ESE 531: Digital Signal Processing

Lec 21: April 13, 2017

Fast Fourier Transform

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Adapted from M. Lustig, EECS Berkeley

Last Time

- Discrete Fourier Transform
  - Linear convolution through circular
  - Linear convolutions through DFT
    - Overlap and add
    - Overlap and save

- Today
  - The Fast Fourier Transform

Circular Convolution

- Circular Convolution:

$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Circular Convolution as Matrix Operation

- Circular Convolution

$$h[n] \otimes x[n] = \sum_{m=0}^{N-1} h[((n-m))_N] x[m]$$

$$\begin{aligned} h[n] \; \textcircled{N} \; x[n] &= \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & & h[2] \\ & & \vdots & \\ h[N-1] & h[N-2] & & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N\text{-}1] \end{bmatrix} \\ &= H_c x \end{aligned}$$

Circular Convolution as Matrix Operation

- Circular Convolution

$$
\begin{aligned}
h[n] \circledN x[n] &= \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & & h[2] \\ & & \vdots & \\ h[N-1] & h[N-2] & & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \\
&= H_c x
\end{aligned}
$$

$$
W_N = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix}
$$

Circular Convolution as Matrix Operation

- Diagonalize

$$W_N H_c W_N^{-1} = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix}$$

- Right-Multiply by W_N

$$W_N H_c = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N$$

- Multiply both sides by x

$$W_N H_c x = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N x$$

Fast Fourier Transform Algorithms

- We are interested in efficient computing methods for the DFT and inverse DFT:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, \ldots, N-1 \\
x[n] &= \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0, \ldots, N-1
\end{aligned}
$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}.$$

Reminder: Inverse DFT via DFT

- Recall that we can use the DFT to compute the inverse DFT:

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

  - Hence, we can just focus on efficient computation of the DFT.

- Straightforward computation of an N-point DFT (or inverse DFT) requires N^2 complex multiplications.

Computation Order

- *Fast Fourier transform algorithms* enable computation of an N-point DFT (or inverse DFT) with the order of just $N \cdot \log_2 N$ complex multiplications.
  - This can represent a huge reduction in computational load, especially for large N.

N	N^2	$N \cdot \log_2 N$	$\frac{N^2}{N \cdot \log_2 N}$
16	256	64	4.0
128	16,384	896	18.3
1,024	1,048,576	10,240	102.4
8,192	67,108,864	106,496	630.2

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16	256	64	4.0
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8,192	67,108,864	106,496	630.2
6×10^6	36×10^{12}	135×10^6	2.67×10^5

* 6Mp image size

Eigenfunction Properties

- Most FFT algorithms exploit the following properties of W_N^{kn}:
  - Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

  - Periodicity in n and k

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

  - Power

$$W_N^2 = W_{N/2}$$

FFT Algorithms via Decimation

- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - Decimation-in-time algorithms decompose x[n] into successively smaller subsequences.
  - Decimation-in-frequency algorithms decompose X[k] into successively smaller subsequences.
- We mostly discuss decimation-in-time algorithms today.

- Note: Assume length of x[n] is power of 2 ($N = 2^{\nu}$). If not, zero-pad to closest power.

Decimation-in-Time FFT

- We start with the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, \ldots, N-1$$

- Separate the sum into even and odd terms:

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

  - These are two DFTs, each with half the number of samples (N/2)

Decimation-in-Time FFT

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$
\begin{aligned}
X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\
&= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk}
\end{aligned}
$$

Note that:

$$
W_N^{2rk} = e^{-j\left(\frac{2\pi}{N}\right)(2rk)} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}
$$

Remember this trick, it will turn up often.

Decimation-in-Time FFT

Hence:

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \\ &\triangleq G[k] + W_N^k H[k], \qquad k = 0, \ldots, N-1 \end{aligned}$$

