ESE 531: Digital Signal Processing

Lec 10: February 14th, 2017

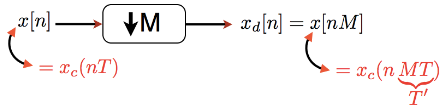
Practical and Non-integer Sampling, Multi-rate Sampling

Lecture Outline

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks

Downsampling

- Definition: Reducing the sampling rate by an integer number



$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$= x_c(nT)$$

$$= x_c(n\underbrace{MT}_{T'})$$

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right)}_{X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}} - \underbrace{\frac{2\pi}{T}}k\right)\right)$$

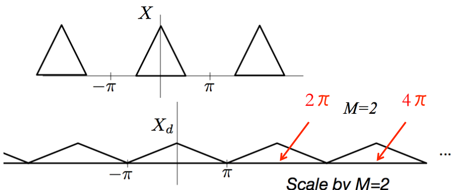
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M

replicate

Example

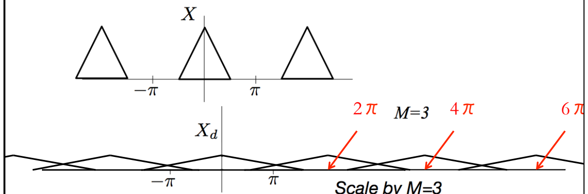
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{w}{M} - \frac{2\pi}{M}i)}\right)$$



Scale by M=2
*Shift by (i=1)*2π/(M=2)*

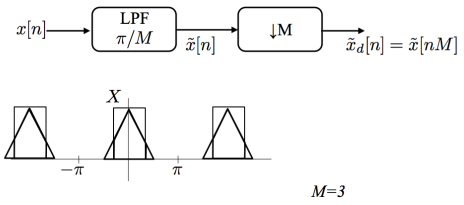
Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{w}{M} - \frac{2\pi}{M}i)}\right)$$

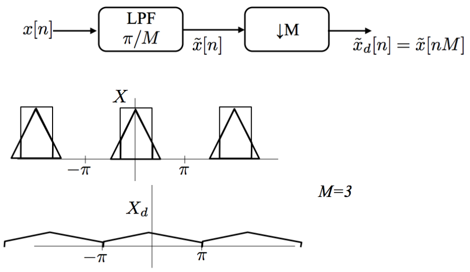


Scale by M=3
*Shift by (i=1)*2π/(M=3)*
*Shift by (i=2)*2π/(M=3)*

Example



Example



Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

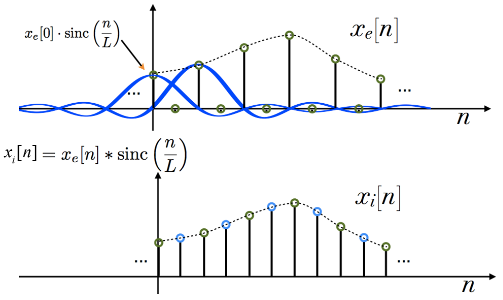
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

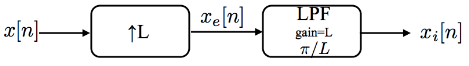
Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

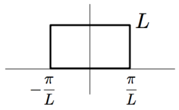


Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ DTFT ⇒



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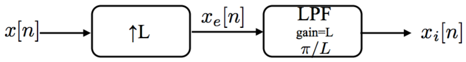
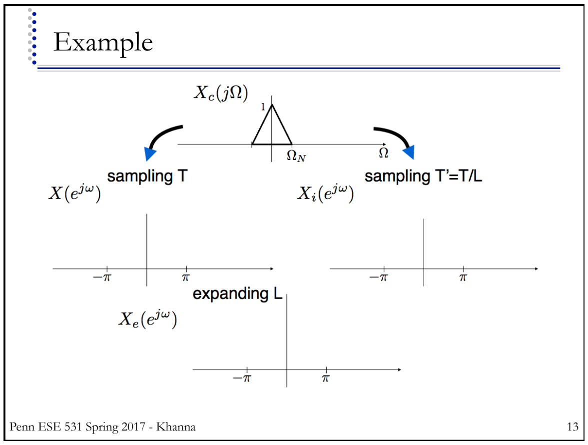


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = \boxed{X(e^{j\omega L})}$$

Compress DTFT by a factor of L!



