## ESE 531: Digital Signal Processing

Lec 19: April 6, 2017

Discrete Fourier Transform



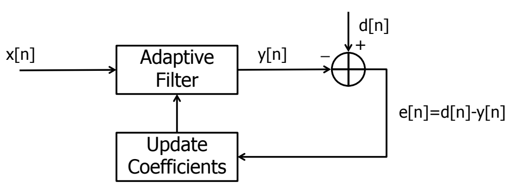
## Today

- Adaptive filtering
  - Blind equalization
- Discrete Fourier Series
- Discrete Fourier Transform (DFT)
- DFT Properties



## Adaptive Filters

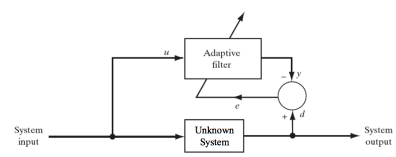
- An adaptive filter is an adjustable filter that processes in time
  - It adapts…






## Adaptive Filter Applications

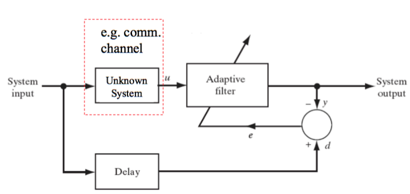
- System Identification






## Adaptive Filter Applications

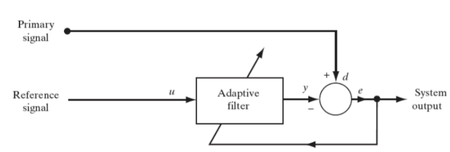
- Identification of inverse system





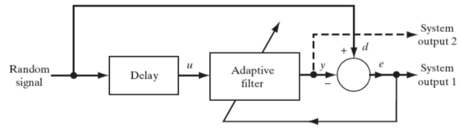

## Adaptive Filter Applications

- Adaptive Interference Cancellation



## Adaptive Filter Applications

- Adaptive Prediction





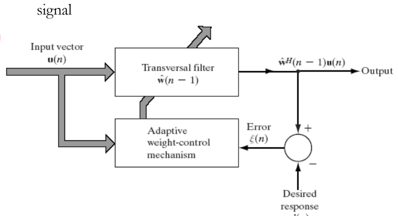

## Stochastic Gradient Approach

- Most commonly used type of Adaptive Filters
- Define cost function as mean-squared error
  - Difference between filter output and desired response
- Based on the method of steepest descent
  - Move towards the minimum on the error surface to get to minimum
  - Requires the gradient of the error surface to be known



## Least-Mean-Square (LMS) Algorithm

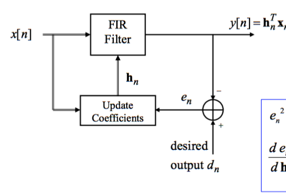
- The LMS Algorithm consists of two basic processes
  - Filtering process
    - Calculate the output of FIR filter by convolving input and taps
    - Calculate estimation error by comparing the output to desired signal
  - 




(a)



## Adaptive FIR Filter: LMS




$\mathbf{x}_n = (x[n], x[n-1]\ldots, x[n-M])^T$

$\mathbf{h}_n = (h_n[0], h_n[1], \ldots, h_n[M])^T$

$e_n^{\,2} = (d[n]-y[n])^2 = (d[n]-\mathbf{h}_n^T\mathbf{x}_n)^2$

$\frac{d\,e_n^{\,2}}{d\,\mathbf{h}_n} = -2(d[n]-\mathbf{h}_n^T\mathbf{x}_n)\mathbf{x}_n = -2e_n\mathbf{x}_n$

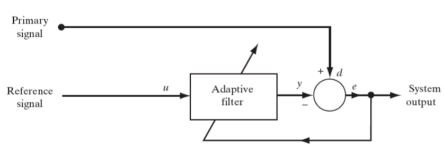
**Coefficient Update:** Move in direction *opposite* to sign of gradient, proportional to magnitude of gradient $\mathbf{h}_{n+1} = \mathbf{h}_n + 2\mu e_n \mathbf{x}_n$