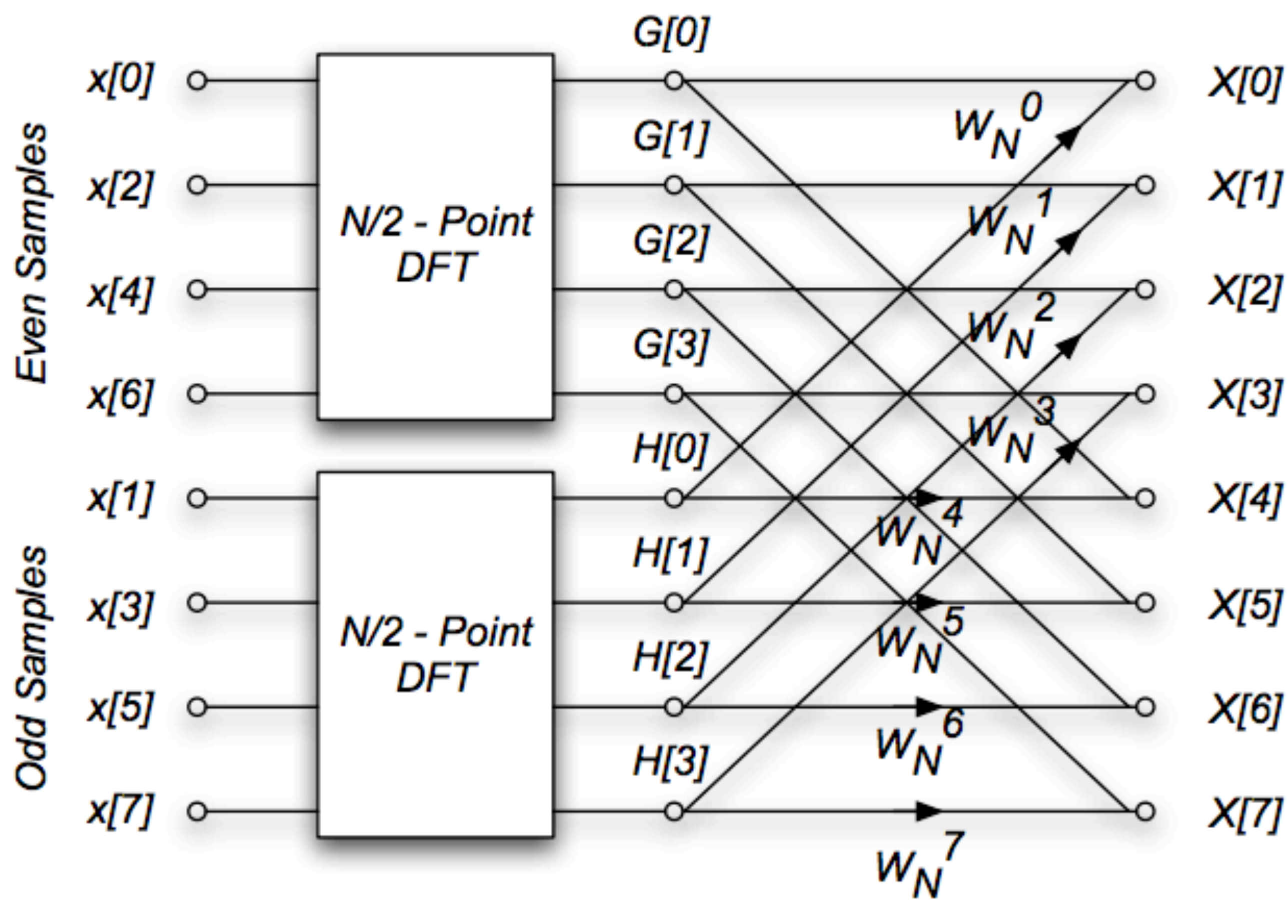
where we have defined:

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \qquad \Rightarrow \text{DFT of even samples}$$

$$H[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \qquad \Rightarrow \text{DFT of odd samples}$$

Decimation-in-Time FFT

An 8 sample DFT can then be diagrammed as



Decimation-in-Time FFT

Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk}$$

$$\begin{aligned} G[k+N/2] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} W_{N/2}^{r(N/2)} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \\ &= G[k] \end{aligned}$$

Decimation-in-Time FFT

- So,

$$\begin{aligned} G[k+(N/2)] &= G[k] \\ H[k+(N/2)] &= H[k] \end{aligned}$$

- The periodicity of G[k] and H[k] allows us to further simplify. For the first N/2 points we calculate G[k] and W_N^kH[k], and then compute the sum

$$X[k] = G[k] + W_N^k H[k] \qquad \forall\{k : 0 \le k < \frac{N}{2}\}.$$

How does periodicity help for $\frac{N}{2} \le k < N$?

Decimation-in-Time FFT

$$X[k] = G[k] + W_N^k H[k] \qquad \forall\{k : 0 \leq k < \frac{N}{2}\}.$$

for $\frac{N}{2} \leq k < N$:

$$W_N^{k+(N/2)} = ?$$

$$X[k + (N/2)] = ?$$

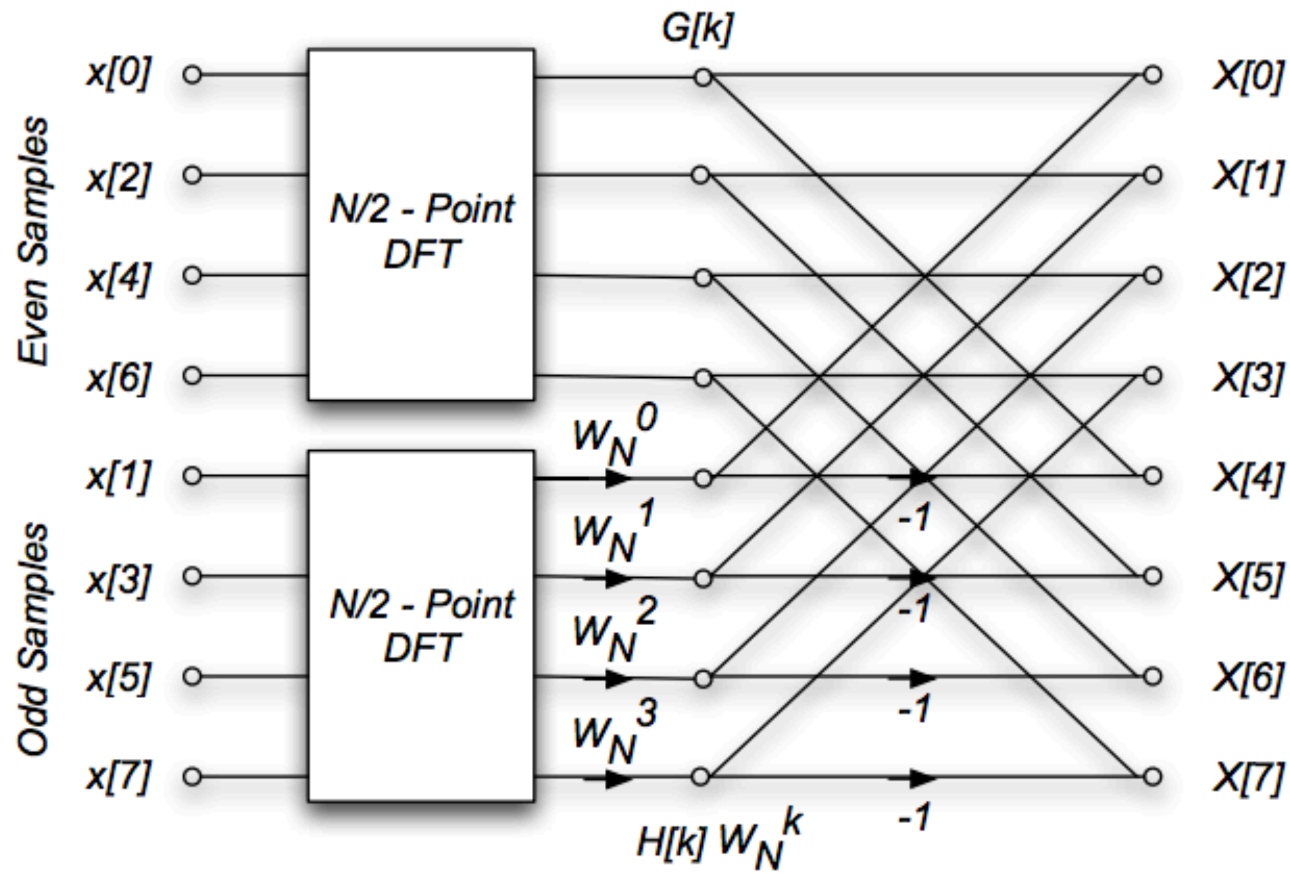
Decimation-in-Time FFT

$$X[k+(N/2)] = G[k] - W_N^k H[k]$$

- We previously calculated G[k] and $W_N^k H[k]$.
- Now we only have to compute their difference to obtain the second half of the spectrum. No additional multiplies are required.

