ESE 531: Digital Signal Processing

Lec 11: February 16th, 2017

Data Converters, Noise Shaping

Lecture Outline

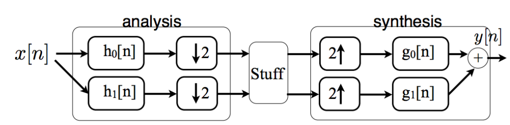
- Multi-Rate Filter Banks (con't)
- Data Converters
  - Anti-aliasing
  - ADC
    - Quantization
  - Practical DAC
- Noise Shaping

Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

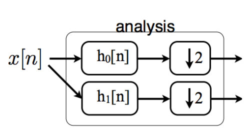
Multi-Rate Filter Banks

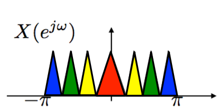
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass





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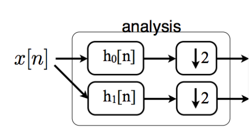

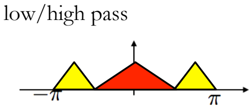

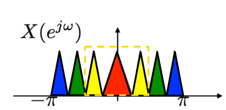

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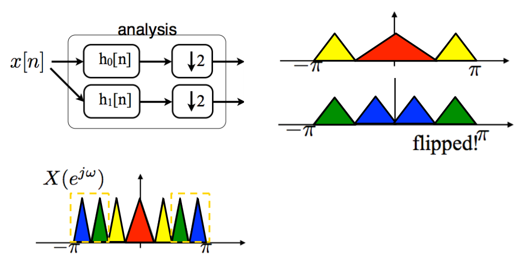

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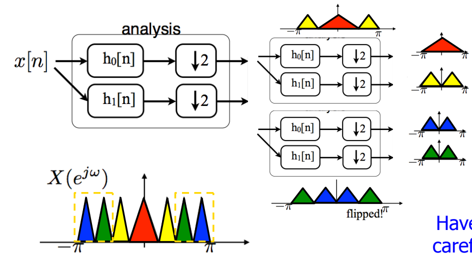
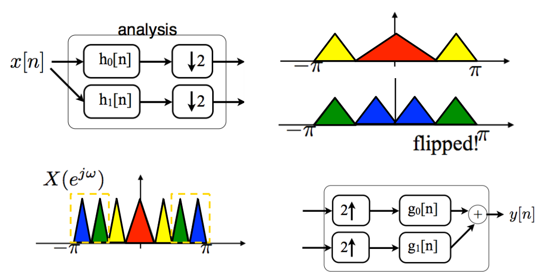


Have to be careful with order!

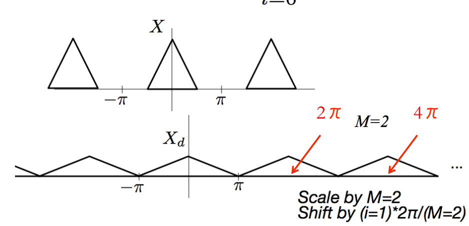
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass



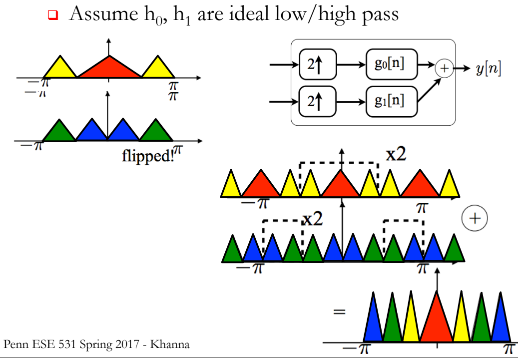
Downsampling Reminder: Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M}-\frac{2\pi}{M}i\right)}\right)$$



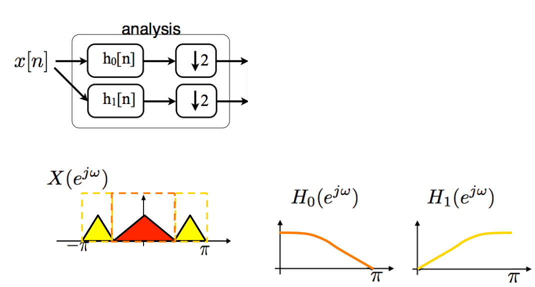
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass



