University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2017 — Final — Wednesday, May 3

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed.
- One two-sided 8.5x11 cheat sheet allowed.
- Final answers here on the question sheet.
- Additional workspace in "blue" book. Note where to find work in "blue" book if relevant.
- Sign Code of Academic Integrity statement at back of "blue" book.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 78.5, Standard Deviation: 13.9

Common z-transform pairs:

Sequence		z-transform
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
$\delta[n-m]$	z^{-m}	All z except 0 (if m>0) or ∞ (if m<0)
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$

Common DTFT pairs:

Sequence	DTFT
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
1	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega+2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-\omega_0+2\pi k)$
$\alpha^n u[n], \|\alpha\|<1$	$\frac{1}{1-\alpha e^{-j\omega}}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega})=\begin{cases}1, & \|\omega\|\le\omega_c\\ 0, & \omega_c<\|\omega\|<\pi\end{cases}$

Trigonometric Identity:

$$e^{jA} = cos(A) + jsin(A)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

1. (30 pts) A continuous-time filter has a system function given by

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

A discrete-time filter is designed using impulse invariance with an impulse response given as samples of the impulse response of the continuous-time filter: $h[n] = Th_a(nT)$, where T is the sampling period. The frequency response of the discrete-time filter is given as:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

(a) Find the impulse response, $h[n]$, and system function, $H(z)$, of the discrete-time filter. Reminder: $\mathcal{L}\{e^{at}\} = \frac{1}{s+a}$ Exam erroneously said $\mathcal{L}\{e^{at}\} = \frac{1}{s+a}$. Correct expression is $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. Exam was graded with incorrect expression with parts (a-c) being out of 20, and everyone got +10 points. . Below are shown the corrected solutions.

$$\begin{aligned}
H_a(s) &= \frac{2}{(s+1)(s+3)} = \frac{1}{s+1} - \frac{1}{s+3} \\
h_a(t) &= (e^{-t} - e^{-3t})u(t) \\
h[n] &= T \cdot h_a(nT) = T(e^{-nT} - e^{-3nT})u[n] \\
H(z) &= \frac{T}{1 - e^{-T}z^{-1}} - \frac{T}{1 - e^{-3T}z^{-1}} \\
&= \frac{T(e^{-T} - e^{-3T})z^{-1}}{(1 - e^{-T}z^{-1})(1 - e^{-3T}z^{-1})}
\end{aligned}$$

(b) What are the poles of the discrete-time filter?

Poles are at $z = e^{-T}$ and $z = e^{-3T}$.

(c) Is the discrete-time filter stable and causal?
Since $T > 0$, both poles are less than 1, therefore ROC includes unit circle. Impulse response is causal. So system is stable and causal.

(d) How should the sampling period, T, be chosen such that $|H(e^{j\pi})| \leq 0.1$? HINT: Think about writing $H(e^{j\omega})$ as a function of $H_a(s)$ for $\omega = \pi$.
Not graded. Intent was to approximate $H(e^{j\omega})|_\pi \approx 2H_a(j\Omega)|_{\Omega=\pi/T}$, since the DTFT is sum of shifted replicas of $H_a(s)$ with most contributions from two closest replicas.

$$\begin{aligned}
H_a(j\Omega) &= \frac{1}{(j\Omega+1)(j\Omega+3)} \\
|H_a(j\Omega)| &= \frac{1}{|j\Omega+1||j\Omega+3|} = \frac{1}{\sqrt{1+\Omega^2}\cdot\sqrt{9+\Omega^2}} \\
|H(e^{j\pi})| &\approx 2|H_a(j\frac{\pi}{T})| \\
0.1 &\geq \frac{2}{\sqrt{1+(\frac{\pi}{T})^2}\cdot\sqrt{9+(\frac{\pi}{T})^2}}
\end{aligned}$$

Solving results in $T < 0.8$s.