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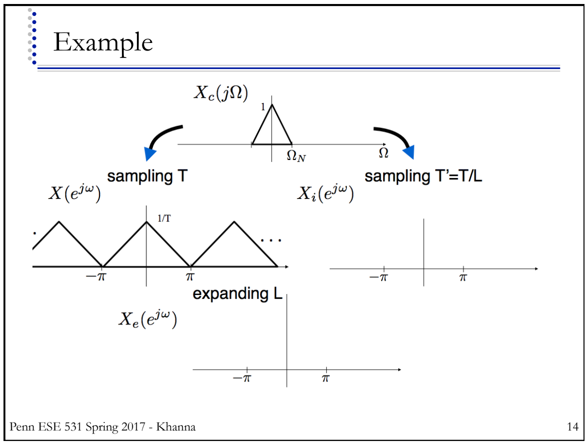

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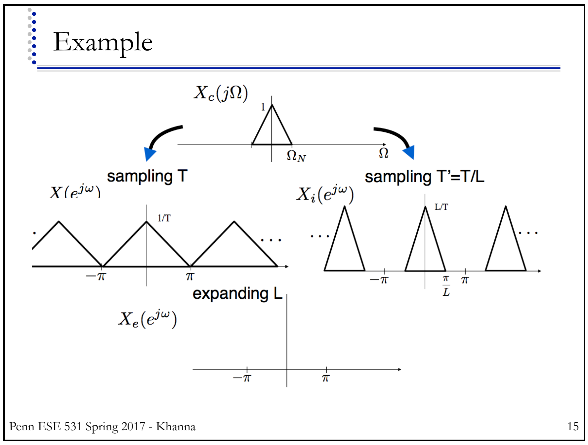

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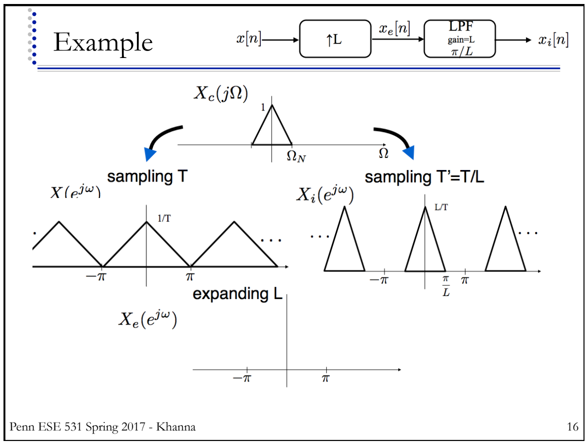

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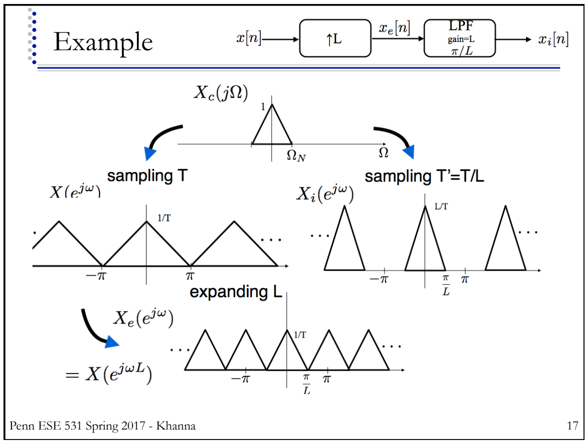

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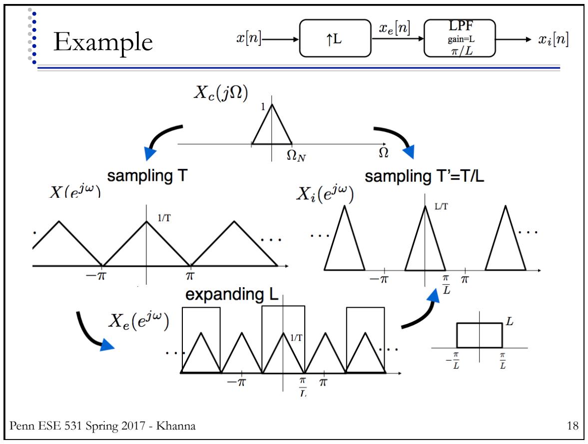
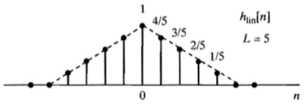


Example

Practical Interpolation

- Interpolate with simple, practical filters
  - Linear interpolation – samples between original samples fall on a straight line connecting the samples
    - Convolve with triangle instead of sinc