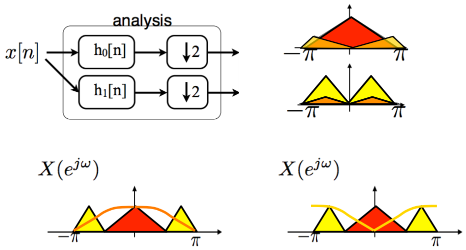
Multi-Rate Filter Banks

- h_0, h_1 are NOT ideal low/high pass

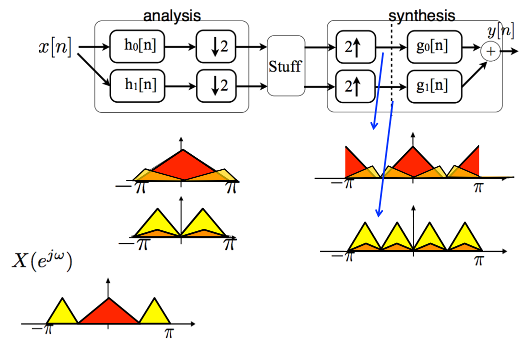


Non Ideal Filters

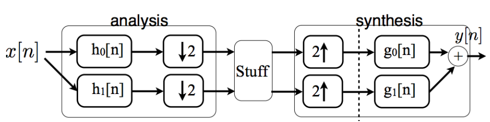
- h_0, h_1 are NOT ideal low/high pass



Non Ideal Filters



Perfect Reconstruction non-Ideal Filters

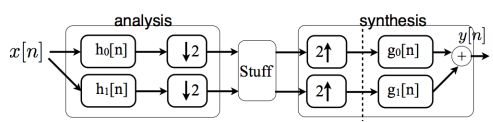


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Perfect Reconstruction non-Ideal Filters

$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

need to cancel!

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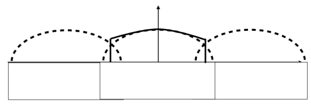
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

ADC

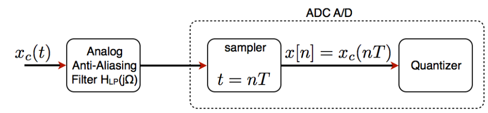
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

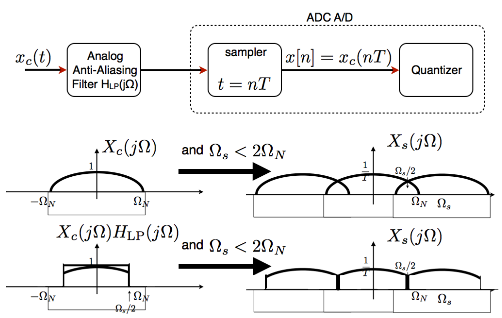


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

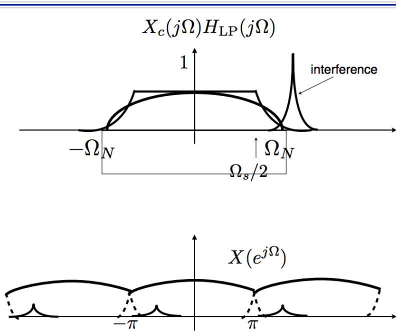
Anti-Aliasing Filter with ADC



Anti-Aliasing Filter with ADC



Non-Ideal Anti-Aliasing Filter



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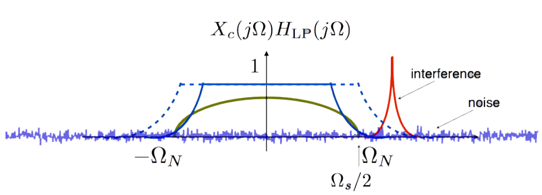
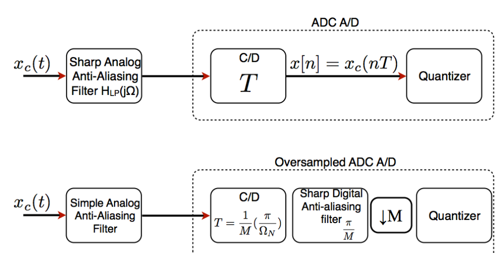
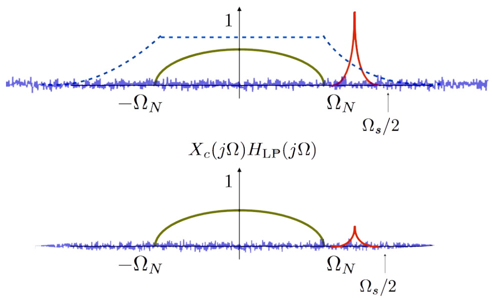