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2. (25 points) Let $H(z)$ be the system function for a stable LTI system and given as:

$$H(z) = \frac{(1-9z^{-2})(1+\frac{1}{3}z^{-1})}{1-\frac{1}{3}z^{-1}}$$

(a) $H(z)$ can be represented as a cascade of a minimum-phase system $H_{min}(z)$ and a unity-gain all-pass system $H_{ap}(z)$. Determine $H_{min}(z)$ and $H_{ap}(z)$ such that $|H_{ap}(z)| = 1$ for any z on the unit circle.

For H(z) we can write the poles and zeros:
Zeros: $z = 3, -3, -\frac{1}{3}$
Poles: $z = \frac{1}{3}$

The zeros at 3 and -3 can't be in the minimum phase system, so they must go in the all-pass system. In order to make the latter all-pass, it must have poles at $\frac{1}{3}$ and $-\frac{1}{3}$. Since these were not part of $H(z)$, they must be cancelled by zeros in the minimum phase system. Inserting the zero at $\frac{1}{3}$ in the minimum phase system cancels the pole that was there.

$$\begin{aligned} H_{min}(z) &= \frac{1}{K}\left(1+\frac{1}{3}z^{-1}\right)^2 \\ H_{ap}(z) &= K\frac{(1-3z^{-1})(1+3z^{-1})}{(1-\frac{1}{3}z^{-1})(1+\frac{1}{3}z^{-1})} \end{aligned}$$

The product of these two functions is the original $H(z)$ given in the problem. Since we need the all-pass system to have unity gain for any z on the unit circle where $z = 1$. This gives $K = -\frac{1}{9}$.

(b) Write an expression for the phase of $H_{min}(e^{j\omega})$. Is the minimum-phase system, $H_{min}(z)$, a generalized linear-phase system?

$$H_{min}(e^{j\omega}) = \frac{1}{K}\left(1+\frac{1}{3}e^{-j\omega}\right)^2 = \frac{1}{K}\left(1+\frac{2}{3}e^{-j\omega}+\frac{1}{9}e^{-j2\omega}\right)$$

$$\angle H_{min}(z) = arctan\left(\frac{-\frac{2}{3}sin(\omega)-\frac{1}{9}sin(2\omega)}{1+\frac{2}{3}cos(\omega)+\frac{1}{9}cos(2\omega)}\right)$$

The phase is not linear, so H_{min} is not GLP.

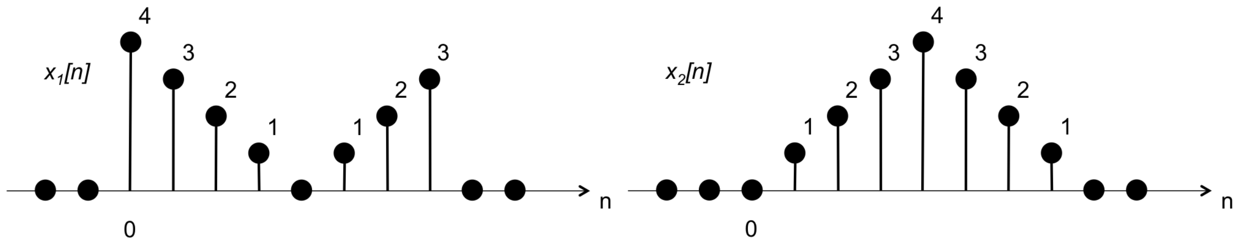
(c) If the minimum-phase system, $H_{min}(z)$, is not a generalized linear-phase system, can $H(z)$ be represented as a cascade of a generalized linear-phase system $H_{lin}(z)$ and an all-pass system $H_{ap2}(z)$? If your answer is yes, determine $H_{lin}(z)$ and $H_{ap2}(z)$. If your answer is no, explain why such representation does not exist.

