z^{-1})$

For $L = 4$: $H_A(-z) = \sum_{n=0}^{3} a^n(-z)^{-n} = \sum_{n=0}^{3}(-1)^n a^n z^{-n} = 1 - az^{-1} + a^2z^{-2} - a^3z^{-3}$

$$-H_A(-z^{-1}) = -\sum_{n=0}^{3}(-1)^n a^n z^n = -1 + az^1 - a^2z^2 + a^3z^3$$

Therefore

$$H_C(z) = 1 - az^{-1} + a^2z^{-2} - a^3z^{-3} - 1 + az^1 - a^2z^2 + a^3z^3$$
$$= a^3z^3 - a^2z^2 + az^1 - az^{-1} + a^2z^{-2} - a^3z^{-3}$$

and $\boxed{h_C = \{a^3, -a^2, a, 0, -a, a^2, -a^3\}}$ for $L = 4$
$\uparrow n = 0$

$$H_C(e^{j\omega}) = H_A(-e^{j\omega}) - H_A(-e^{-j\omega}) = H_A(e^{j(\pi+\omega)}) - H_A(e^{-j(\pi+\omega)})$$
$$= H_A(e^{j(\pi+\omega)}) - \underbrace{H_A^*(e^{j(\pi+\omega)})}_{\text{because } h_A \text{ real}} = j\,\mathrm{Im}\left\{H_A(e^{j(\pi+\omega)})\right\} = \underbrace{\mathrm{Im}\left\{H_A(e^{j(\pi+\omega)})\right\}}_{\text{real}} e^{j\pi/2}$$

The phase response of System C is therefore *generalized linear phase*. This follows also from the symmetry of h_C

Problem 2

The FIR filter must have 6 zeros (and 6 poles at the origin). For a real filter, if $z_0 = j\sqrt{2}$ is a zero then $z_0^* = -j\sqrt{2}$ is also a zero; in addition for gen. linear phase FIR filter,

$z_0^{-1} = \frac{-j}{\sqrt{2}}$ and $(z_0^*)^{-1} = \frac{j}{\sqrt{2}}$ are also zeros.

The other two zeros must be either (i) a *real* reciprocal pair, or must be (ii) a *complex conjugate reciprocal* pair *on the unit circle*. In any case, in general the two other zeros z_5 and $z_6 = 1/z_5$ must be a *reciprocal* pair.

Therefore the overall system function must be of the form

$$H(z) = \frac{K}{z^6}(z - j\sqrt{2})(z + j\sqrt{2})(z + j/\sqrt{2})(z - j/\sqrt{2})(z - z_5)(z - 1/z_5)$$

$$= \frac{K}{z^6}(z^2 + 2)(z^2 + 1/2)(z - z_5)(z - 1/z_5)$$

The frequency response at $\omega = 0$ (i.e. at z=1) is

$$H(e^{j0}) = H(1) = K\frac{9}{2}\frac{(z_5 - 1)^2}{-z_5} = 9 \quad \text{so that} \quad \underline{K(z_5 - 1)^2 = -2z_5}$$

At $\omega = \pi$ we have $H(-1) = K\frac{9}{2}\frac{(z_5 + 1)^2}{z_5} = 9$ so that $\underline{K(z_5 + 1)^2 = 2z_5}$

From these two conditions, we get $K(z_5 - 1)^2 = -K(z_5 + 1)^2$ so that

$K[(z_5 - 1)^2 + (z_5 + 1)^2] = 0$ or $2z_5^2 + 2 = 0 \quad \Rightarrow \boxed{z_5 = j}$ and from

$K(j + 1)^2 = 2j$ we get $\boxed{K = 1}$

Therefore we have

$$H(z) = \frac{1}{z^6}(z^2 + 1/2)(z^2 + 2)(z - j)(z + j) = \frac{(z^4 + \frac{5}{2}z^2 + 1)(z^2 + 1)}{z^6} = 1 + \tfrac{7}{2}z^{-2} + \tfrac{7}{2}z^{-4} + z^{-6}$$

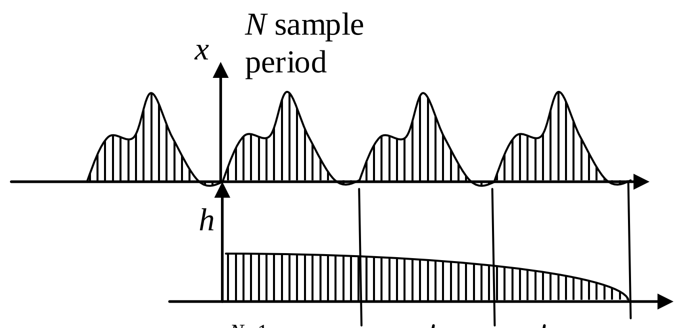
and $\boxed{h = \{h[0], h[1], ..., h[6]\} = \{1, 0, \tfrac{7}{2}, 0, \tfrac{7}{2}, 0, 1\}}$ (Type I filter)