- Problem: Hard to implement sharp analog filter
- Solution: Crop part of the signal and suffer from noise and interference

Oversampled ADC



Oversampled ADC



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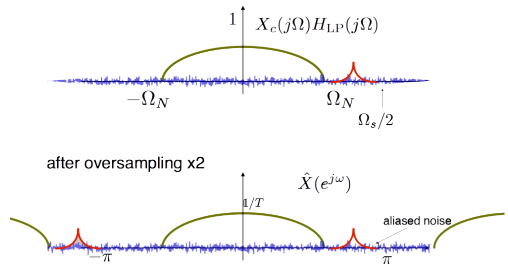

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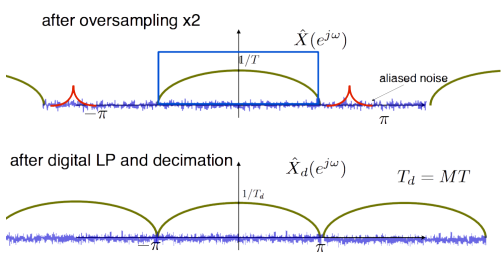
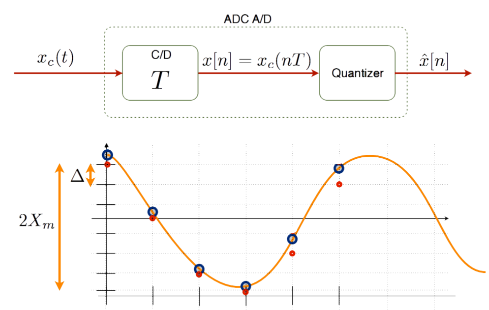


Sampling and Quantization

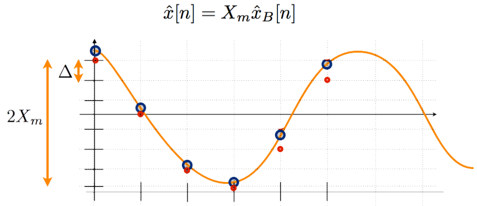


Sampling and Quantization

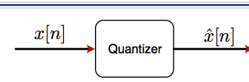
for 2's complement with B+1 bits $-1 \le \hat{x}_B[n] < 1$

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$

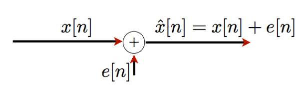
$$\hat{x}[n] = X_m \hat{x}_B[n]$$



Quantization Error



- Model quantization error as noise



- In that case:

$$-\Delta/2 \le e[n] < \Delta/2$$

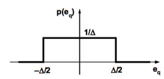
$$(-X_m - \Delta/2) < x[n] \le (X_m - \Delta/2)$$

Effect of Quantization Error on Signal

- Quantization error is a deterministic function of the signal
  - Consequently, the effect of quantization strongly depends on the signal itself
- Unless, we consider fairly trivial signals, a deterministic analysis is usually impractical
  - More common to look at errors from a statistical perspective
  - "Quantization noise"
- Two aspects
  - How much noise power (variance) does quantization add to our samples?
  - How is this noise distributed in frequency?

Quantization Error Statistics

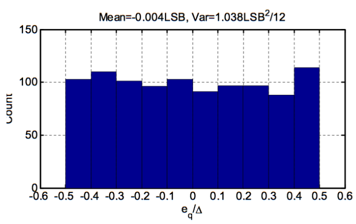
- Crude assumption: $e_q(x)$ has uniform probability density
- This approximation holds reasonably well in practice when
  - Signal spans large number of quantization steps
  - Signal is "sufficiently active"
  - Quantizer does not overload



Mean $\bar{e} = \int_{-\Delta/2}^{+\Delta/2} \frac{e}{\Delta} de = 0$

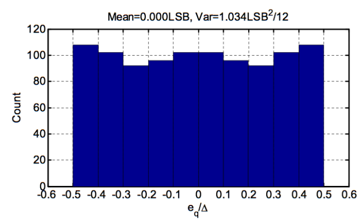
Variance $\overline{e^2} = \int_{-\Delta/2}^{+\Delta/2} \frac{e^2}{\Delta} de = \frac{\Delta^2}{12}$