Stochastic Gradient Algorithm



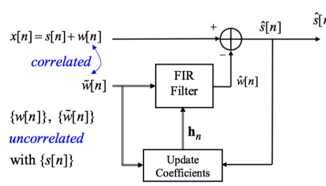
## Adaptive Filter Applications

- Adaptive Interference Cancellation






## Adaptive Interference Cancellation




$\hat{s}[n] = x[n] - \mathbf{h}_n^T\tilde{\mathbf{w}}_n$

$= s[n] + \{w[n] - \mathbf{h}_n^T\tilde{\mathbf{w}}_n\}$

*uncorrelated*

Minimize *instantaneous power* $\hat{s}^2[n]$ for best interference cancellation

$\frac{d(\hat{s}[n])^2}{d\,\mathbf{h}_n} = -2\hat{s}[n]\tilde{\mathbf{w}}_n$

$\mathbf{h}_{n+1} = \mathbf{h}_n + 2\mu\,\hat{s}[n]\tilde{\mathbf{w}}_n$

## Stability of LMS

- The LMS algorithm is convergent in the mean square if and only if the step-size parameter satisfy

$$0 < \mu < \frac{2}{\lambda_{max}}$$

- Here $\lambda_{max}$ is the largest eigenvalue of the correlation matrix of the input data
- More practical test for stability is

$$0 < \mu < \frac{2}{\text{input signal power}}$$

- Larger values for step size
  - Increases adaptation rate (faster adaptation)
  - Increases residual mean-squared error



## Discrete Fourier Series



## Reminder: Eigenvalue (DTFT)

- $x[n]=e^{j\omega n}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$
$$= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$
$$= e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$= H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency $\omega$
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase



## Discrete Fourier Series

- Definition:
  - Consider N-periodic signal:

$$\tilde{x}[n+N] = \tilde{x}[n] \quad \forall n$$

  - Frequency-domain also periodic in N:

$$\tilde{X}[k+N] = \tilde{X}[k] \quad \forall k$$

  - "~" indicates periodic signal/spectrum



## Discrete Fourier Series

- Define:

$$W_N \triangleq e^{-j2\pi/N}$$

- DFS:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$

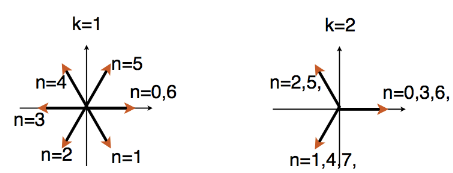
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}$$



## Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

- Properties of WN:
  - $W_N^0 = W_N^N = W_N^{2N} = ... = 1$
  - $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$
- Example: $W_N^{kn}$ (N=6)



## Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!



## Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$ Inverse DFT, synthesis

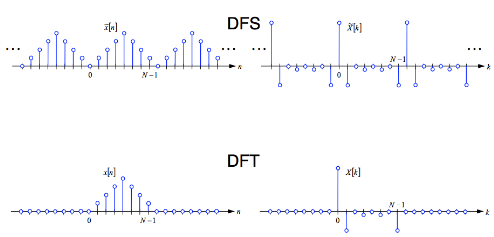
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$ DFT, analysis

- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

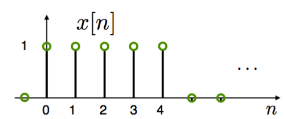


## DFS vs. DFT






## Example $W_N \triangleq e^{-j2\pi/N}$



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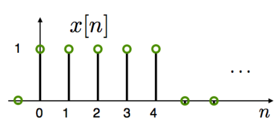



Take N=5

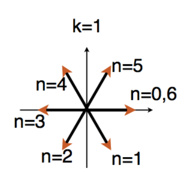
$$X[k] = \begin{cases} \sum_{n=0}^{4} W_5^{nk} & k = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$
$$= 5\delta[k]$$