$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise,} \end{cases}$$



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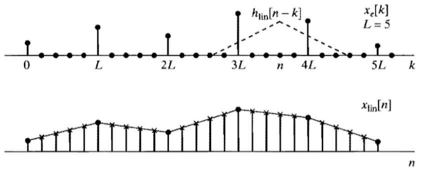

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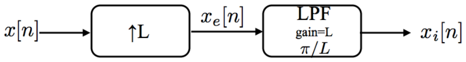

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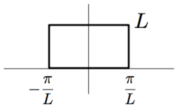
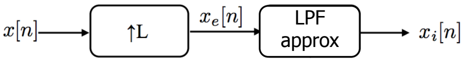


Linear Interpolation -- Frequency Domain

$$x_i[n] = x_e[n] * h_{lin}[n]$$



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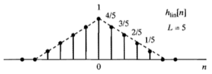
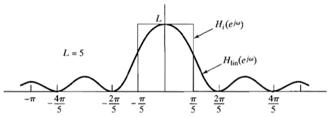


DTFT ⇒



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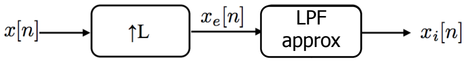

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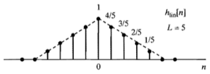

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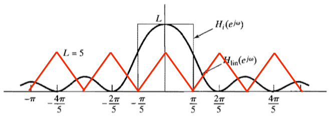

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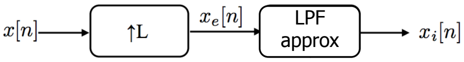

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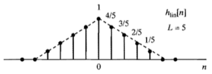

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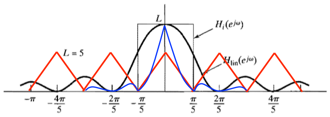
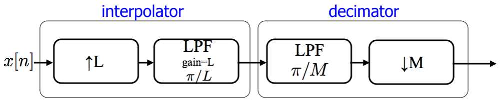


Non-integer Sampling

- T'=TM/L
  - Upsample by L, then downsample by M



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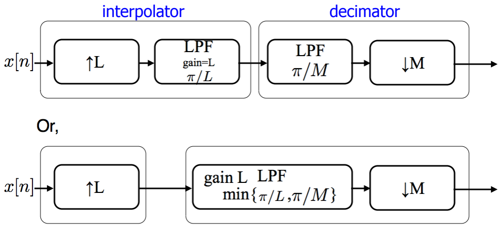
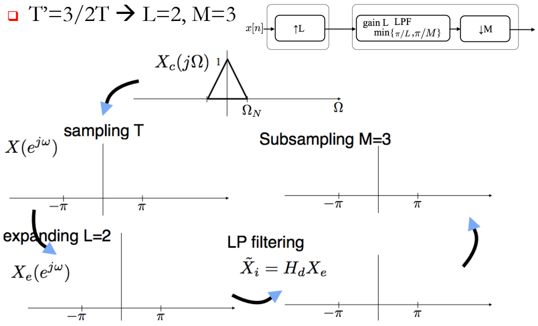


Example

- T'=3/2T → L=2, M=3



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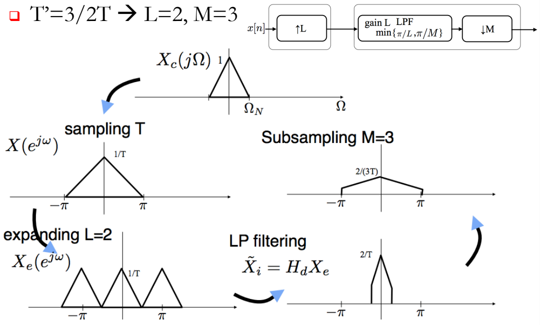
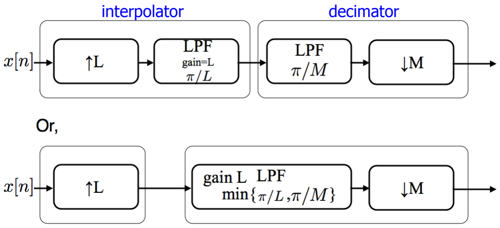


Non-integer Sampling

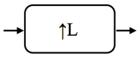
- T'=TM/L
  - Downsample by M, then upsample by L?



Multi-Rate Signal Processing

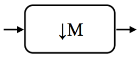
- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π /101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations



"expander"

Upsampling
-expanding in time
-compressing in frequency



not LTI!

