ESE 531: Digital Signal Processing

Lec 13: February 23st, 2017

Frequency Response of LTI Systems

Penn ESE 531 Spring 2017 – Khanna
Adapted from M. Lustig, EECS Berkeley

Lecture Outline

- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

Frequency Response of LTI System

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$ defines magnitude and phase change at each frequency

Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

- And a phase response

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

Phase Response

- Limit the range of the phase response

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] \leq \pi.$$

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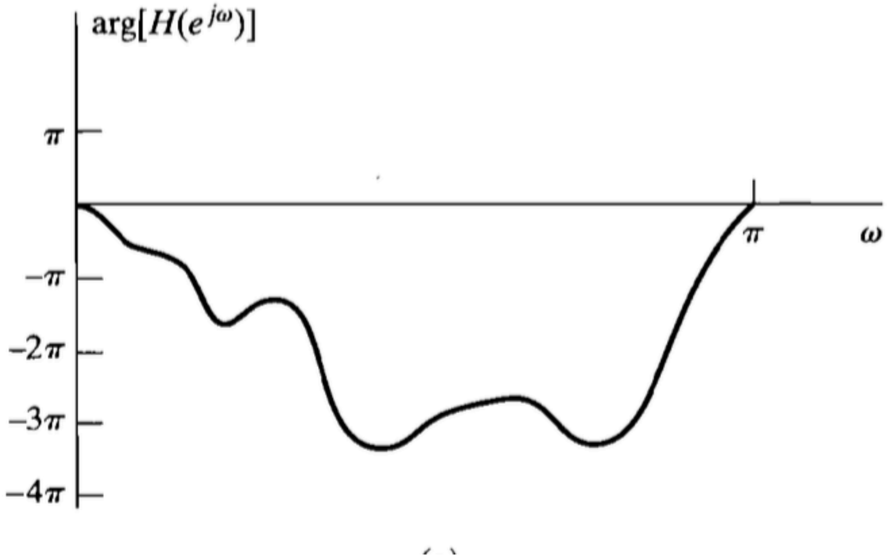

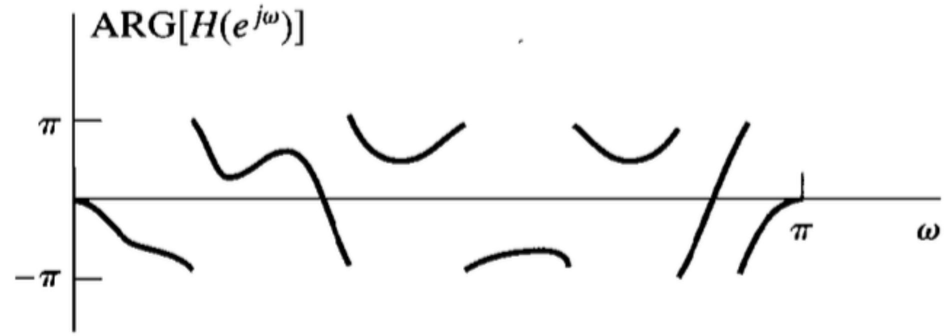

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

- More later…

Linear Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = |\frac{b_0}{a_0}| \cdot \frac{\prod_{k=0}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=0}^{N} |1 - d_k e^{-j\omega}|}$$

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Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

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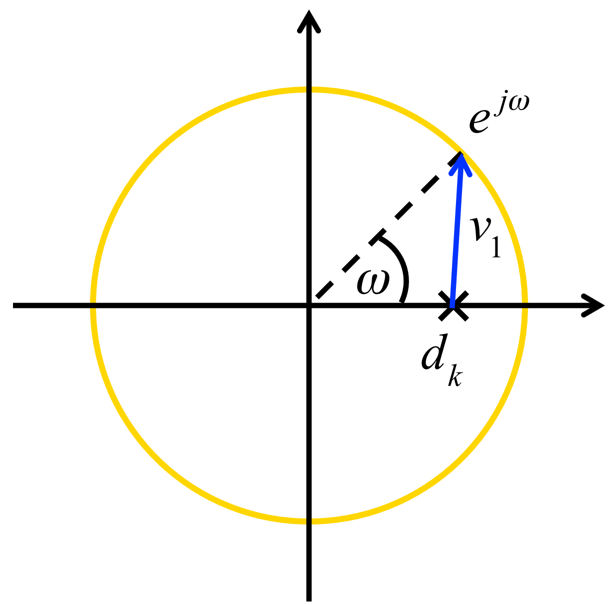
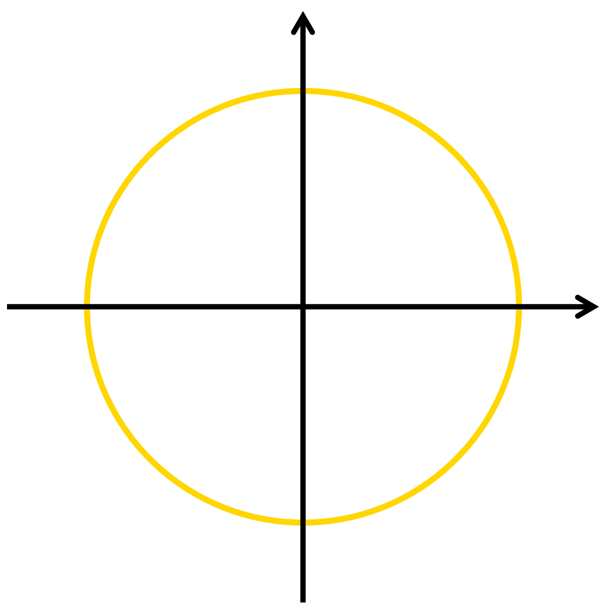


Magnitude Response Example

$$H(z) = 0.05\frac{1+z^{-1}}{1-0.9z^{-1}}$$

$$|H(z)| = 0.05\frac{|v_2|}{|v_1|}$$



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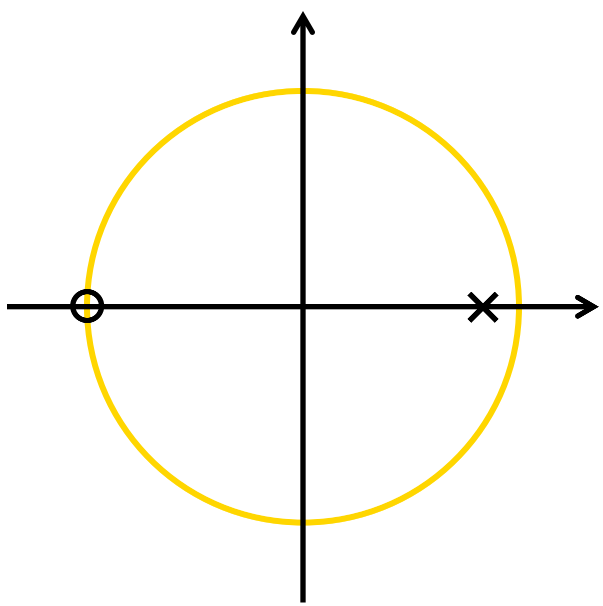

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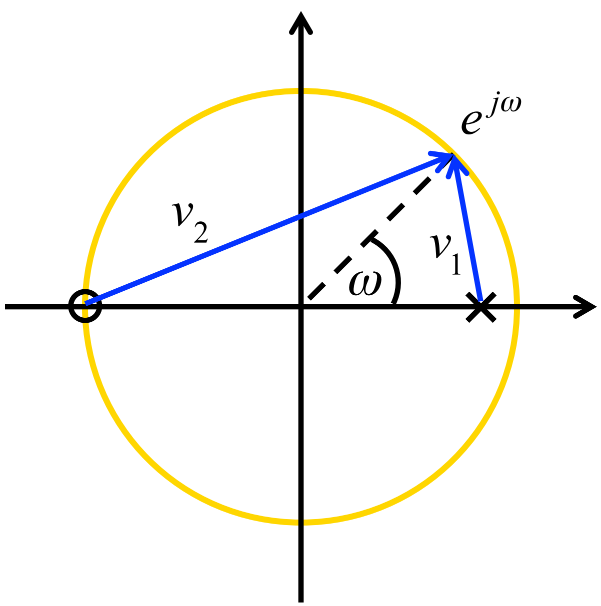

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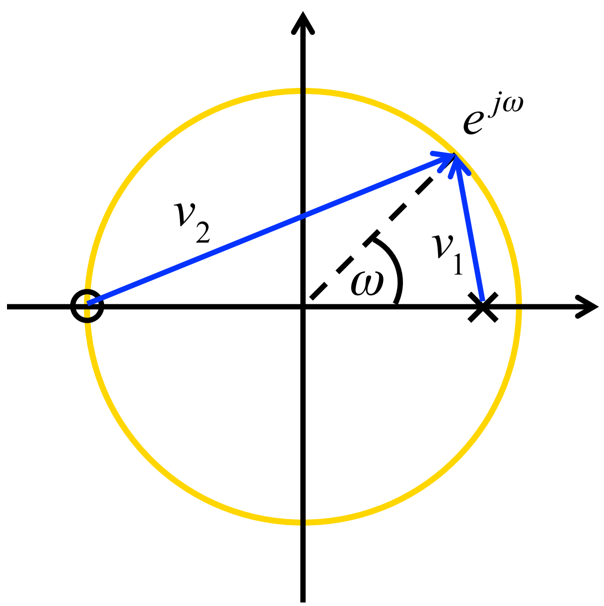

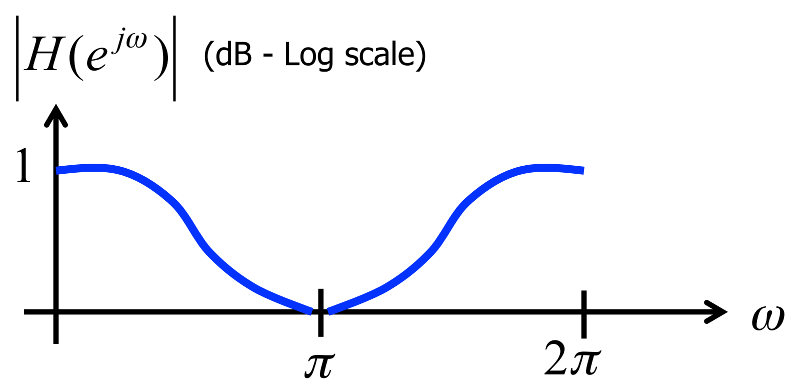

Simple Low Pass Filter

$$H_{LP}(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}} \qquad |\alpha| < 1$$

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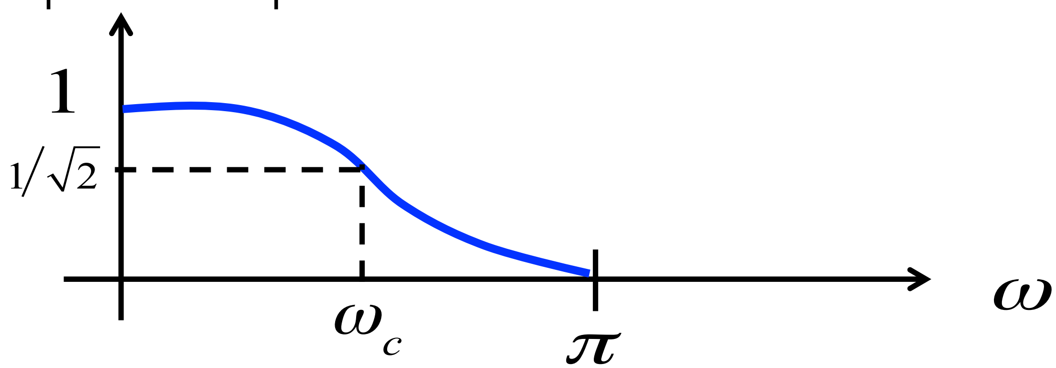
$\left|H(e^{j\omega})\right|$ (dB - Log scale)



w_c is the 3dB cutoff frequency $\quad \alpha = \dfrac{1-\sin(\omega_c)}{\cos(\omega_c)}$