Reality Check



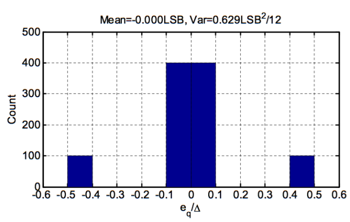
- Shown below is a histogram of e_q in an 8-bit quantizer
  - Input sequence consists of 1000 samples with Gaussian distribution, 4σ=FSR

Reality Check



- Same as before, but now using a sinusoidal input signal with f_{sig}/f_s=101/1000

Reality Check



- Same as before, but now using a sinusoidal input signal with f_{sig}/f_s=100/1000
- What went wrong?

Analysis

$$v_{sig}(n) = \cos\left(2\pi \cdot \frac{f_{sig}}{f_s} \cdot n\right)$$

- Signal repeats every m samples, where m is the smallest integer that satisfies $m \cdot \frac{f_{sig}}{f_s} = \text{integer}$

$$m \cdot \frac{101}{1000} = \text{integer} \Rightarrow m=1000$$

$$m \cdot \frac{100}{1000} = \text{integer} \Rightarrow m=10$$

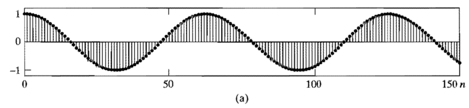
- This means that in the last case $e_q(n)$ consists at best of 10 different values, even though we took 1000 samples

Noise Model for Quantization Error

- Assumptions:
  - Model e[n] as a sample sequence of a stationary random process
  - e[n] is not correlated with x[n]
  - e[n] not correlated with e[m] where m≠n (white noise)
  - e[n] ~ U[-Δ/2, Δ/2] (uniform pdf)
- Result:
- Variance is: $\sigma_e^2 = \frac{\Delta^2}{12}$, or $\sigma_e^2 = \frac{2^{-2B} X_m^2}{12}$ since $\Delta = 2^{-B} X_m$
- Assumptions work well for signals that change rapidly, are not clipped, and for small Δ

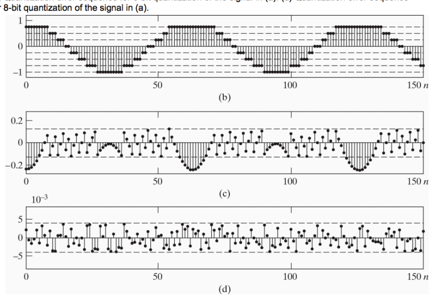
Quantization Noise

- **Figure 4.57** Example of quantization noise. (a) Unquantized samples of the signal x[n] = 0.99cos(n/10).



Quantization Noise

Figure 4.57 (*continued*) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a). (d) Quantization error sequence for 8-bit quantization of the signal in (a).



Signal-to-Quantization-Noise Ratio

- For uniform B+1 bits quantizer

$$\begin{aligned} SNR_Q &= 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) \\ &= 10\log_{10}\left(\frac{12\cdot 2^{2B}\sigma_x^2}{X_m^2}\right) \end{aligned}$$

$$\text{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right)$$

Quantizer range (X_m); rms of amp (σ_x)

Signal-to-Quantization-Noise Ratio

$$\text{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right)$$

Quantizer range (X_m); rms of amp (σ_x)

- Improvement of 6dB with every bit
- The range of the quantization must be adapted to the rms amplitude of the signal
  - Tradeoff between clipping and noise!
  - Often use pre-amp
  - Sometimes use analog auto gain controller (AGC)

Signal-to-Quantization-Noise Ratio

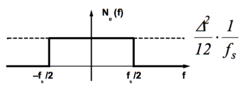
- Assuming full-scale sinusoidal input, we have

$$\text{SQNR} = \frac{P_{sig}}{P_{qnoise}} = \frac{\frac{1}{2}\left(\frac{2^B\Delta}{2}\right)^2}{\frac{\Delta^2}{12}} = 1.5\times 2^{2B} = 6.02B + 1.76 \text{ dB}$$