"compressor"

Downsampling
-compressing in time
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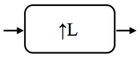
Interchanging Operations - Expander



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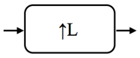

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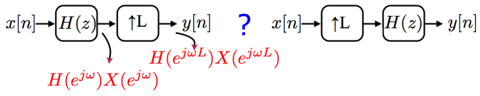
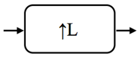


Interchanging Operations - Expander



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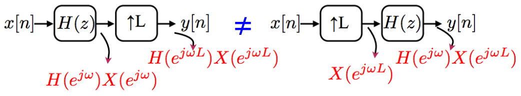

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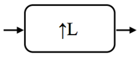

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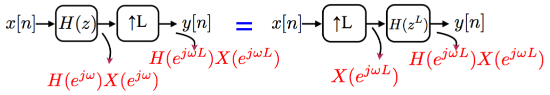
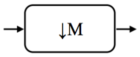


Interchanging Operations - Compressor



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"compressor"

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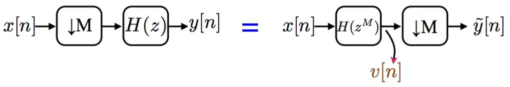

Interchanging Operations - Compressor

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad = \quad x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow \tilde{y}[n]$$

$v[n]$

$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)\right)$$

Interchanging Operations - Compressor

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After compressing

$$V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

Interchanging Operations - Summary

Filter and expander | Expander and expanded filter*

$$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

Compressor and filter | Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

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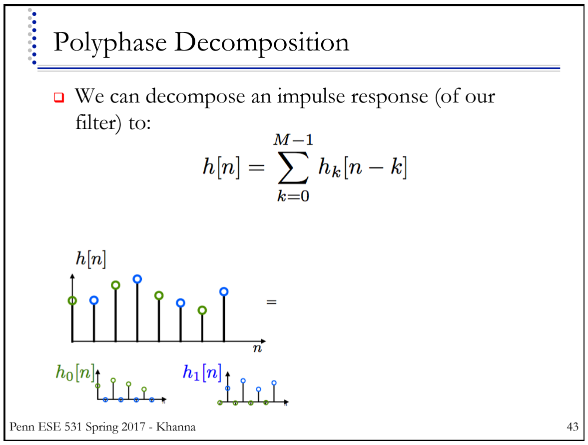
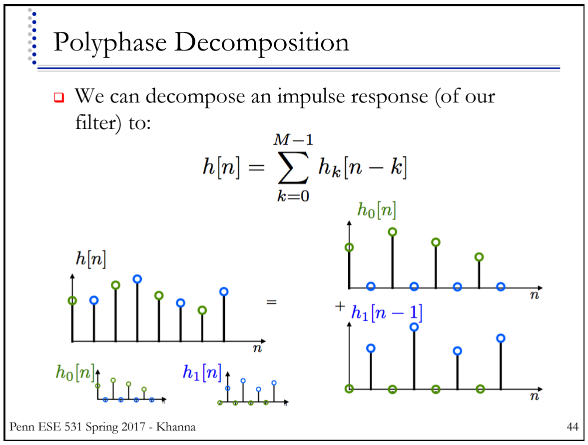


Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

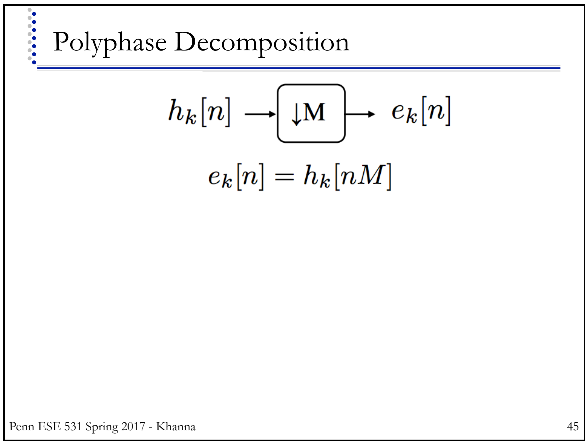
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$



Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

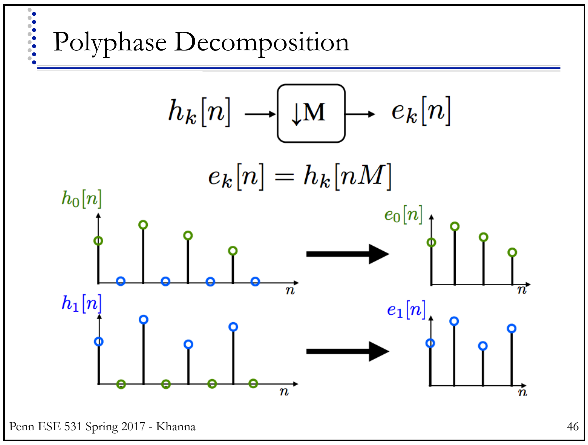
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$