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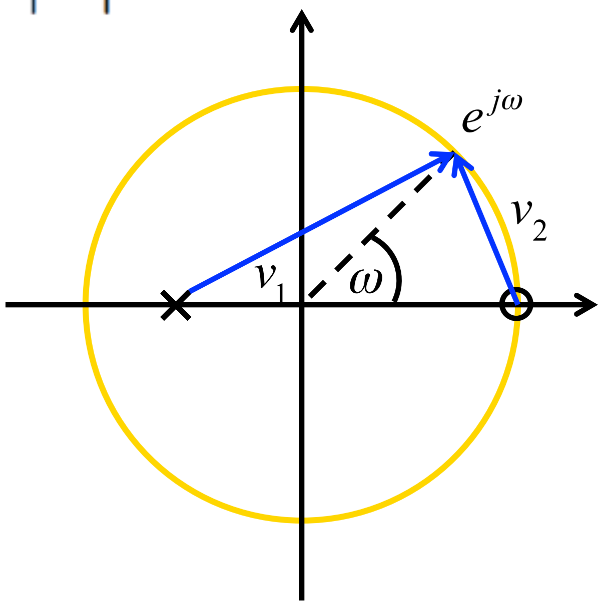

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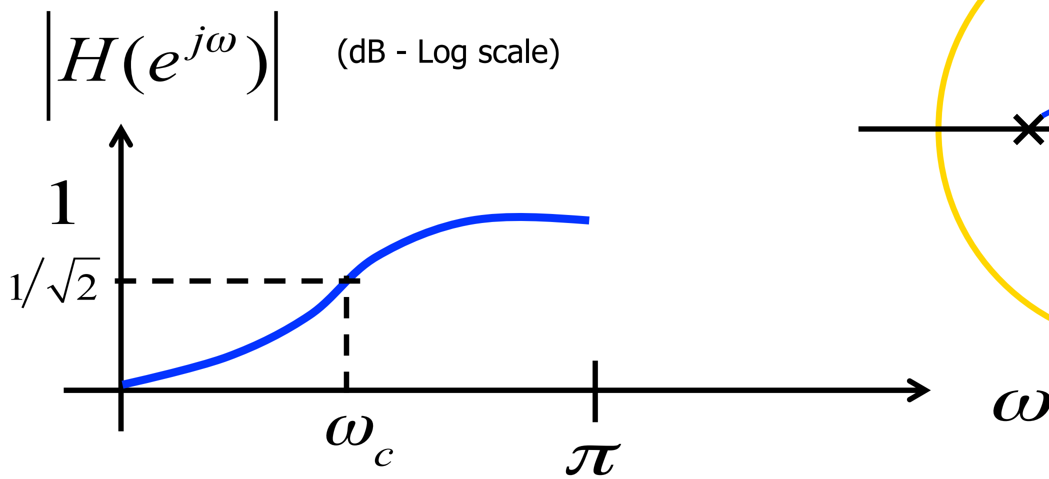

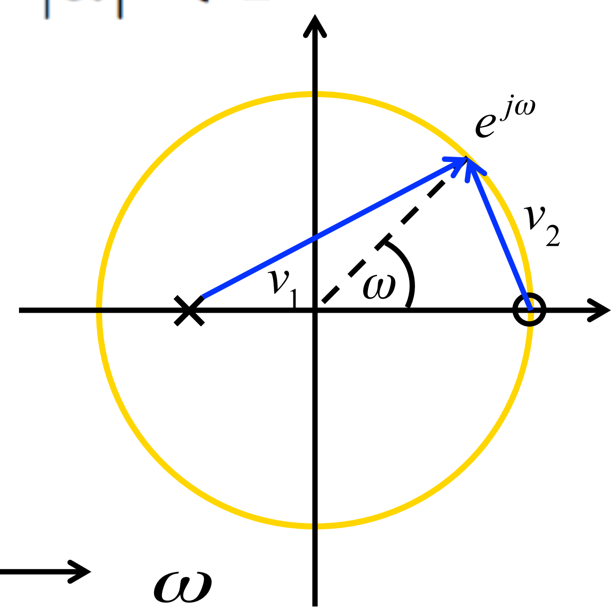

w_c is the 3dB cutoff frequency $\qquad \alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$

Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$

$$\cos(\omega_0) = \beta$$

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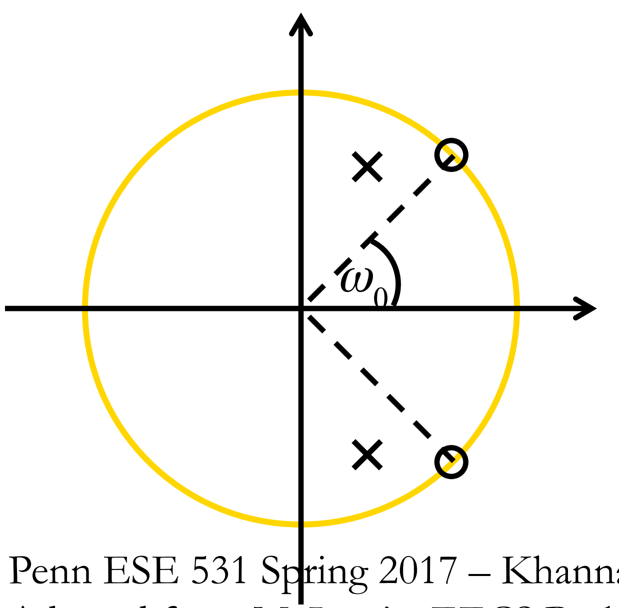

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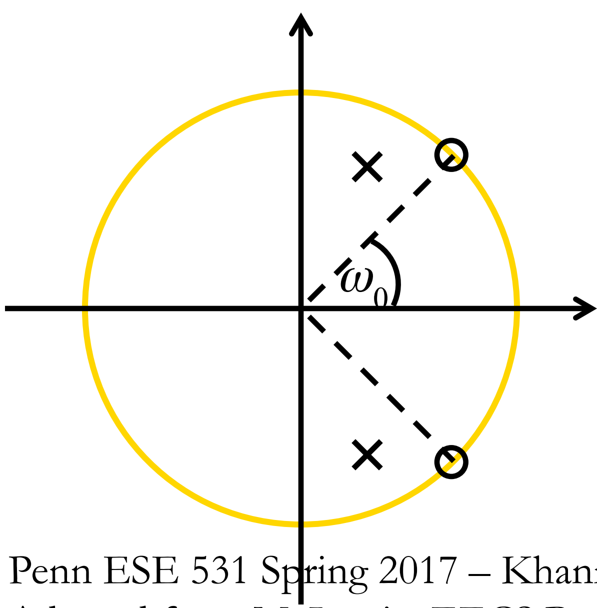

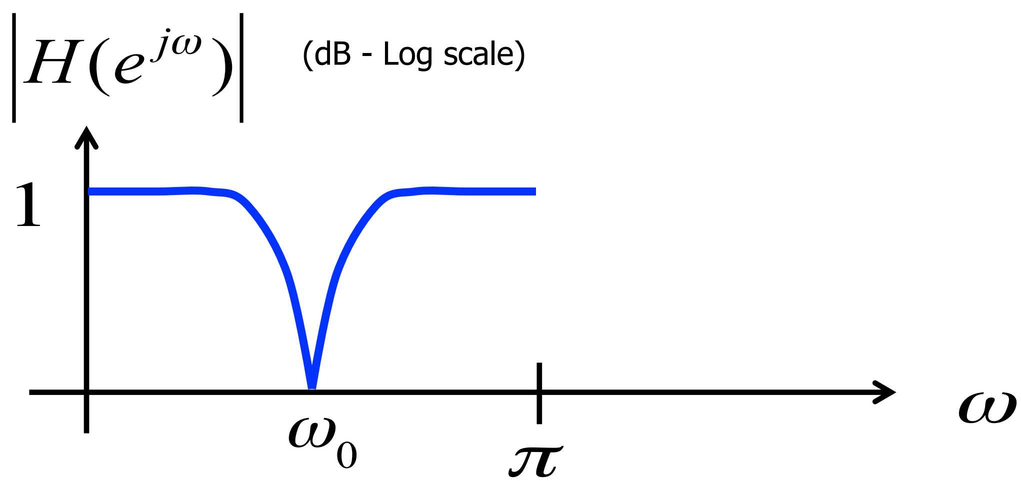

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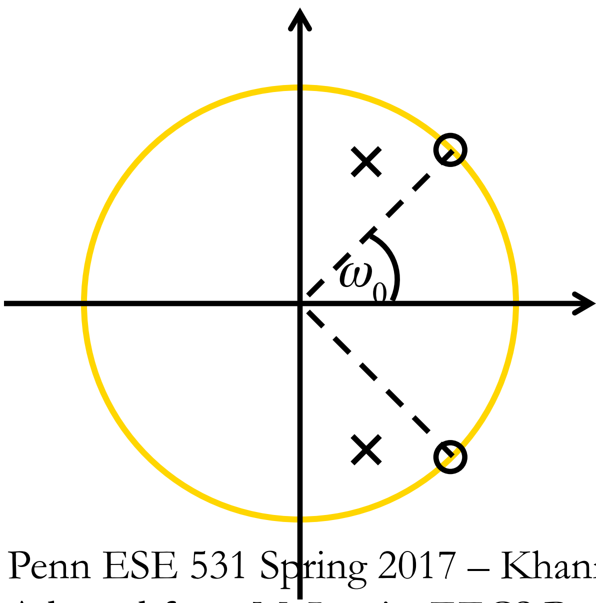
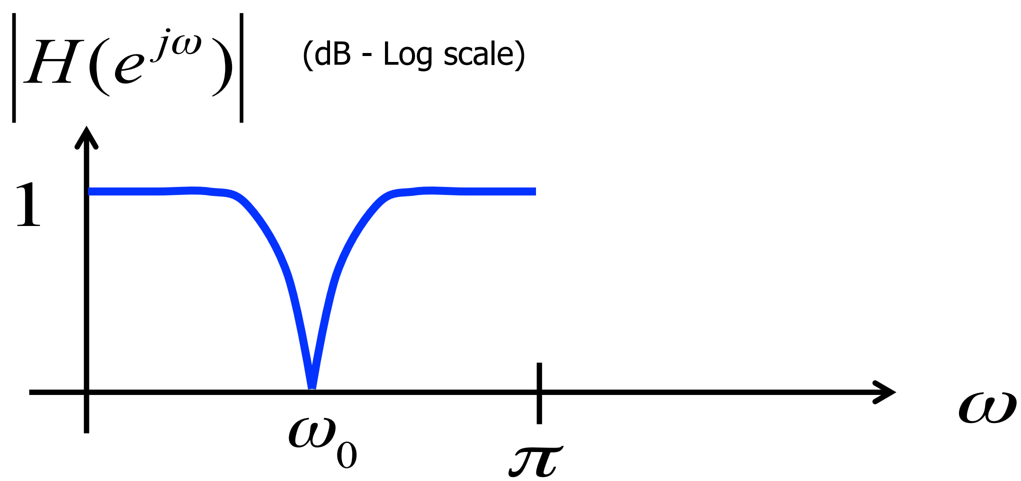
Note:

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$





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Note: As $\alpha \to 1$ poles approach zeros