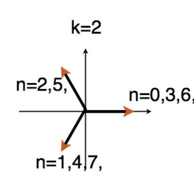
"5-point DFT"



## Discrete Fourier Series $W_N \triangleq e^{-j2\pi/N}$

- Properties of WN:
  - $W_N^0 = W_N^N = W_N^{2N} = \ldots = 1$
  - $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$
- Example: $W_N^{kn}$ (N=6)



## Example

$W_N \triangleq e^{-j2\pi/N}$

- Q: What if we take N=10?
- A: $X[k] \;=\; \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



## Example

$W_N \triangleq e^{-j2\pi/N}$

- Q: What if we take N=10?
- A: $X[k] \;=\; \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.


$$X[k] \;=\; \begin{cases} \sum_{n=0}^{4} W_{10}^{nk} & k = 0, 1, 2, \cdots, 9 \\ 0 & \text{otherwise} \end{cases}$$

"10-point DFT"



## Example

- Now, sum from n=0 to 9

$$X[k] \;=\; \sum_{n=0}^{9} W_{10}^{nk}$$



## Example

- Now, sum from n=0 to 9

$$\begin{aligned} X[k] \;&=\; \sum_{n=0}^{9} W_{10}^{nk} \\ &=\; \sum_{n=0}^{4} W_{10}^{nk} \\ &=\; e^{-j\frac{4\pi}{10}k} \frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)} \end{aligned}$$

"10-point DFT"



## DFT vs. DTFT

- For finite sequences of length N:
  - The N-point DFT of x[n] is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \qquad 0 \le k \le N-1$$

  - The DTFT of x[n] is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \qquad -\infty < \omega < \infty$$



## DFT vs. DTFT

- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \le k \le N-1$$

## DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk} = e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$

"10-point DFT"



## DFT vs DTFT

- Back to example

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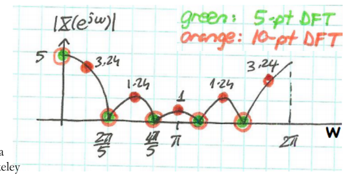




## DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk} = e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$

Use `fftshift` to center around dc




## DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$



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$$N \cdot x^*[n] = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^* = N\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)^*$$



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$$\begin{aligned} N\cdot x^*[n] &= N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^* \\ &= N\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)^* \\ &= \sum_{k=0}^{N-1} X^*[k]W_N^{kn} \\ &= \mathcal{DFT}\{X^*[k]\}. \end{aligned}$$



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## DFT and Inverse DFT

- So

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## DFT and Inverse DFT

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$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$



## DFT and Inverse DFT

- So

$$\mathcal{DFT}\{X^*[k]\} = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

- Implement IDFT by:
  - Take complex conjugate
  - Take DFT
  - Multiply by 1/N
  - Take complex conjugate



## DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

**DFT:**

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

## DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1]\end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)}\end{pmatrix}\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1]\end{pmatrix}$$

IDFT:

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1]\end{pmatrix} = \frac{1}{N}\begin{pmatrix} W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)}\end{pmatrix}\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1]\end{pmatrix}$$



## DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

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$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1]\end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)}\end{pmatrix}\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1]\end{pmatrix}$$

IDFT:

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$N^2$ complex multiples



## DFT as Matrix Operator

- Can write compactly as

$$\mathbf{X} = \mathbf{W}_N \, \mathbf{x}$$
$$\mathbf{x} = \frac{1}{N} \mathbf{W}_N^* \, \mathbf{X}$$



## Properties of the DFT

- Properties of DFT inherited from DFS
- Linearity

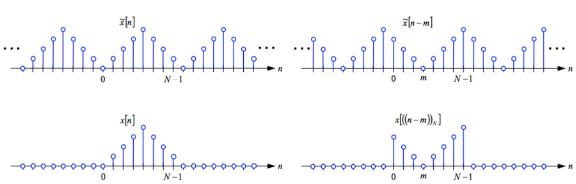
$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

$$x[((n-m))_N] \leftrightarrow X[k] e^{-j(2\pi/N)km} = X[k] W_N^{km}$$



## Circular Shift






## Properties of DFT

- Circular frequency shift

$$x[n] e^{j(2\pi/N)nl} = x[n] W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