B (Number of Bits)	SQNR
8	50dB
12	74dB
16	98dB
20	122dB

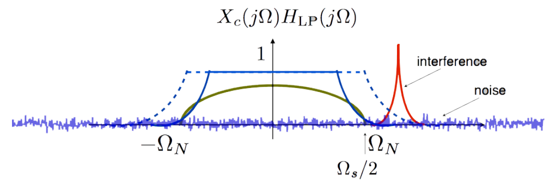
Quantization Noise Spectrum

- If the quantization error is "sufficiently random", it also follows that the noise power is uniformly distributed in frequency



- References
  - W. R. Bennett, "Spectra of quantized signals," Bell Syst. Tech. J., pp. 446-72, July 1988.
  - B. Widrow, "A study of rough amplitude quantization by means of Nyquist sampling theory," IRE Trans. Circuit Theory, vol. CT-3, pp. 266-76, 1956.

Non-Ideal Anti-Aliasing Filter



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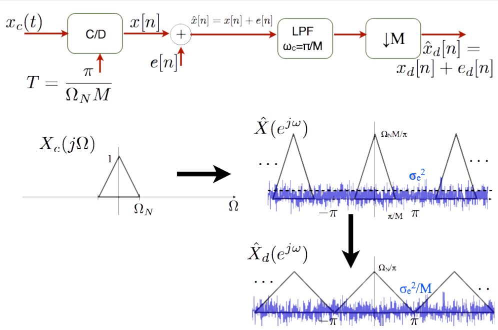
- Solution: Crop part of the signal and suffer from noise and interference

Quantization Noise with Oversampling



Quantization Noise with Oversampling

- Energy of $x_d[n]$ equals energy of x[n]
  - No filtering of signal!
- Noise variance is reduced by factor of M

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + 10\log_{10} M$$

- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Practical DAC

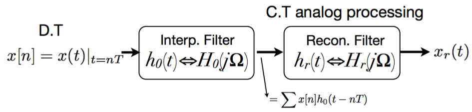
Practical DAC

D.T $\quad x[n] = x(t)|_{t=nT} \rightarrow$ [sinc pulse generator] $\rightarrow$ C.T

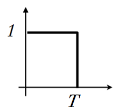
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

- Scaled train of sinc pulses
- Difficult to generate sinc → Too long!

Practical DAC



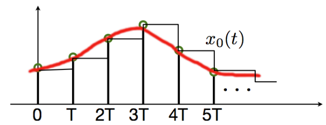
- $h_0(t)$ is finite length pulse → easy to implement
- For example: zero-order hold



$$H_0(j\Omega) = Te^{-j\Omega\frac{T}{2}}\mathrm{sinc}(\frac{\Omega}{\Omega_s})$$

Practical DAC

Zero-Order-Hold interpolation



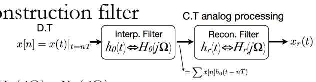
$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT) = h_0(t) * x_s(t)$$

Taking a FT:

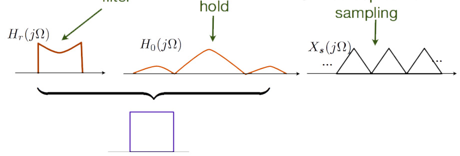
$$X_(j\Omega) = H_0(j\Omega)X_s(j\Omega)$$
$$= H_0(j\Omega)\frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))$$

Practical DAC

- Output of the reconstruction filter

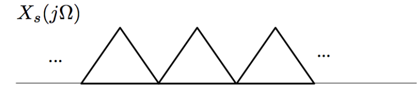


$$X_r(j\Omega) = H_r(j\Omega) \cdot H_0(j\Omega) \cdot X_s(j\Omega)$$
$$= \underbrace{H_r(j\Omega)}_{\text{recon filter}} \cdot \underbrace{Te^{-j\Omega\frac{T}{2}}\mathrm{sinc}(\frac{\Omega}{\Omega_s})}_{\text{from zero-order hold}} \cdot \underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))}_{\text{Shifted copies from sampling}}$$



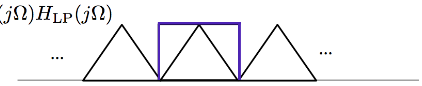
Practical DAC

$X_s(j\Omega)$



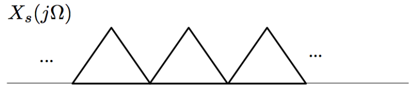
Ideally:

$X_s(j\Omega)H_{LP}(j\Omega)$

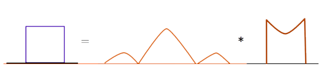


Practical DAC

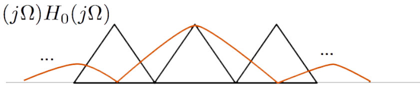
$X_s(j\Omega)$



Practically:

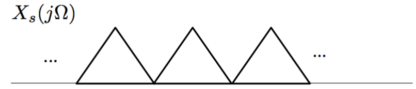


$X_s(j\Omega)H_0(j\Omega)$



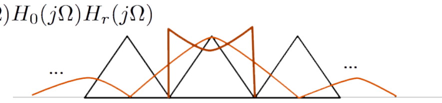
Practical DAC

$X_s(j\Omega)$

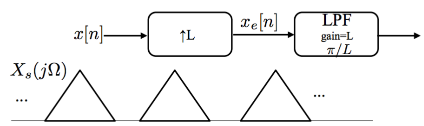


Practically:

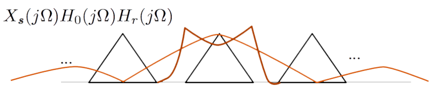
$X_s(j\Omega)H_0(j\Omega)H_r(j\Omega)$



Practical DAC with Upsampling

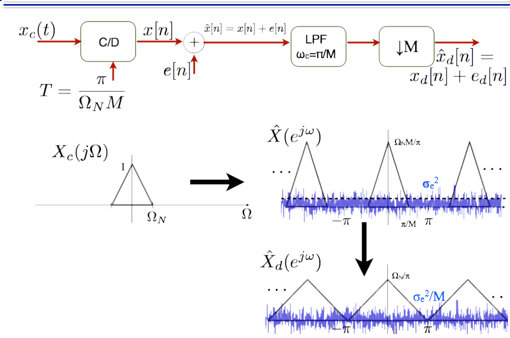


Practically:



Noise Shaping

Quantization Noise with Oversampling



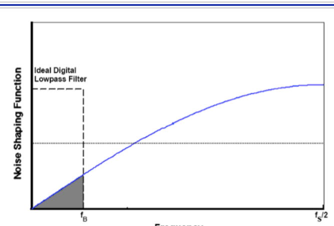
Quantization Noise with Oversampling

- Energy of $x_d[n]$ equals energy of x[n]
  - No filtering of signal!
- Noise variance is reduced by factor of M

$$\text{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + 10\log_{10} M$$

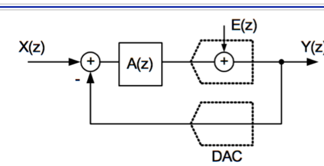
- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Noise Shaping



- Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- Key: Feedback

Noise Shaping Using Feedback



$$Y(z) = E(z) + A(z)X(z) - A(z)Y(z)$$

$$= E(z)\frac{1}{1+A(z)} + X(z)\frac{A(z)}{1+A(z)}$$

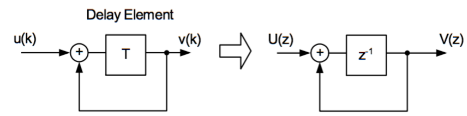
$$= \underbrace{E(z)H_E(z)}_{\text{Noise Transfer Function}} + \underbrace{X(z)H_X(z)}_{\text{Signal Transfer Function}}$$

Noise Shaping Using Feedback

$$Y(z)=E(z)\underbrace{\frac{1}{1+A(z)}}_{\substack{Noise\\Transfer\\Function}}+X(z)\underbrace{\frac{A(z)}{1+A(z)}}_{\substack{Signal\\Transfer\\Function}}$$

- Objective
  - Want to make STF unity in the signal frequency band
  - Want to make NTF "small" in the signal frequency band
- If the frequency band of interest is around DC (0...f_B) we achieve this by making |A(z)|>>1 at low frequencies
  - Means that NTF << 1
  - Means that STF ≅ 1