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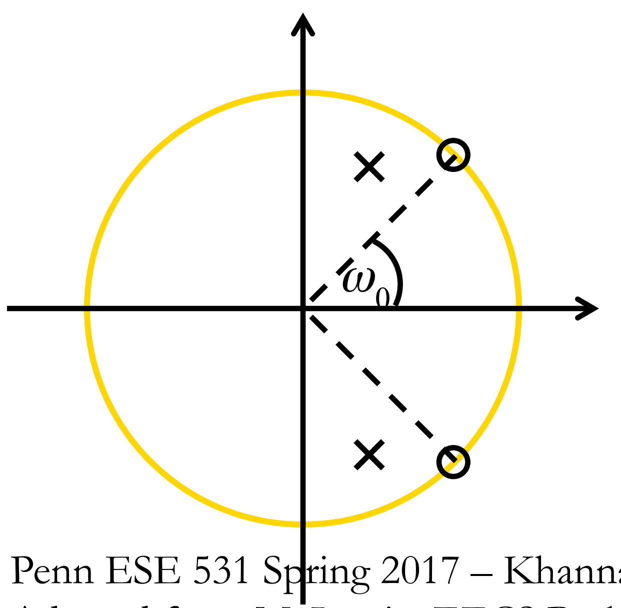

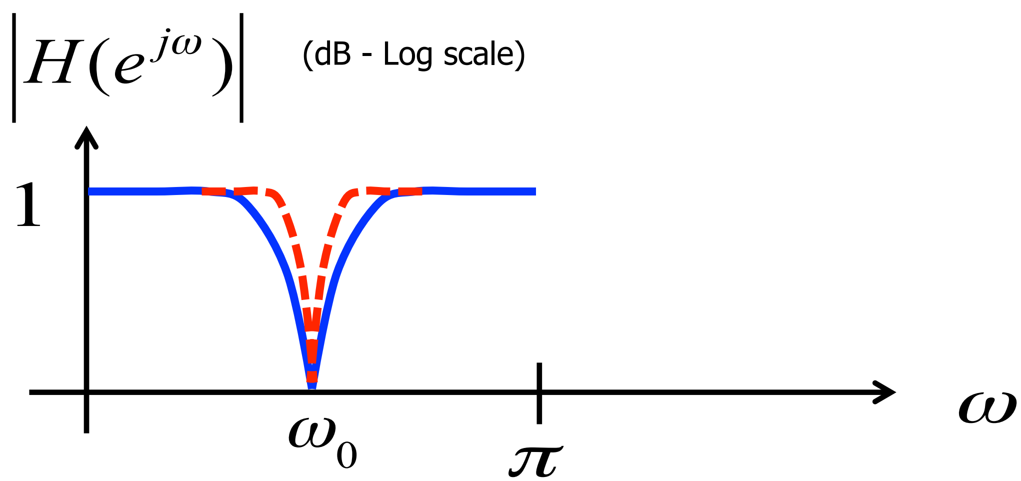

Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

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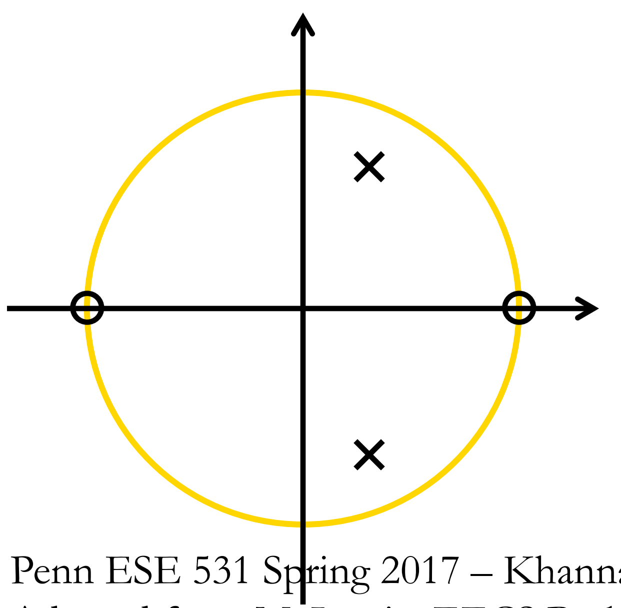

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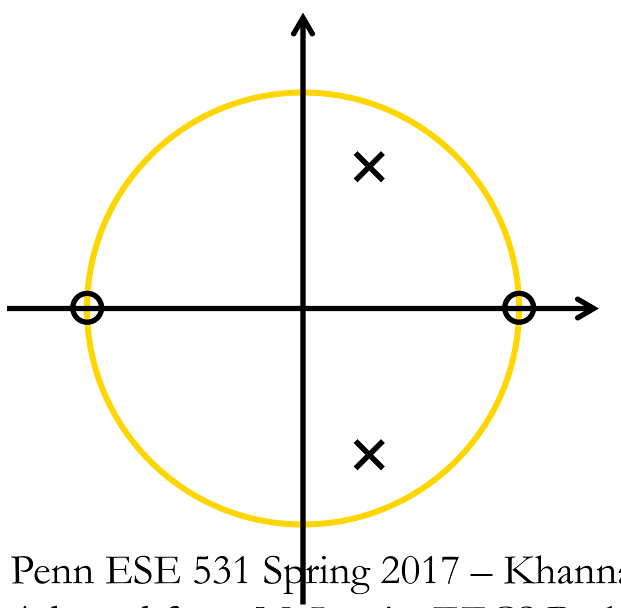

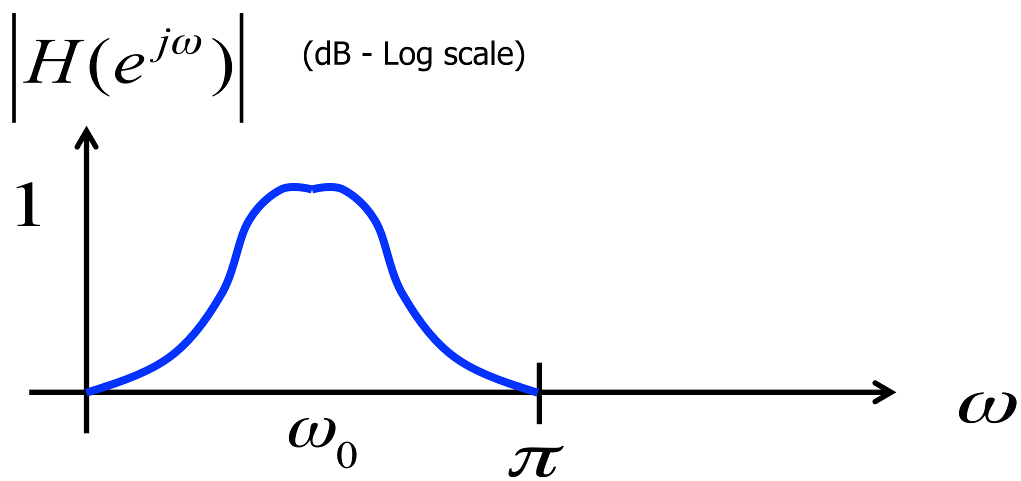

$$\cos(\omega_0) = \beta$$

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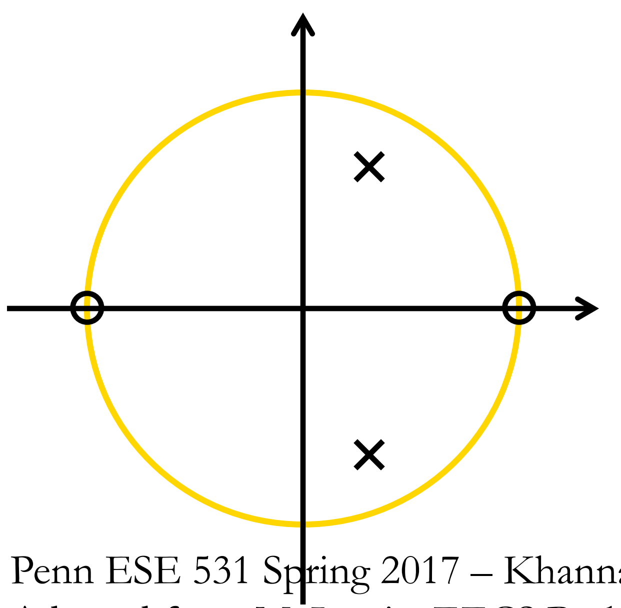
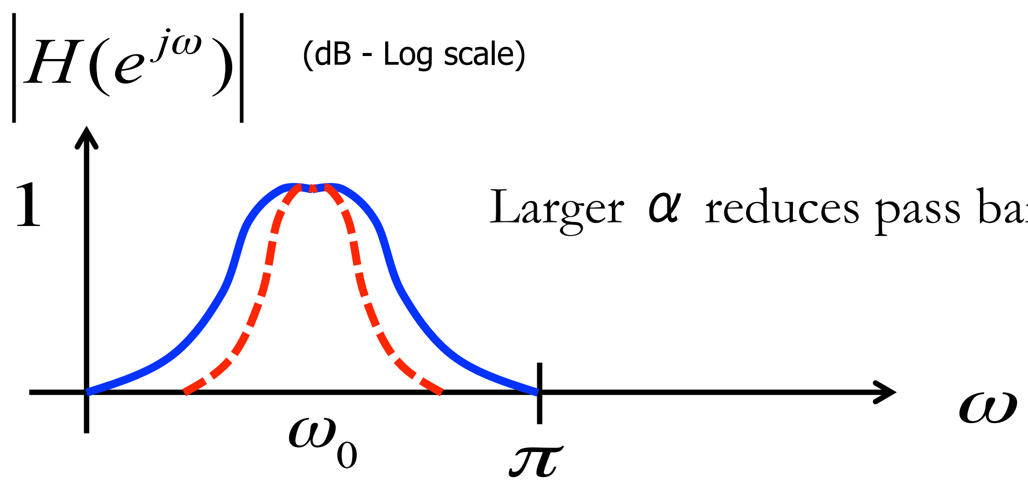