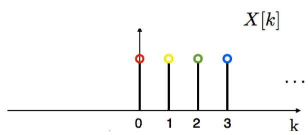
- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

## Example: Conjugate Symmetry

4-point DFT

–Symmetry






$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$



## Example: Conjugate Symmetry

4-point DFT

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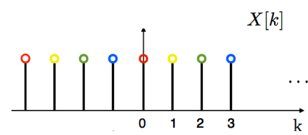




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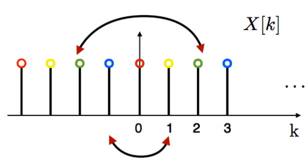




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## Example: Conjugate Symmetry

4-point DFT

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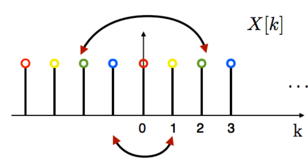
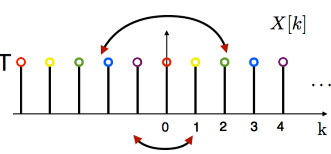



5-point DFT

–Symmetry






$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$



## Example

4-point DFT

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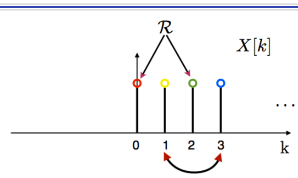
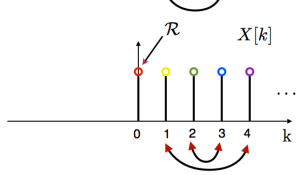



5-point DFT

–Symmetry






## Properties of the DFS/DFT

| Discrete Fourier Series | | | Discrete Fourier Transform | | |
|---|---|---|---|---|---|
| **Property** | ***N*-periodic sequence** | ***N*-periodic DFS** | **Property** | ***N*-point sequence** | ***N*-point DFT** |
| | $\tilde{x}[n]$<br>$\tilde{x}_1[n],\ \tilde{x}_2[n]$ | $\tilde{X}[k]$<br>$\tilde{X}_1[k],\ \tilde{X}_2[k]$ | | $x[n]$<br>$x_1[n],\ x_2[n]$ | $X[k]$<br>$X_1[k],\ X_2[k]$ |
| Linearity | $a\tilde{x}_1[n]+b\tilde{x}_2[n]$ | $a\tilde{X}_1[k]+b\tilde{X}_2[k]$ | Linearity | $ax_1[n]+bx_2[n]$ | $aX_1[k]+bX_2[k]$ |
| Duality | $\tilde{X}[n]$ | $N\,\tilde{x}[-k]$ | Duality | $X[n]$ | $N\,x[((-k))_N]$ |
| Time Shift | $\tilde{x}[n-m]$ | $W_N^{km}\tilde{X}[k]$ | Circular Time Shift | $x[((n-m))_N]$ | $W_N^{km}X[k]$ |
| Frequency Shift | $W_N^{-ln}\tilde{x}[n]$ | $\tilde{X}[k-l]$ | Circular Frequency Shift | $W_N^{-ln}x[n]$ | $X[((k-l))_N]$ |
| Periodic Convolution | $\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ | $\tilde{X}_1[k]\tilde{X}_2[k]$ | Circular Convolution | $\sum_{m=0}^{N-1}x_1[m]x_2[((n-m))_N]$ | $X_1[k]X_2[k]$ |
| Multiplication | $\tilde{x}_1[n]\tilde{x}_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l]$ | Multiplication | $x_1[n]x_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1}X_1[l]X_2[((k-l))_N]$ |
| Complex Conjugation | $\tilde{x}^*[n]$ | $\tilde{X}^*[-k]$ | Complex Conjugation | $x^*[n]$ | $X^*[((-k))_N]$ |