Polyphase Decomposition

$$h_k[n] \longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$



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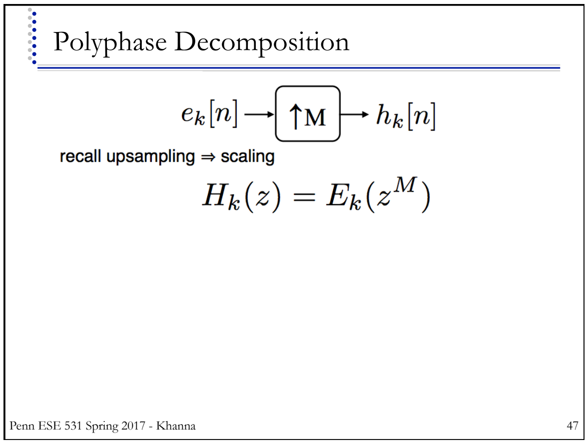
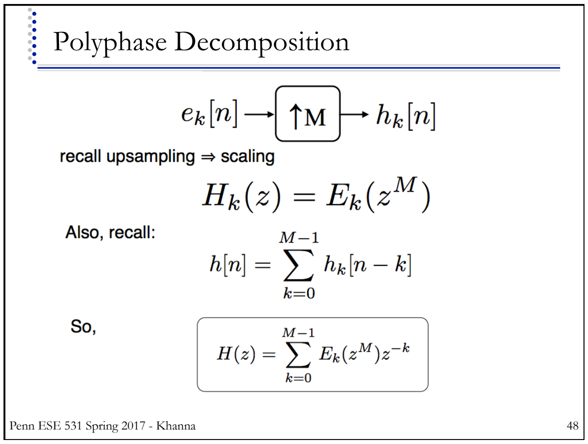


Polyphase Decomposition

$$e_k[n] \longrightarrow \boxed{\uparrow \mathrm{M}} \longrightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$



Polyphase Decomposition

$$e_k[n] \longrightarrow \boxed{\uparrow \mathrm{M}} \longrightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

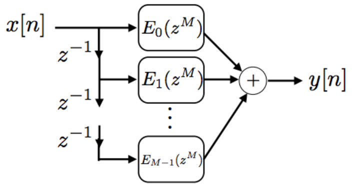
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

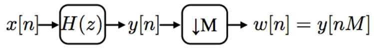
$$\boxed{H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}}$$

Polyphase Decomposition

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$

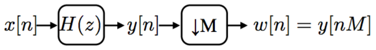


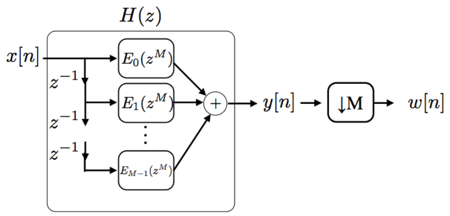
Polyphase Implementation of Decimation



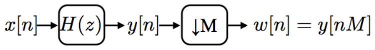
- Problem:
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N mults/unit time