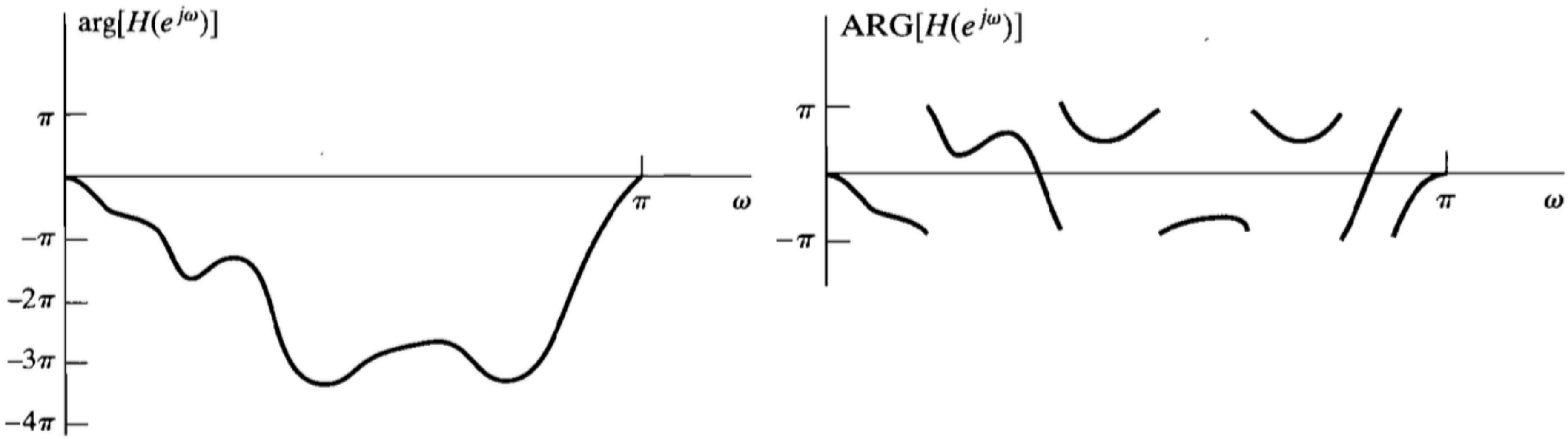
Larger α reduces pass band

$\cos(\omega_0) = \beta$

Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \le \pi.$$



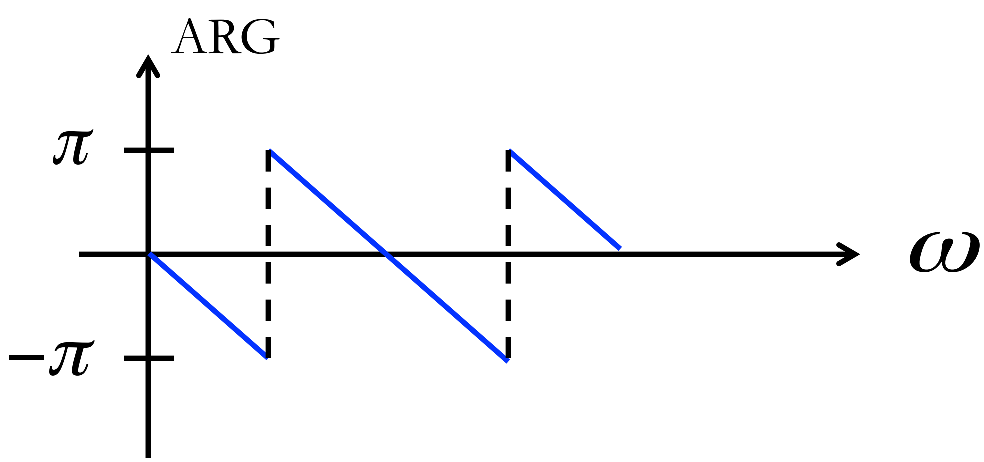
Phase Response Example

$$H(e^{j\omega}) = e^{-j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

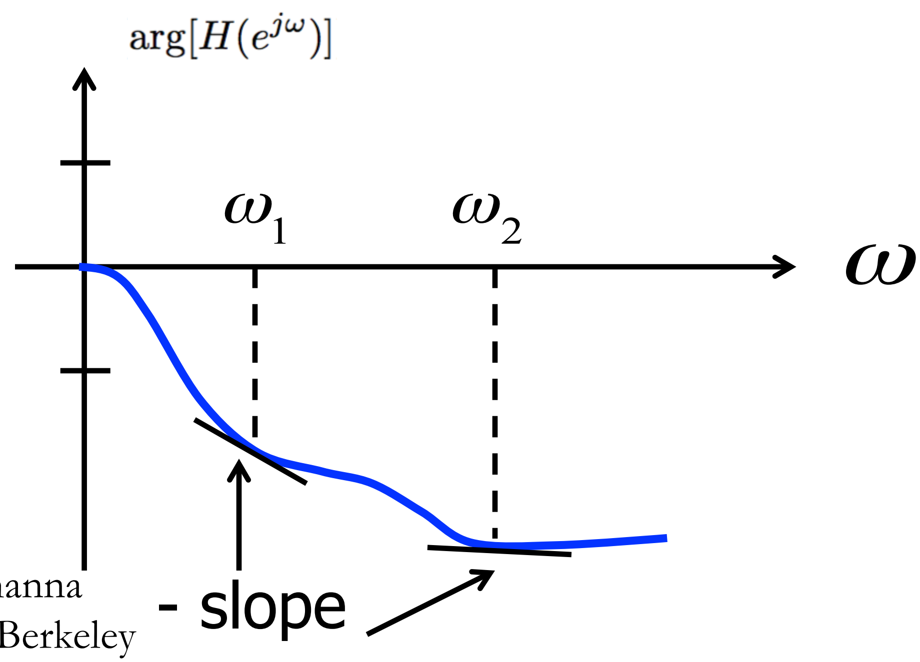
ARG is the wrapped phase
arg is the unwrapped phase



Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



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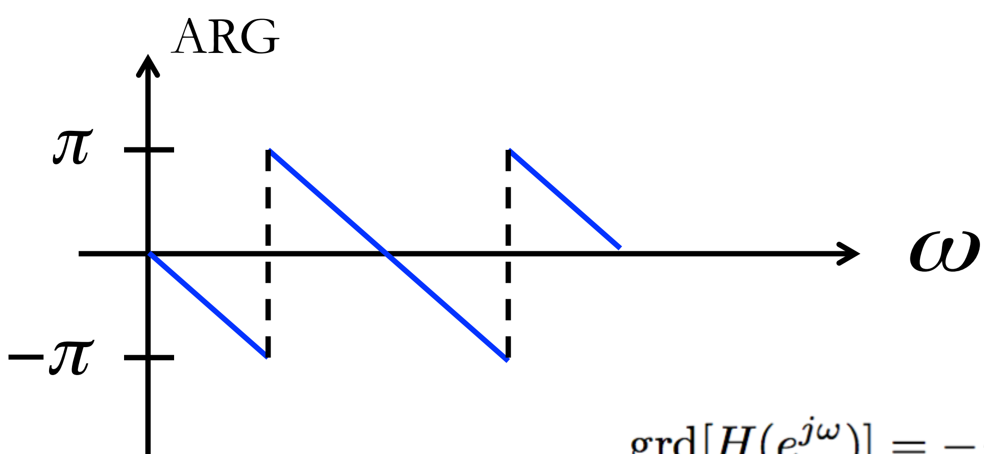

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For linear phase system, group delay is n_d

Group Delay

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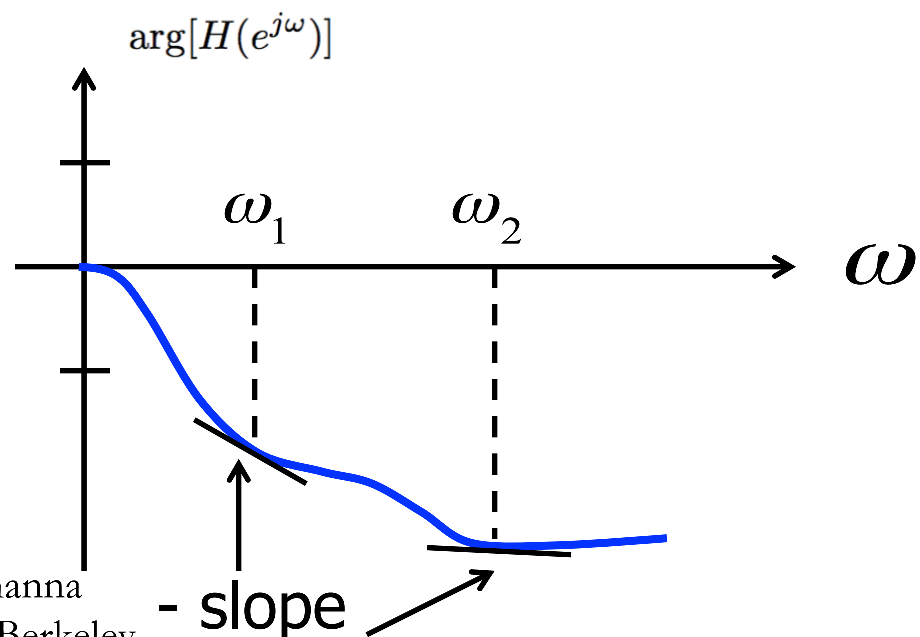

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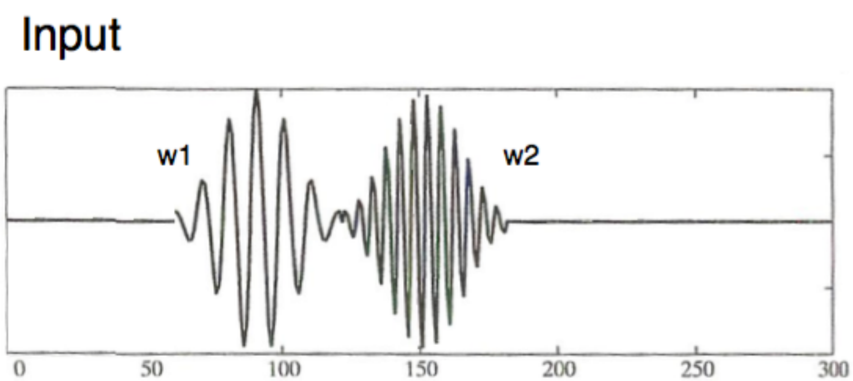