## Properties (Continued)

| | | | | | |
|---|---|---|---|---|---|
| Time-Reversal and Complex Conjugation | $\tilde{x}^*[-n]$ | $\tilde{X}^*[k]$ | Time-Reversal and Complex Conjugation | $x^*[((-n))_N]$ | $X^*[k]$ |
| Real Part | $\mathrm{Re}\{\tilde{x}[n]\}$ | $\tilde{X}_{ep}[k] = \frac{1}{2}\left(\tilde{X}[k] + \tilde{X}^*[-k]\right)$ | Real Part | $\mathrm{Re}\{x[n]\}$ | $X_{ep}[k] = \frac{1}{2}\left(X[k] + X^*[((-k))_N]\right)$ |
| Imaginary Part | $j\,\mathrm{Im}\{\tilde{x}[n]\}$ | $\tilde{X}_{op}[k] = \frac{1}{2}\left(\tilde{X}[k] - \tilde{X}^*[-k]\right)$ | Imaginary Part | $j\,\mathrm{Im}\{x[n]\}$ | $X_{op}[k] = \frac{1}{2}\left(X[k] - X^*[((-k))_N]\right)$ |
| Even Part | $\tilde{x}_{ep}[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right)$ | $\mathrm{Re}\{\tilde{X}[k]\}$ | Even Part | $x_{ep}[n] = \frac{1}{2}\left(x[n] + x^*[((-n))_N]\right)$ | $\mathrm{Re}\{X[k]\}$ |
| Odd Part | $\tilde{x}_{op}[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right)$ | $j\,\mathrm{Im}\{\tilde{X}[k]\}$ | Odd Part | $x_{op}[n] = \frac{1}{2}\left(x[n] - x^*[((-n))_N]\right)$ | $j\,\mathrm{Im}\{X[k]\}$ |
| Symmetry for Real Sequence | $\tilde{x}[n] = \tilde{x}^*[n]$ | $\tilde{X}[k] = \tilde{X}^*[-k]$ <br> $\begin{cases} \mathrm{Re}\{\tilde{X}[k]\} = \mathrm{Re}\{\tilde{X}[-k]\} \\ \mathrm{Im}\{\tilde{X}[k]\} = -\mathrm{Im}\{\tilde{X}[-k]\} \end{cases}$ <br> $\begin{cases} \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$ | Symmetry for Real Sequence | $x[n] = x^*[n]$ | $X[k] = X^*[((-k))_N]$ <br> $\begin{cases} \mathrm{Re}\{X[k]\} = \mathrm{Re}\{X[((-k))_N]\} \\ \mathrm{Im}\{X[k]\} = -\mathrm{Im}\{X[((-k))_N]\} \end{cases}$ <br> $\begin{cases} \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \angle X[k] = -\angle X[((-k))_N] \end{cases}$ |
| Parseval's Identity | $\sum_{n=0}^{N-1} \tilde{x}_1[n]\tilde{x}_2^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}_1[k]\tilde{X}_2^*[k]$ <br> $\sum_{n=0}^{N-1} \lvert\tilde{x}[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert\tilde{X}[k]\rvert^2$ | | Parseval's Identity | $\sum_{n=0}^{N-1} x_1[n]x_2^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2^*[k]$ <br> $\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$ | |



## Circular Convolution

- Circular Convolution:

$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

For two signals of length N

**Note: Circular convolution is commutative**

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$



## Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$



## Multiplication

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$



## Linear Convolution

- Next….
  - Using DFT, circular convolution is easy
  - But, linear convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Use DFT to do linear convolution



## Big Ideas

- Adaptive filtering
  - Use LMS algorithm to update filter coefficients for applications like system ID, channel equalization, and signal prediction
- Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution

## Admin

- HW 7 out now
  - Due tonight

- Project posted after class tonight
  - Work in groups of up to 2
    - Can work alone if you want
    - Use Piazza to find partners
    - Report your groups to me by 4/11 by email
      - taniak@seas.upenn.edu