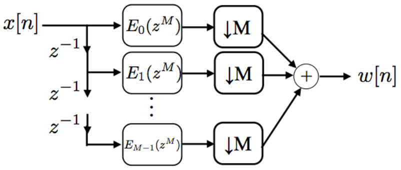
Polyphase Implementation of Decimation



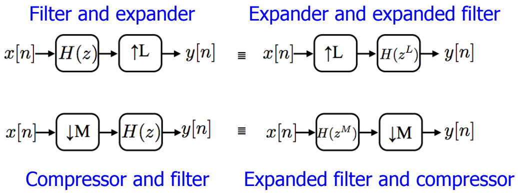


Polyphase Implementation of Decimation





Interchanging Operations - Summary

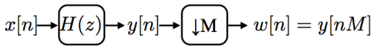

Filter and expander

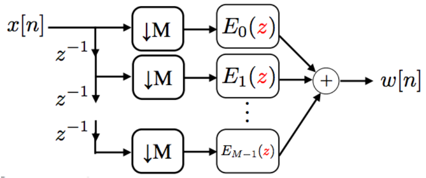
Expander and expanded filter

Compressor and filter

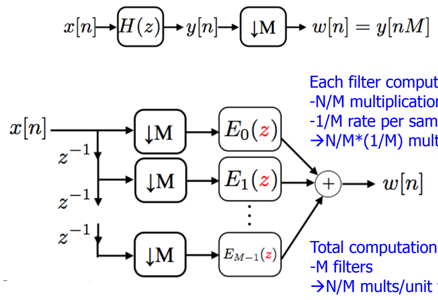
Expanded filter and compressor

Polyphase Implementation of Decimation





Polyphase Implementation of Decimation



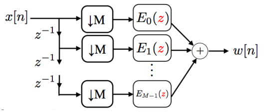
Each filter computation:
-N/M multiplications
-1/M rate per sample
→N/M*(1/M) mults/unit time

Total computation:
-M filters
→N/M mults/unit time

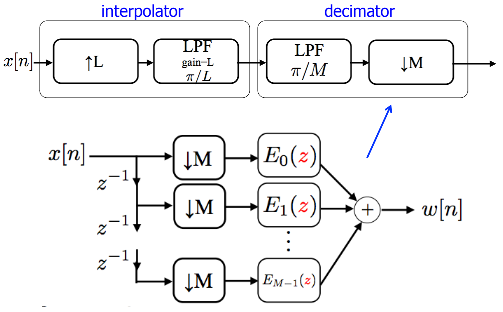
Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter $\pi/101$ ($$$$$)
  - Downsample by M=101

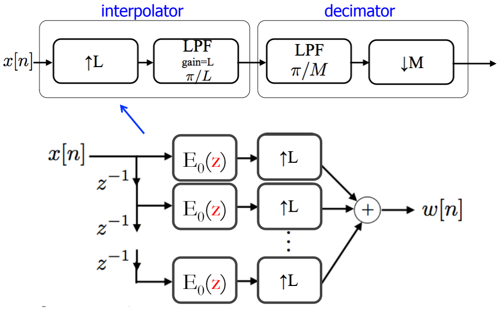
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering



Polyphase Implementation of Decimator



Polyphase Implementation of Interpolation



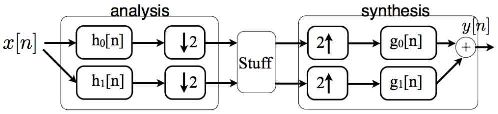
Multi-Rate Filter Banks

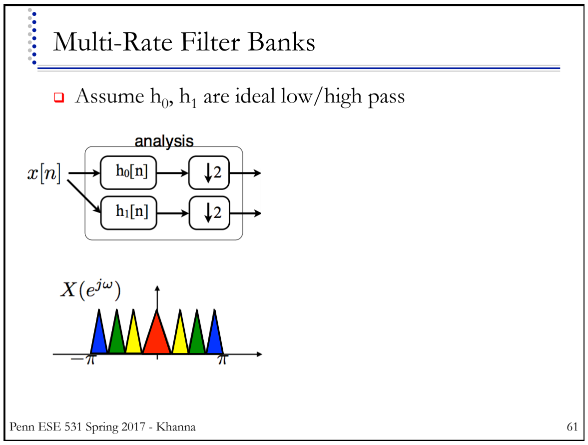
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

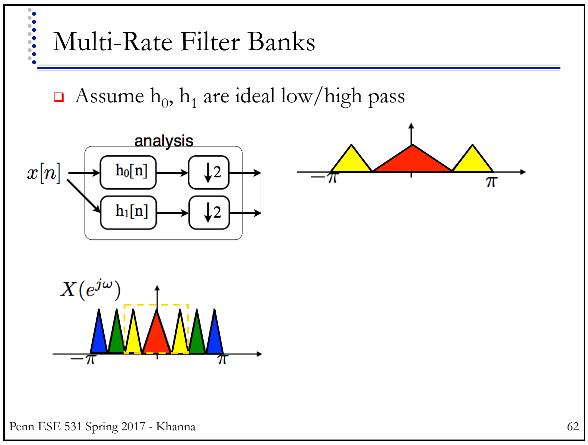
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π





Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass



Multi-Rate Filter Banks

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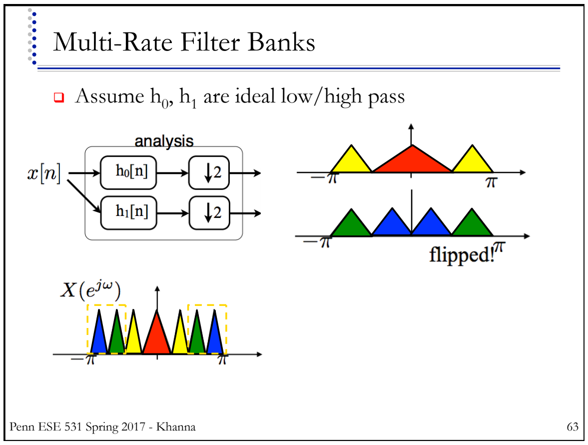

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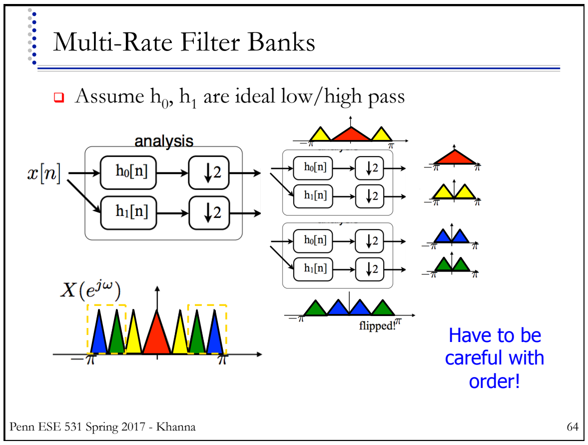
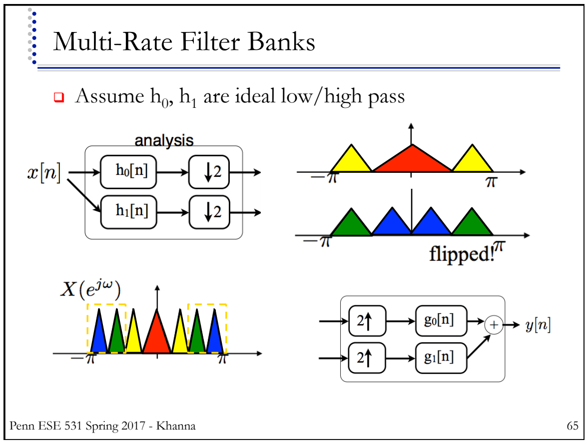


Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass

Have to be careful with order!



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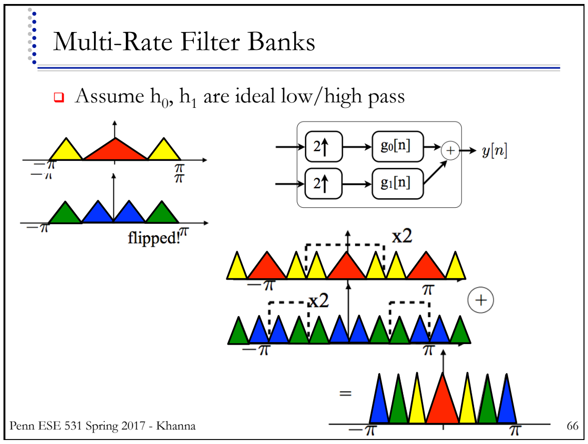

Multi-Rate Filter Banks

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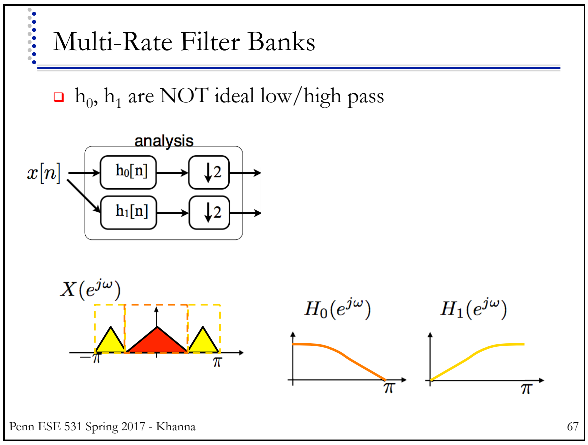
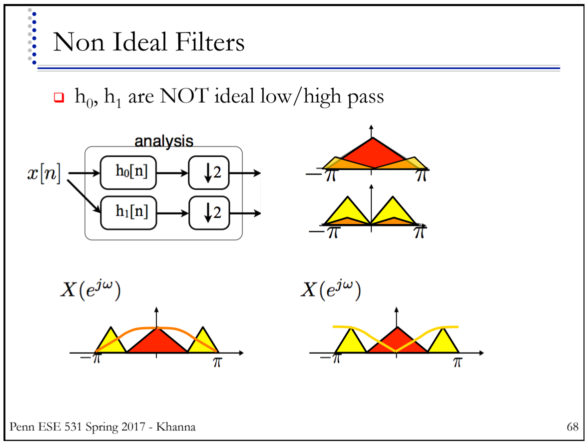