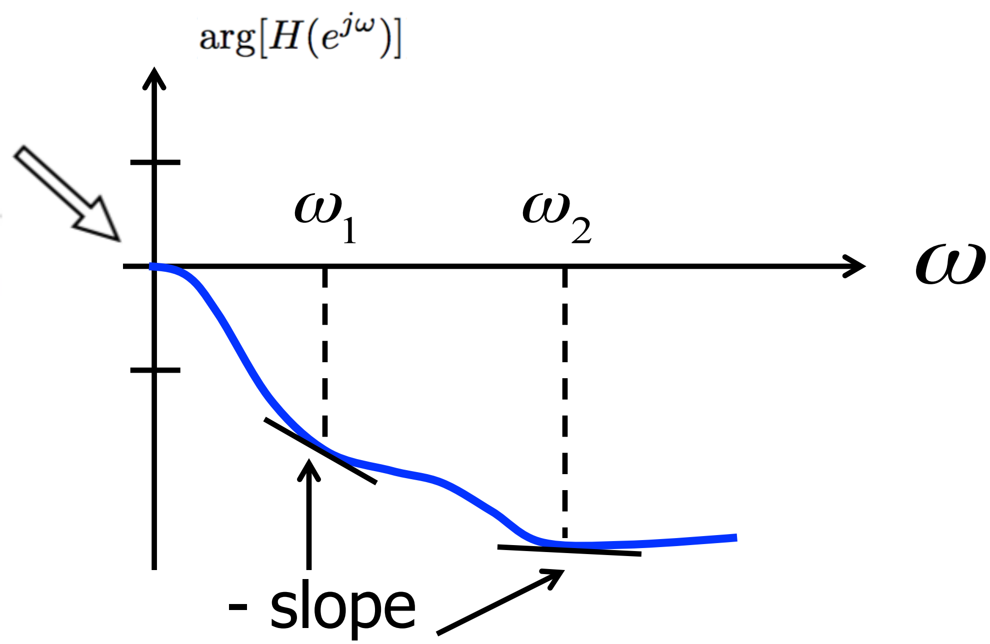

Group Delay

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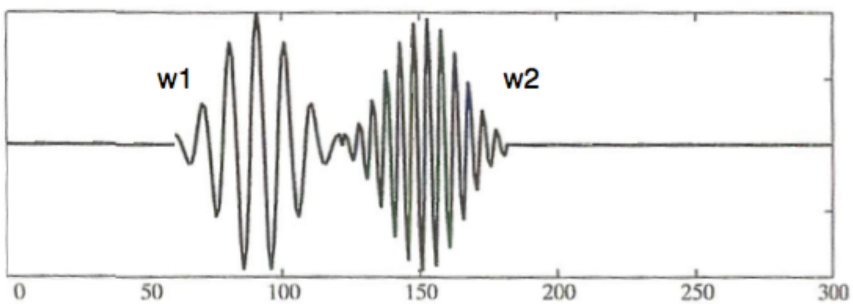
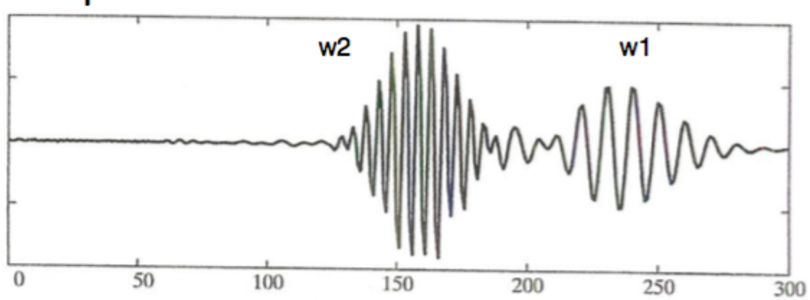
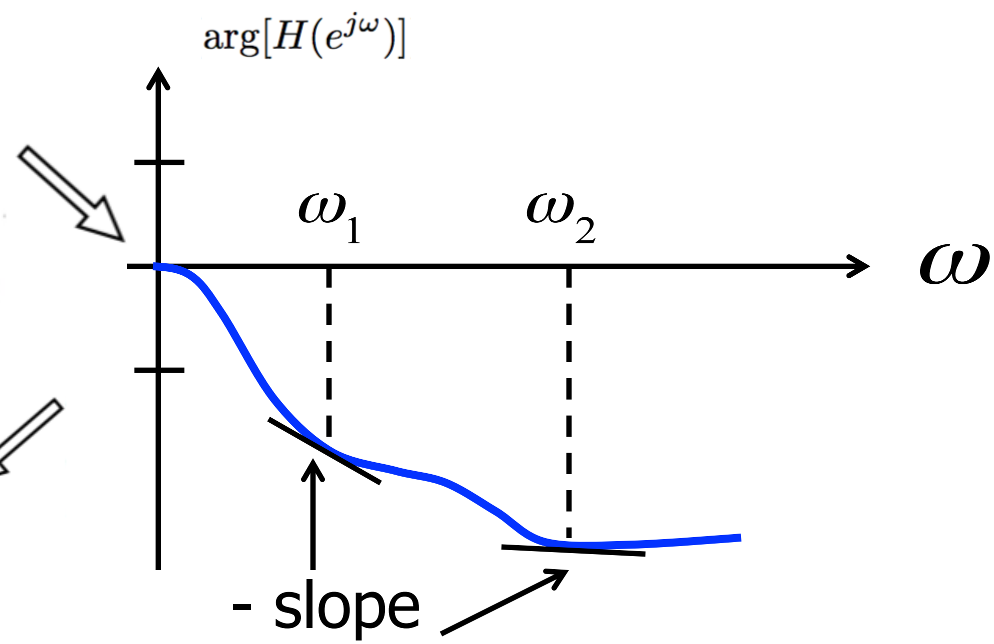
Input



Output





Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \qquad H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

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arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^{M} \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \arg[1 - d_k e^{-j\omega}]$$

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Group Delay Math

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- Look at each factor:

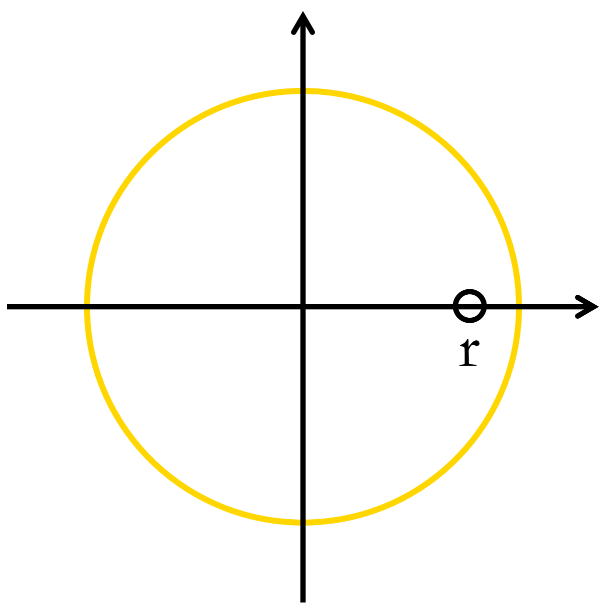
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}]$$

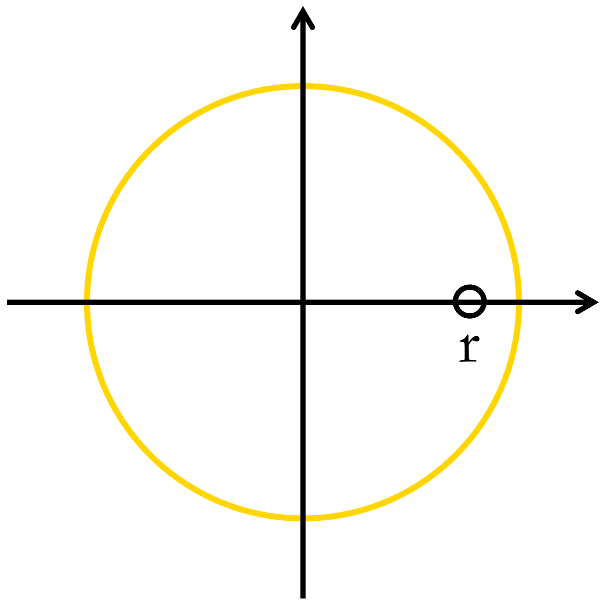


Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \arg[e^{j\omega}-r] - \arg[e^{j\omega}]$$

$$\underbrace{\arg[e^{j\omega}-r]}_{\varphi} \qquad \underbrace{\arg[e^{j\omega}]}_{\omega}$$



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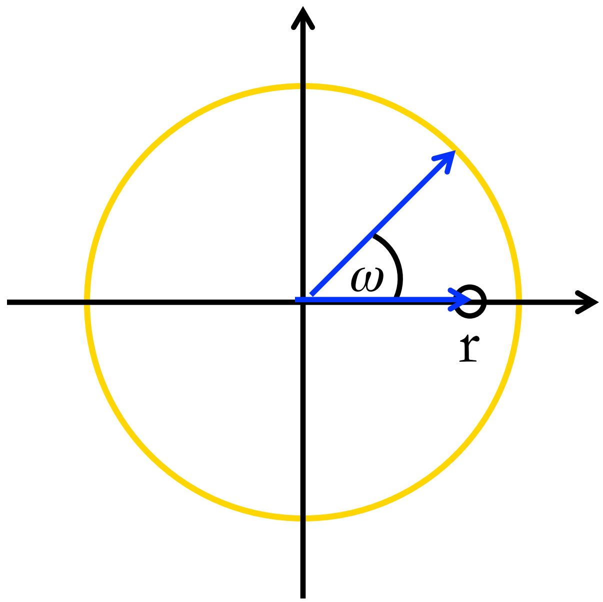

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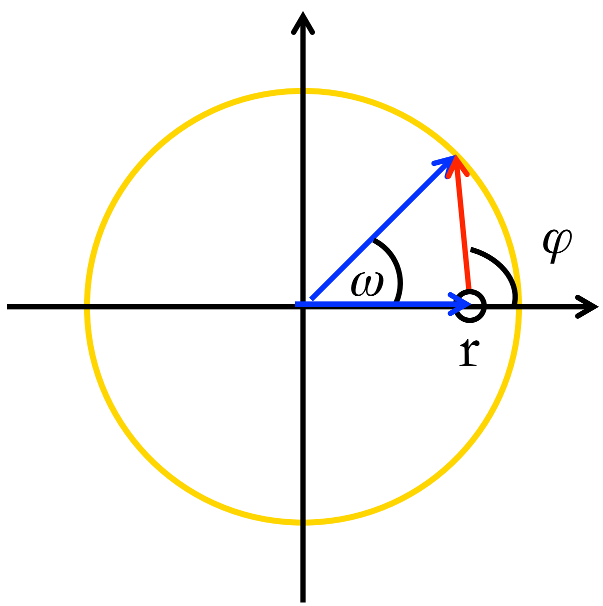

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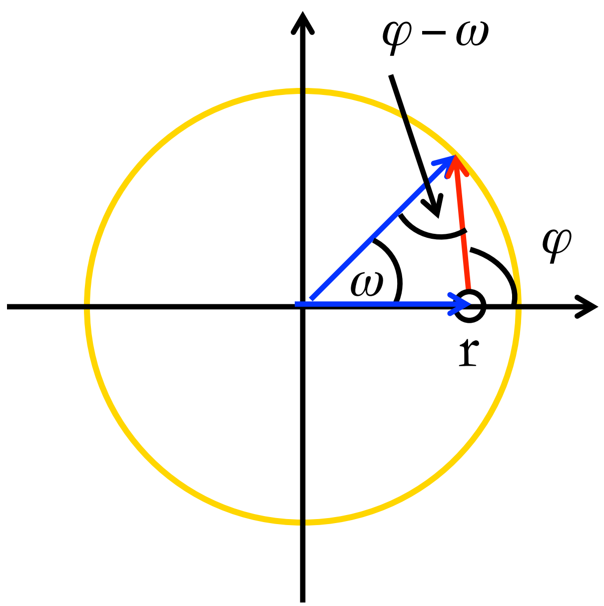
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



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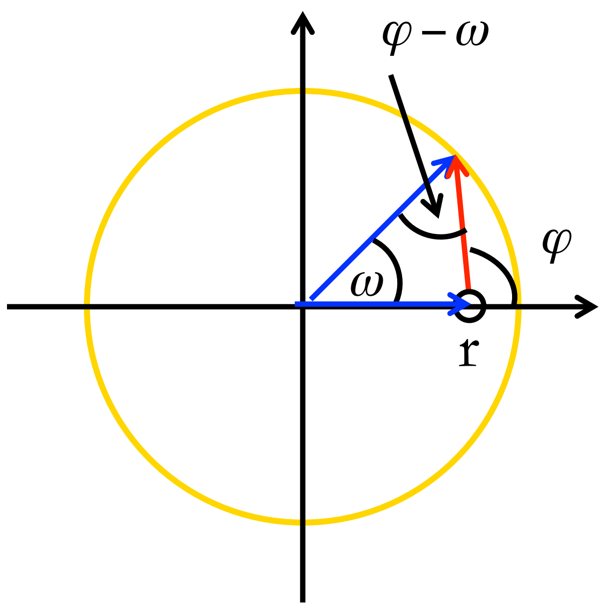