Discrete Time Integrator

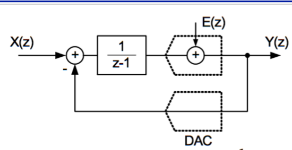


$$v(k)=u(k-1)+v(k-1) \qquad V(z)=z^{-1}U(z)+z^{-1}V(z)$$

$$\frac{V(z)}{U(z)}=\frac{z^{-1}}{1-z^{-1}}=\frac{1}{z-1} \qquad z=e^{j\omega T}$$

- "Infinite gain" at DC (ω=0, z=1)

First Order Sigma-Delta Modulator

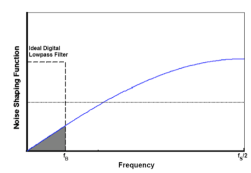


$$Y(z)=E(z)\frac{1}{1+\frac{1}{z-1}}+X(z)\frac{\frac{1}{z-1}}{1+\frac{1}{z-1}}$$
$$=E(z)\left(1-z^{-1}\right)+X(z)z^{-1}$$

- Output is equal to delayed input plus filtered quantization noise

NTF Frequency Domain Analysis

$$H_e(z)=1-z^{-1}$$
$$H_e(j\omega)=\left(1-e^{-j\omega T}\right)=2e^{-j\omega T/2}\left(\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{2}\right)$$
$$=2e^{-j\frac{\omega T}{2}}\left(j\,sin\left(\frac{\omega T}{2}\right)\right)=2\,sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T-\pi}{2}}$$
$$|H_e(f)|=2|sin(\pi fT)|=2\left|sin\left(\pi\frac{f}{f_s}\right)\right|$$



- "First order noise Shaping"
  - Quantization noise is attenuated at low frequencies, amplified at high frequencies

In-Band Quantization Noise

- Question: If we had an ideal digital lowpass, what is the achieved SQNR as a function of oversampling ratio?
- Can integrate shaped quantization noise spectrum up to f_B and compare to full-scale signal

$$P_{qnoise}=\int_0^{f_B}\frac{\Delta^2}{12}\cdot\frac{2}{f_s}\cdot\left[2\,sin\left(\pi\frac{f}{f_s}\right)\right]^2 df$$
$$\cong\int_0^{f_B}\frac{\Delta^2}{12}\cdot\frac{2}{f_s}\cdot\left[2\pi\frac{f}{f_s}\right]^2 df$$
$$\cong\frac{\Delta^2}{12}\cdot\frac{\pi^2}{3}\left[\frac{2f_B}{f_s}\right]^3=\frac{\Delta^2}{12}\cdot\frac{\pi^2}{3}\cdot\frac{1}{M^3}$$

In-Band Quantization Noise

- Assuming a full-scale sinusoidal signal, we have

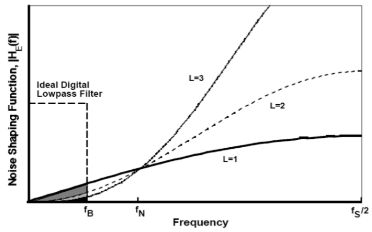
$$SQNR\cong\frac{P_{sig}}{P_{qnoise}}=\frac{\frac{1}{2}\left(\frac{\left(2^B-1\right)\Delta}{2}\right)^2}{\frac{\Delta^2}{12}\cdot\frac{\pi^2}{3}\cdot\frac{1}{M^3}}=1.5\times\left(2^B-1\right)^2\times\underbrace{\frac{3}{\pi^2}\times M^3}_{\substack{Due\ to\ noise\\shaping\ \&\\digital\ filter}}$$
$$\cong 1.76+6.02B-5.2+30\log(M)\quad[dB]\quad(for\ large\ B)$$

- Each 2x increase in M results in 8x SQNR improvement
  - Also added ½ bit resolution

Higher Order Noise Shaping

- Lth order noise transfer function

$$H_E(z) = \left(1 - z^{-1}\right)^L$$



Big Ideas

- Multi-Rate Filter Banks (con't)
  - Operating on different frequency bands at lower sampling rates
- Data Converters
  - Oversampling to reduce interference and quantization noise → increase ENOB (effective number of bits)
  - Practical DACs use practical interpolation and reconstruction filters with oversampling
- Noise Shaping
  - Use feedback to reduce oversampling factor

Admin

- HW 4 extended to Tuesday at midnight
  - Typo in code in MATLAB problem, corrected handout
  - See Piazza for more information
- New tentative HW schedule posted