Problem 3

(a) $y[n]=\sum_{k=0}^{K-1}x[n-k]h[k]$; since x is preiodic of period N, we anticipate the same for y

Consider $y[n+N]=\sum_{k=0}^{K-1}x[n+N-k]h[k]=\sum_{k=0}^{K-1}x[n-k]h[k]=y[n]$

Therfore y is periodic of period N



(b) $K=mN$

We see that the result of the convolution in each *output* periodic block of length N is a circular convolution of one period of the input $\tilde{x}=\{x[0],...,x[N-1]\}$ *with* $\left\{\tilde{h}[n]\right\}_{n=0}^{N-1}=\left\{\sum_{p=0}^{m-1}h[n+pN]\right\}_{n=0}^{N-1}$

This can be implemented by multiplying the DFT's of $\tilde{x}$ and of the $\left\{\sum_{p=0}^{m-1}h[n+pN]\right\}_{n=0}^{N-1}$, each of which requires $\frac{N}{2}\log N$ multiplications (using the FFT), followed by an IDFT. Thus total number of multiplies is $\left(\frac{3N}{2}\log N\right)+N$ to compute the output block of N samples; the rest of the output is just this block repeated.

- For the given numbers, this is a total of $1536\times 10+1024=\boxed{16384 \text{ multiplications}}$
- Regular *convolution* requires mN multiplies for each output point, so for a total of N output points we need mN^2 multiplies.
- Even regular *circular* convolution (not using FFT) requires a total of N^2 multiplies which is $>10^6$ here.

Problem 4

(a)

$e^{j2\pi 7/12} = e^{j2\pi 14/24}$, and $e^{j2\pi 3/8} = e^{j2\pi 9/24}$

We can compute the DTFT values on 24 equi-spaced points around unit circle, by computing the DFT of a sequence of length 24. Therefore in this case we can append 14 zeros to the given x sequence and obtain its DFT. Then the 10th and 15th output coefficients $X[9]$ and $X[14]$ will be the desired values.

With zeros appended, DFT is $X[k] = \sum_{n=0}^{23} x[n] e^{-j\frac{2\pi}{24}kn} = \sum_{n=-\infty}^{\infty} x[n] \left(e^{j\frac{2\pi}{24}k} \right)^{-n} = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{24}k}$, $k = 0,1,...,23$

(b) We are now computing the DTFT on (i) M uniformly spaced locations on the full unit circle (starting at $\omega = 0$), and also at (ii) a set of M locations offset from these by a small angle $2\pi / N$

-- The first set of M DTFT values can be obtained by taking the M-point DFT of an M-point sequence x_1 derived from x as follows:

$$x_1[m] = \sum_{r=0}^{\infty} x[m + Mr], \quad m = 0,1,...,M-1, \text{ where } x[k] = 0 \text{ for } k > N-1.$$

No multiplications are needed in forming this finite sequence from x.

--For the DTFT at $\omega = \frac{2\pi}{M}m + \frac{2\pi}{N}$, $m = 0,1,...,M-1$ we want

$$X\left(e^{j\left(\frac{2\pi}{M}m + \frac{2\pi}{N}\right)} \right) = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{M}m + \frac{2\pi}{N}\right)n} = \sum_{n=0}^{N-1} \underbrace{x[n] e^{-j\frac{2\pi}{N}n}}_{y[n]} e^{-j\frac{2\pi}{M}m\,n}, \quad m = 0,1,...,M-1$$

Thus with $y[n] \triangleq x[n] e^{-j\frac{2\pi}{N}n}$, $n = 0,1,...,N-1$, we are now looking for the DTFT of y at M uniformly spaced locations on the full unit circle (starting at $\omega = 0$). This can again be found by taking the M-point DFT of an M-point sequence y_1 derived from y as follows:

$$y_1[n] = \sum_{r=0}^{\infty} y[n + Mr], \quad n = 0,1,...,M-1, \text{ where } y[k] = 0 \text{ for } k > N-1.$$

No multiplications are needed in forming this finite sequence from y.

Therefore we need a total of 2 M-point FFT's, and N multiplications (to obtain sequence y from x). The total number of multiplications is then approximately

$$2\frac{M}{2}\log_2 M + N = \boxed{M \log_2 M + N}$$

Using a single FFT on the original sequence x, since N is divisible by M, we would need an N-point FFT with total number of multiplications $\frac{N}{2}\log_2 N$. This can be substantially larger than the result above; e.g. for $N=2^{14}=16{,}384$ and $M=2^8=256$, we need approximately 18,416 vs. 114,688