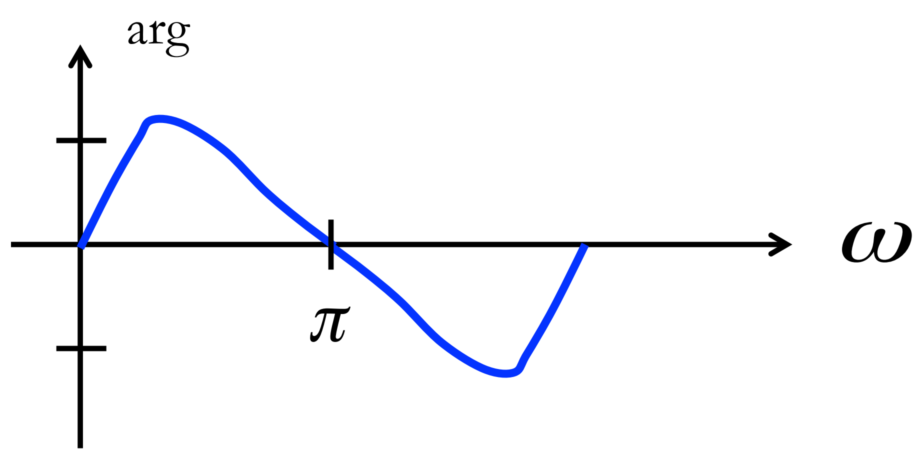

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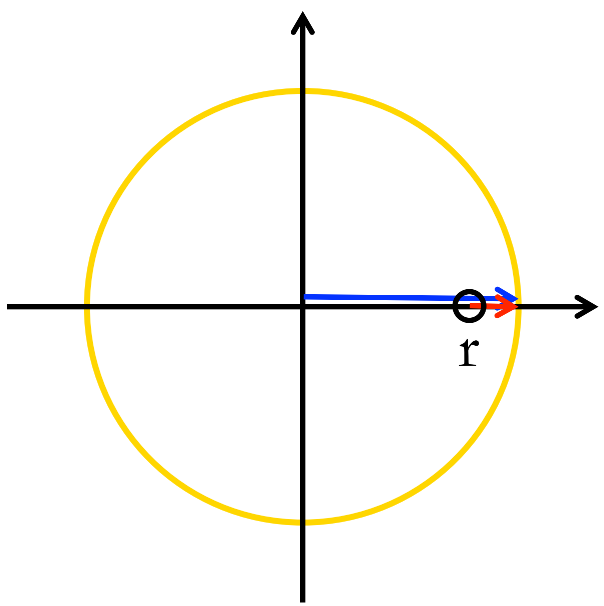
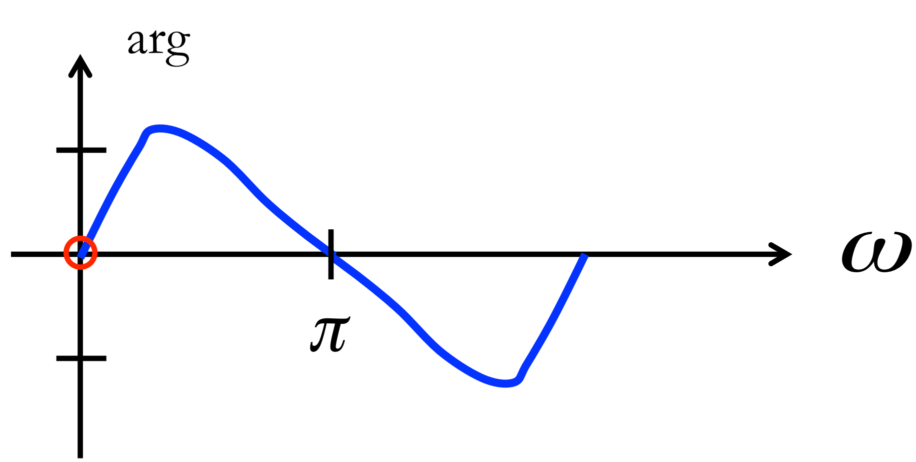
$\omega = 0$





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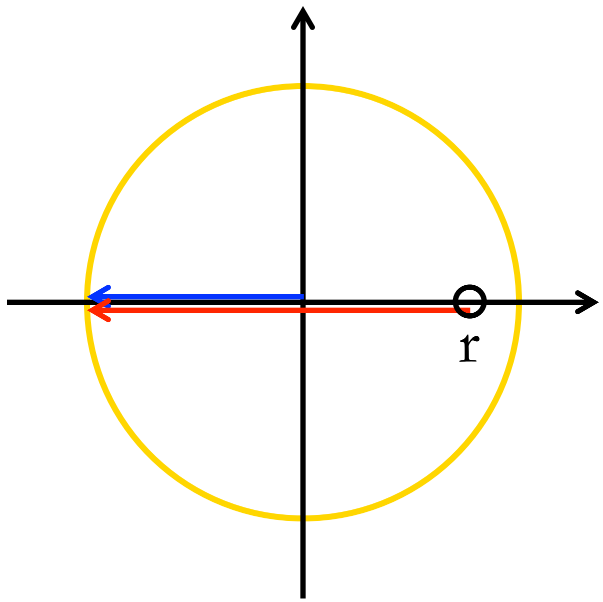
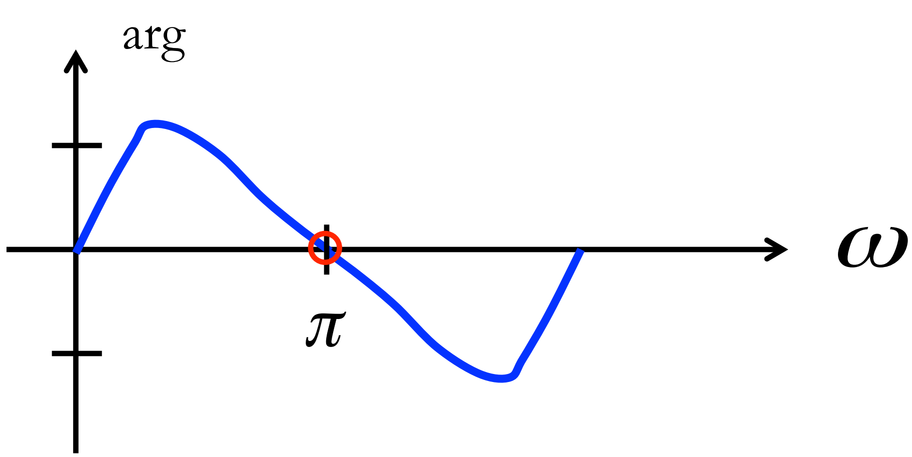
$\omega = \pi$





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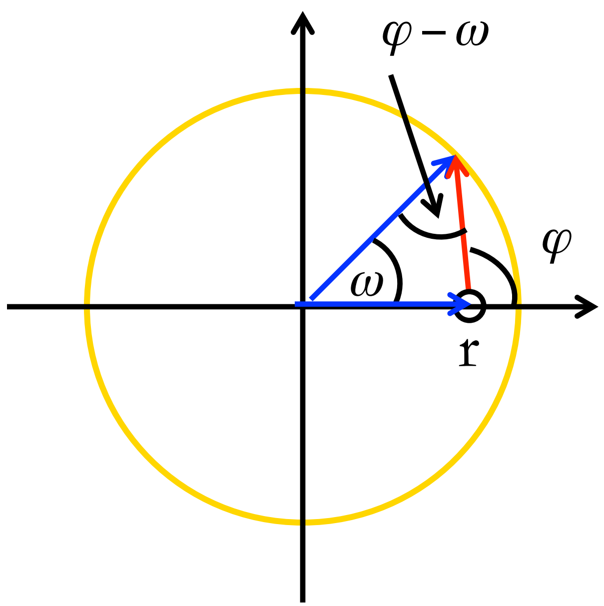

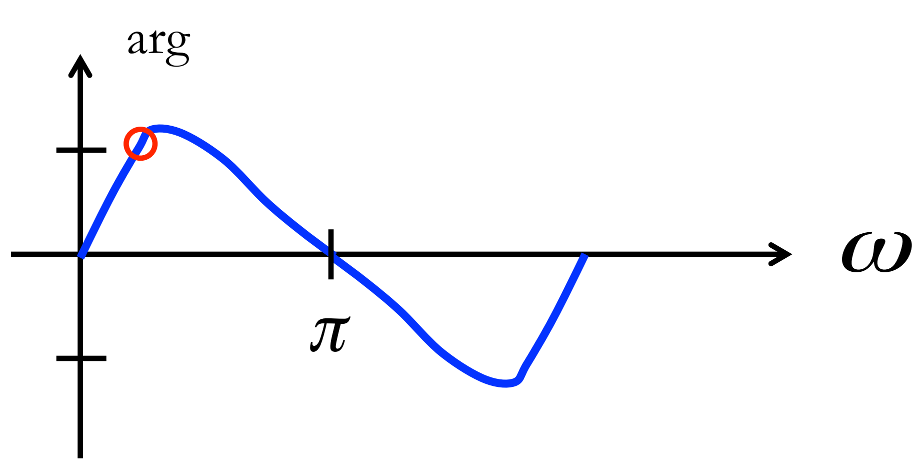

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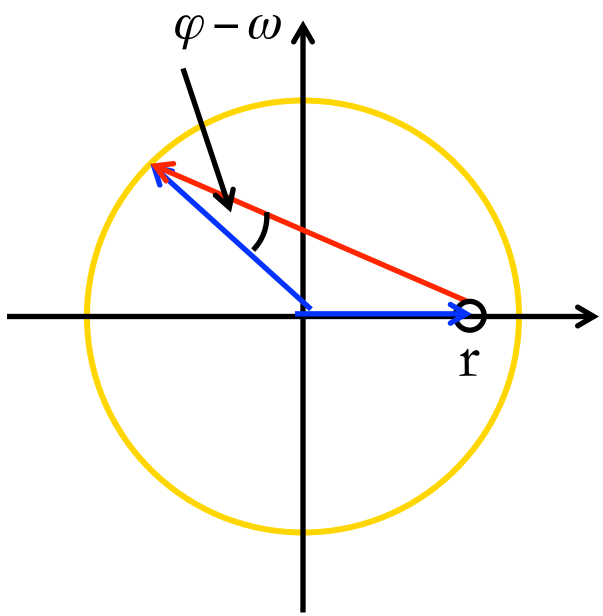

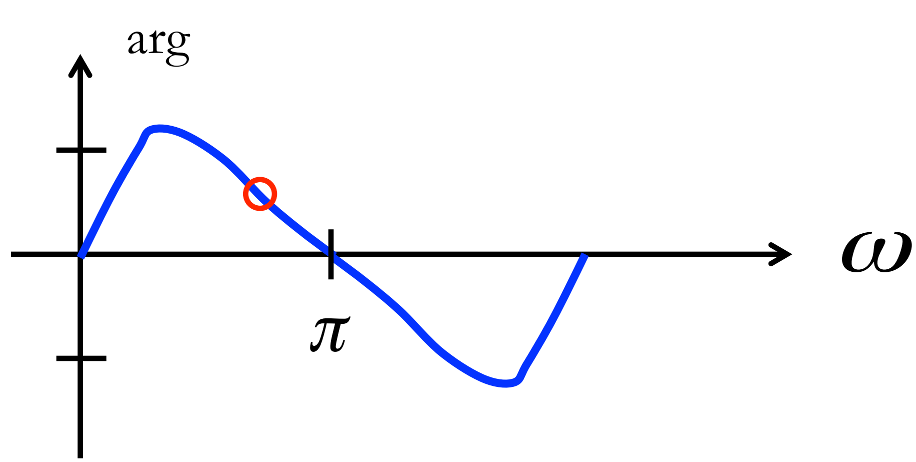

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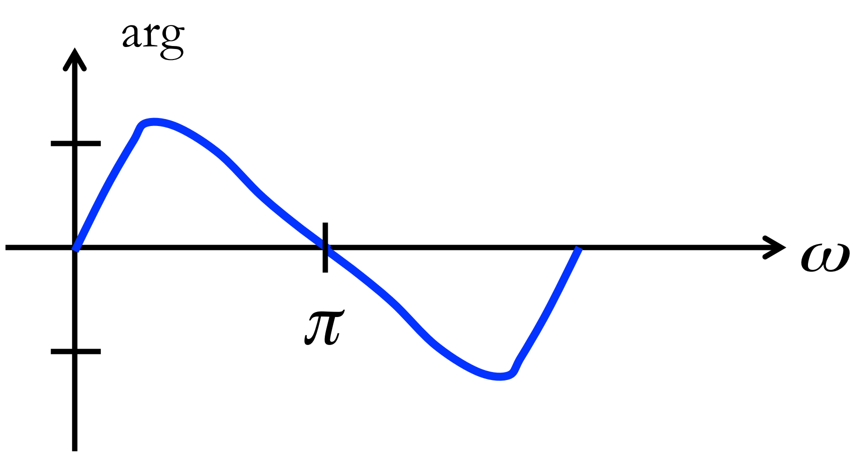
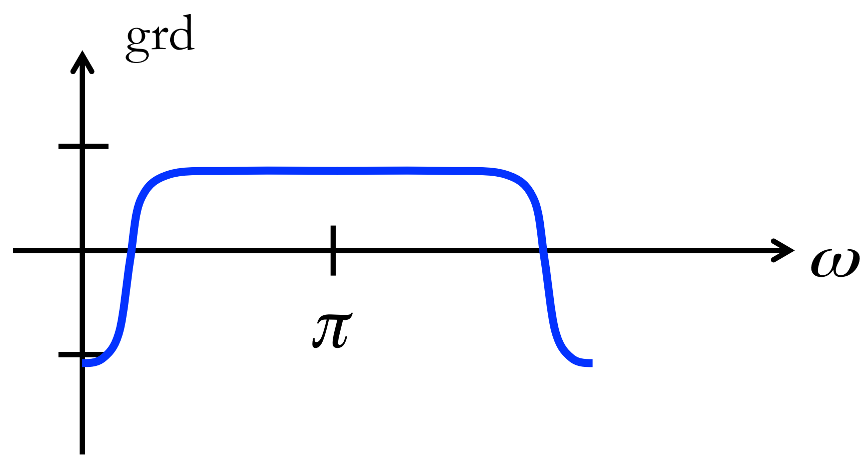




Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-j\omega}]$$

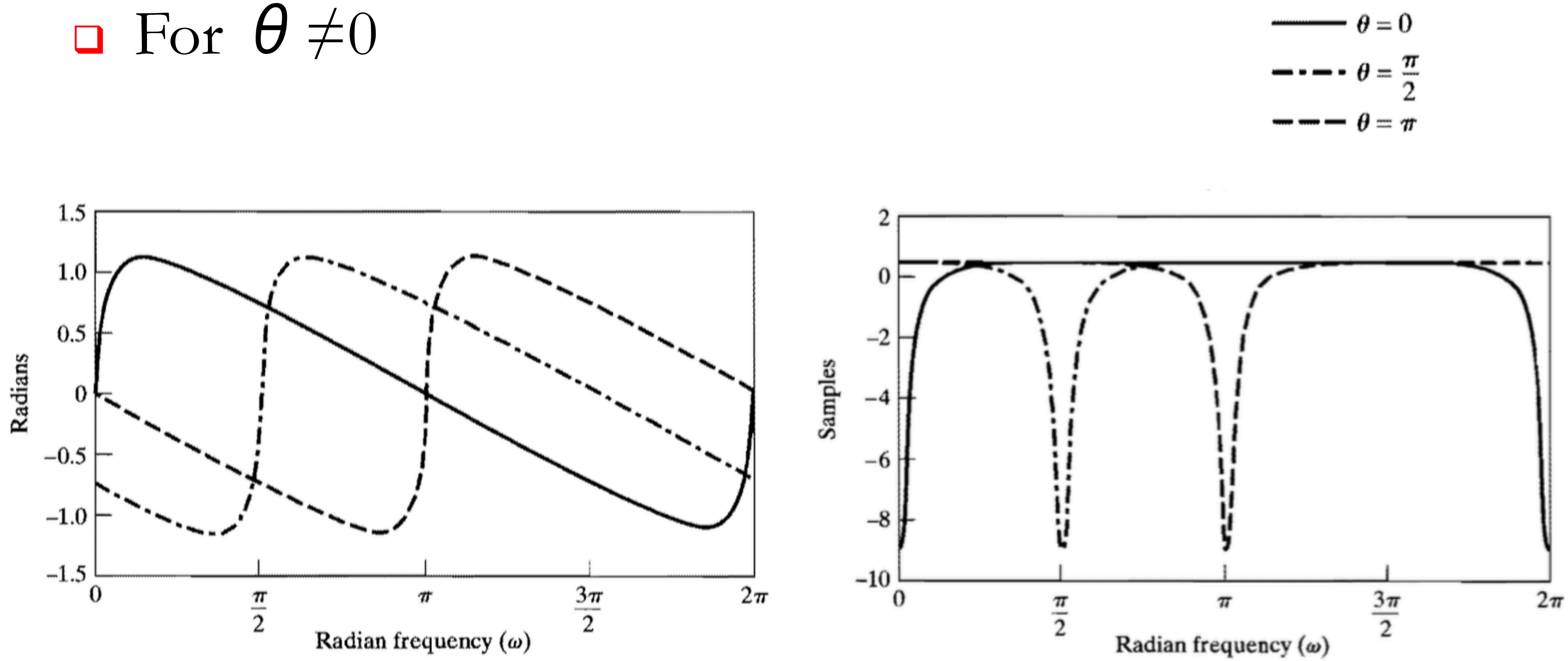
- Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

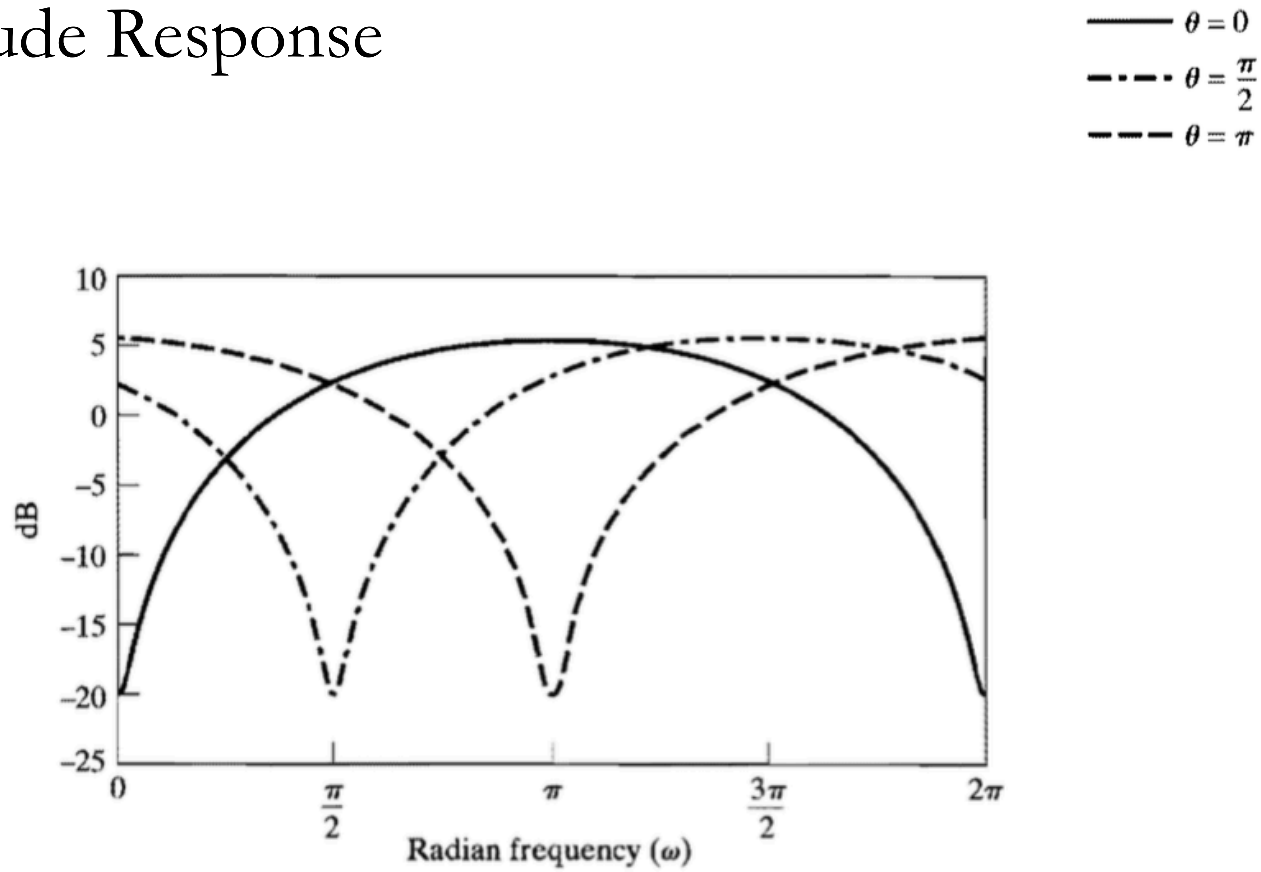
Example: Zero on Real Axis

- For $\theta \neq 0$



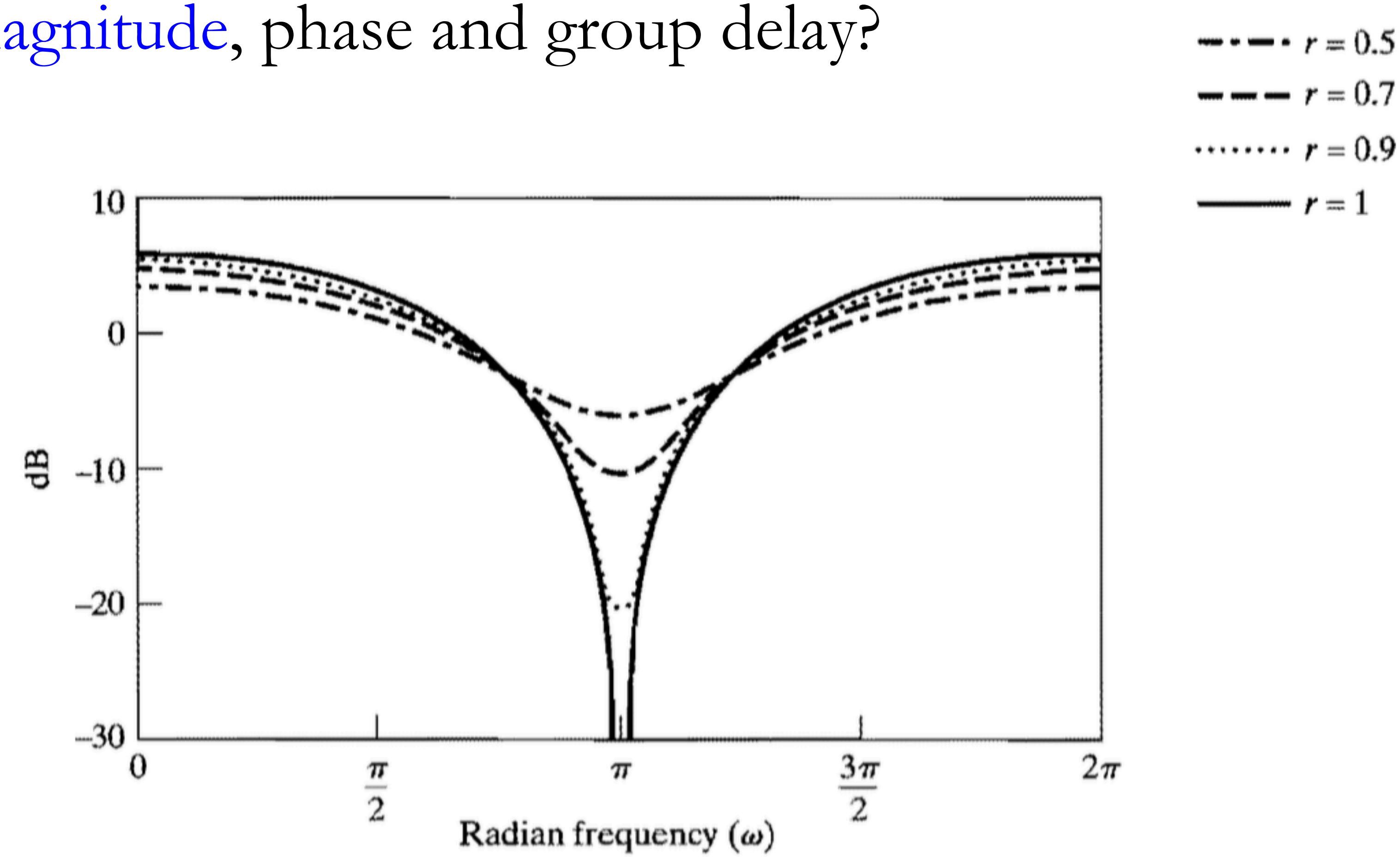
Example: Zero on Real Axis

- Magnitude Response



Example: Zero on Real Axis

- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?