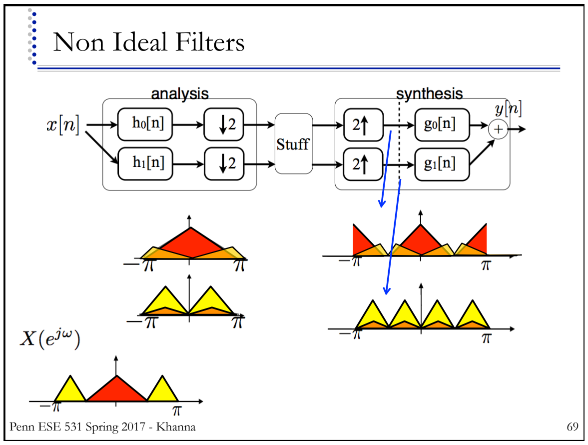
Multi-Rate Filter Banks

- h_0, h_1 are NOT ideal low/high pass

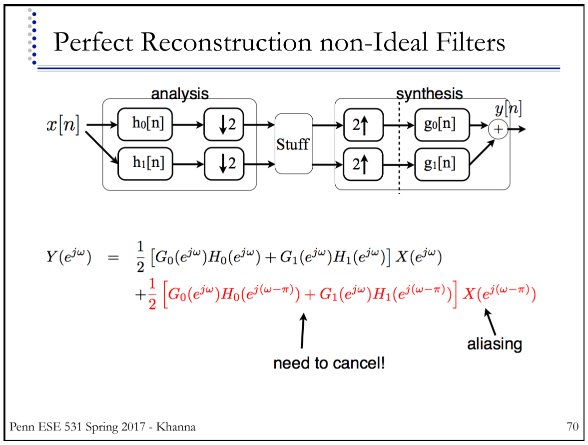


Non Ideal Filters

- h_0, h_1 are NOT ideal low/high pass



Non Ideal Filters

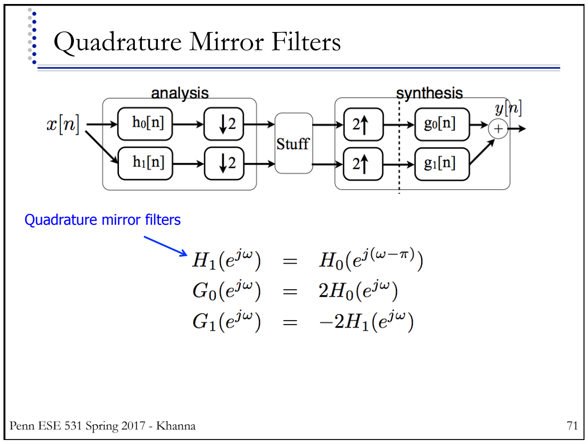


Perfect Reconstruction non-Ideal Filters

$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

need to cancel!

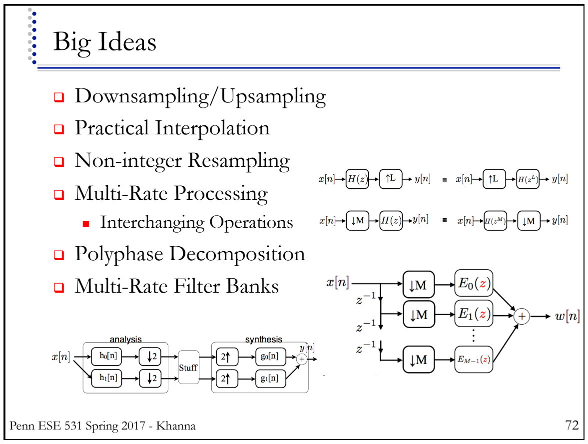
aliasing



Quadrature Mirror Filters

Quadrature mirror filters

$$\begin{aligned} H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\ G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\ G_1(e^{j\omega}) &= -2H_1(e^{j\omega}) \end{aligned}$$



Big Ideas

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks

Admin

- HW 4 due Friday
  - Typo in code in MATLAB problem, corrected handout
  - See Piazza for more information