Yes, let the zeros at $z = -3$ and $z = -\frac{1}{3}$ be part of $H_{lin}(z)$:

$$\begin{aligned} H_{lin}(z) &= (1 + 3z^{-1}) \left(1 + \frac{1}{3} z^{-1}\right) \\ H_{ap2}(z) &= \frac{1 - 3z^{-1}}{1 - \frac{1}{3} z^{-1}} \end{aligned}$$

This is equivalent to $H_{lin}(z) = 1 + \frac{10}{3} z^{-1} + z^{-2}$, which has an impulse response with even symmetry, and therefore has linear phase.

3. (25 points) Let $x_1[n]$ and $x_2[n]$ be the two 8-point shown below:



(a) Let $X_1[k]$ and $X_2[k]$ be 8-point DFTs of $x_1[n]$ and $x_2[n]$ respectively. Write an expression for $X_2[k]$ in terms of $X_1[k]$. Hint: Think about how to write $x_2[n]$ in terms of $x_1[n]$.

$x_2[n]$ is $x_1[n]$ circularly shifted by 4:

$$\begin{aligned} x_2[n] &= x_1[[n+4]]_8 = x_1[[n-4]]_8 \\ X_2[k] &= X_1[k]e^{j\omega_0 k} \end{aligned}$$

$$\omega_0 = \frac{4}{8} \cdot 2\pi = \pi$$

$$\begin{aligned} X_2[k] &= X_1[k]e^{j\pi k} \\ X_2[k] &= X_1[k](-1)^k \end{aligned}$$

(b) Find the sequence $h[n]$ which satisfies the relation $x_2[n] = x_1[n] \, ⑧ \, h[n]$ where ⑧ denotes the 8-point circular convolution.

$x_2[n]$ is $x_1[n]$ circularly shifted by 4, so $h[n] = \delta[n-4]$. Using time shift property:

$$\begin{aligned}
X_2[k] &= X_1[k]e^{j\pi k} \\
H[k] &= e^{j\pi k} \\
h[n] &= \delta[n-n_0], \text{ where } \frac{2\pi k n_0}{N} = \pi k \rightarrow n_0 = 4 \\
h[n] &= \delta[n-4]
\end{aligned}$$

4. (20 points) Let $x[n]$ be a signal of length N, $n = 0, 1, ..., N-1$. Let $y[n]$ be the $2N$ point signal created by repeating $x[n]$:

$$y[n] = \begin{cases} x[n] & \text{for } n = 0, 1, ..., N-1, \\ x[n-N] & \text{for } n = N, ..., 2N-1 \end{cases}$$

(a) Find a simple expression for the $2N$-point DFT of $y[n]$ in terms of the DFT of $x[n]$. Specify the length of the DFT of $x[n]$ in your expression.

N-point DFT of x[n]:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

2N-point DFT of y[n]:

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{2N-1} x[n] e^{-j\frac{2\pi}{2N}kn} \\
Y[k] &= \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=N}^{2N-1} y[n] e^{-j\frac{2\pi}{2N}kn} \\
Y[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=N}^{2N-1} x[n-N] e^{-j\frac{2\pi}{2N}kn} \\
Y[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}k(n+N)} \\
Y[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} + \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} e^{-j\pi k} \\
Y[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} + (-1)^k \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{2N}kn} \\
Y[k] &= \begin{cases} 2X[\frac{k}{2}] & \text{for } k = 0, 2, 4, ..., 2N-1, \\ 0 & \text{for } k = 1, 3, 5, ..., 2N-1 \end{cases}
\end{aligned}$$

(b) Approximate the minimum number of multiplications needed to calculate the 2^8-point DFT of $y[n]$. Describe the computations done to obtain the DFT. (I.e What DFT did you perform?)

For the 2^8-point DFT of y, we compute the 2^7-point DFT of X, which needs $\frac{2^7}{2} log_2(2^7) = 448$ multiplications. If you include the multiplication by 2, total multiplications are $448 + 2^7 = 576$.