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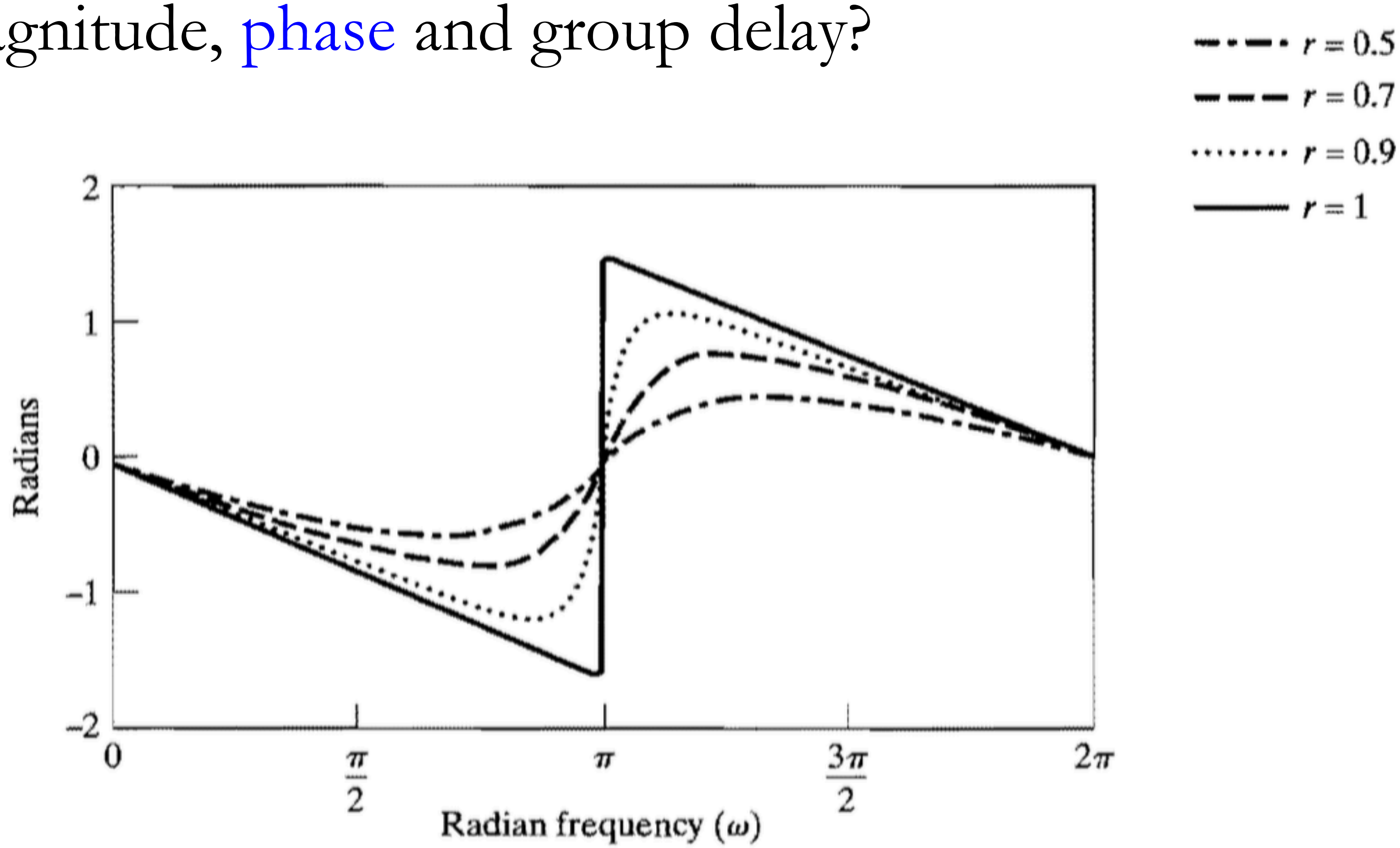

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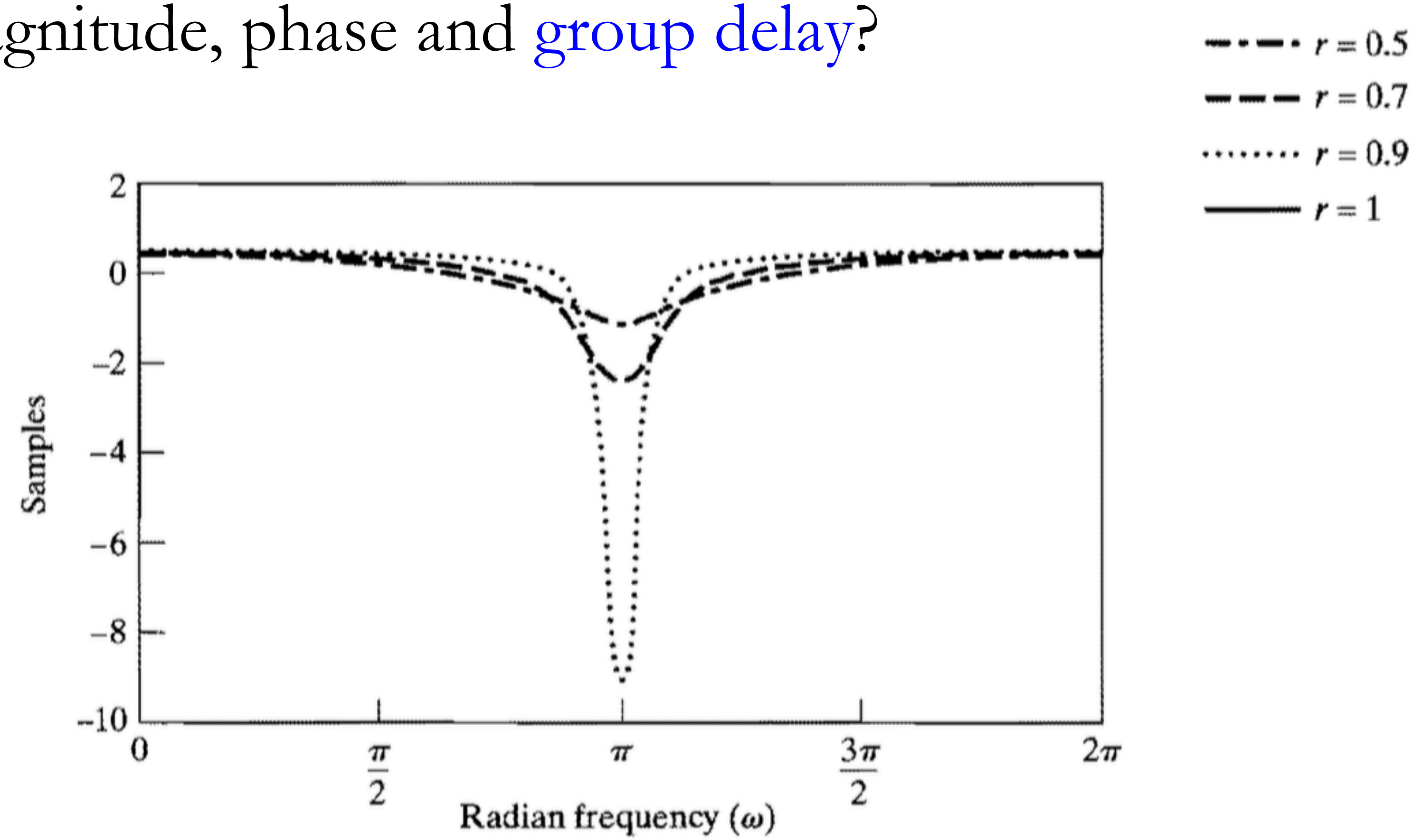
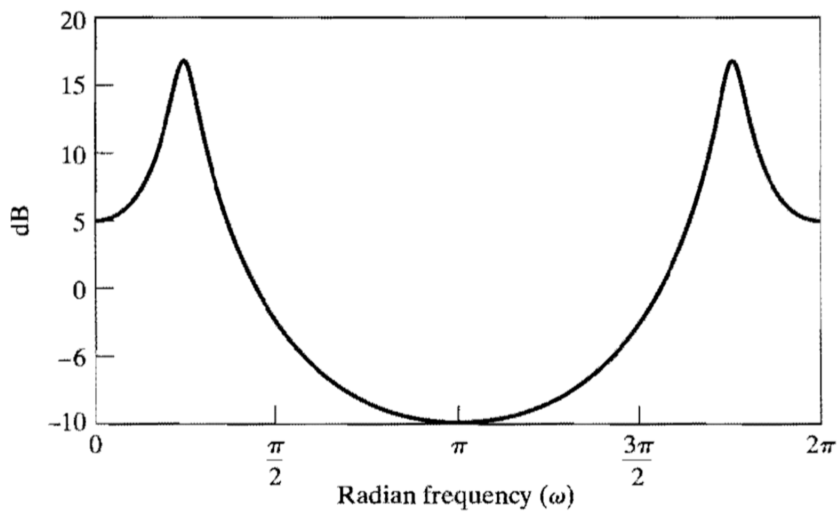


2^{nd} Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

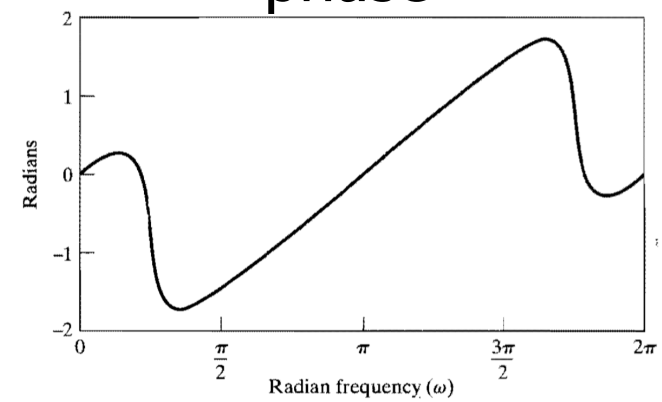
magnitude



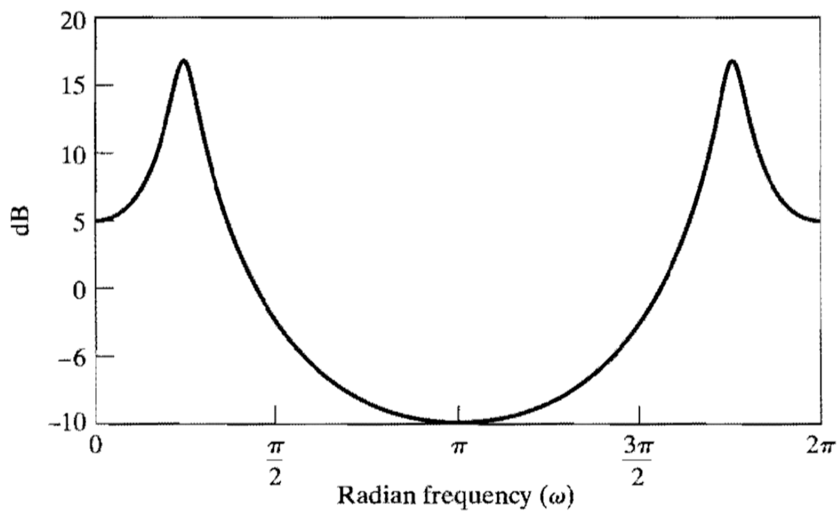
2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

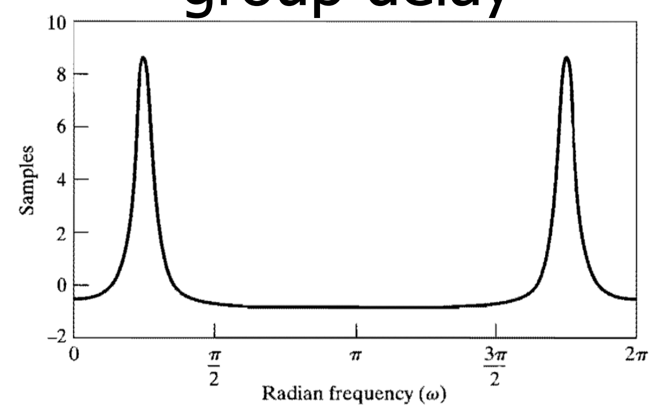
phase



magnitude



group delay



Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

Admin

- HW 5
  - Due Friday 3/3

- Homework solutions to be posted soon