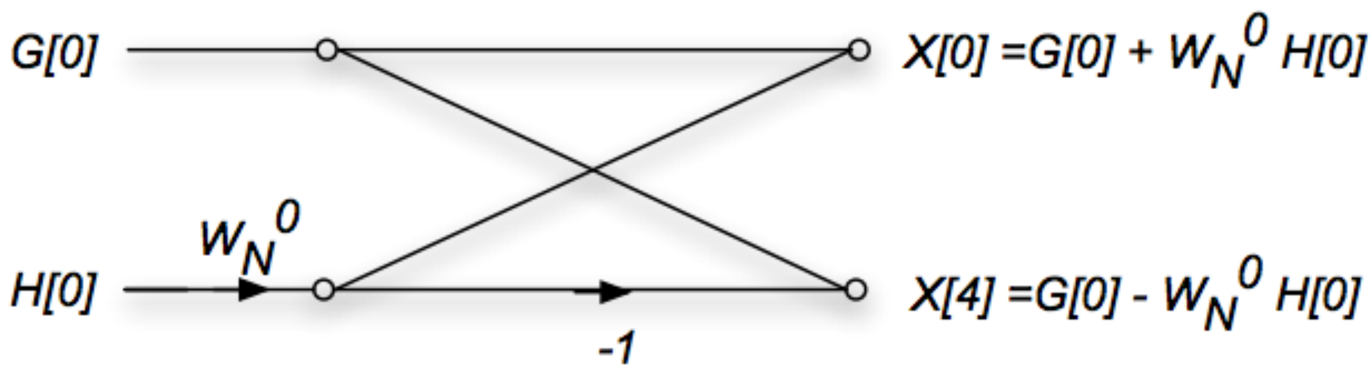
Decimation-in-Time FFT

The N-point DFT has been reduced two $N/2$-point DFTs, plus $N/2$ complex multiplications. The 8 sample DFT is then:



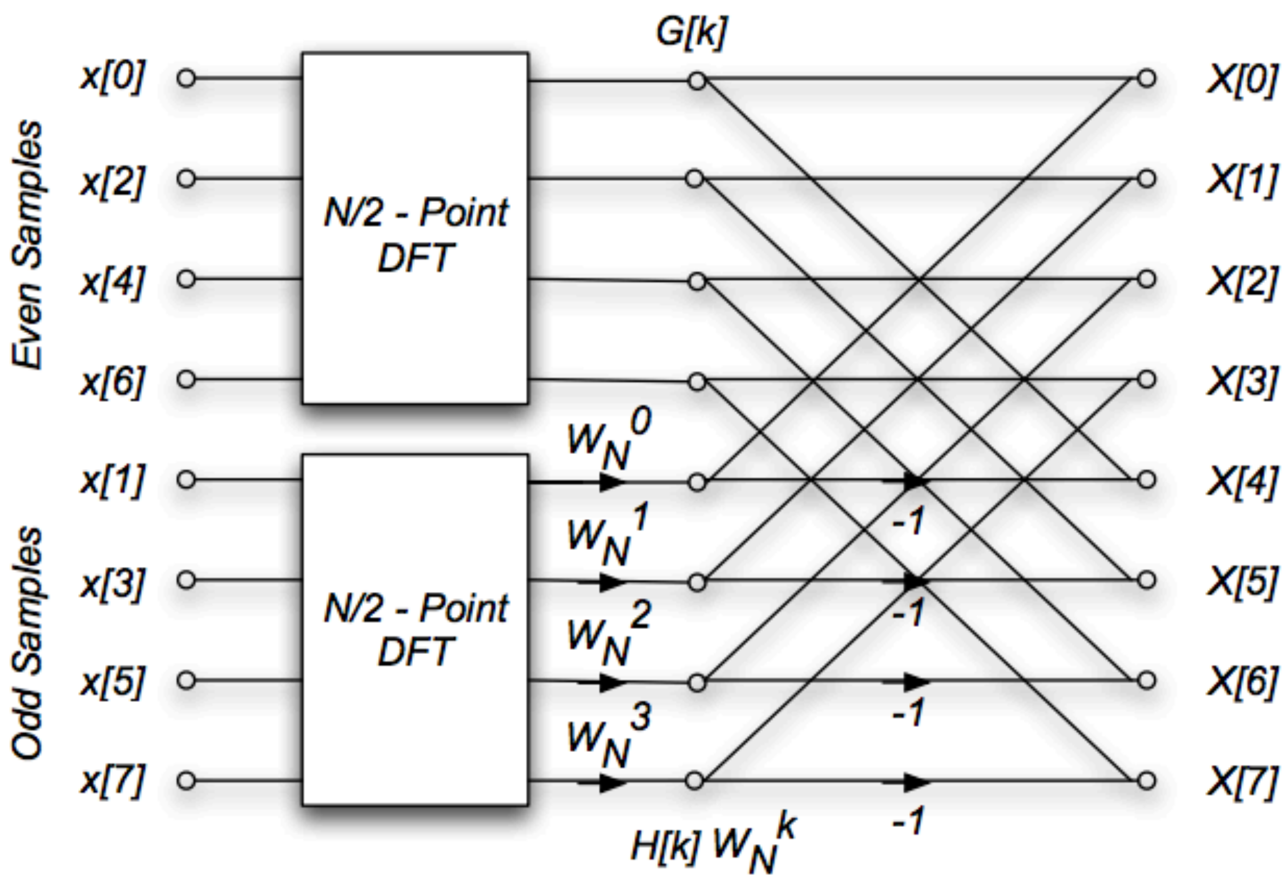
Decimation-in-Time FFT

- Note that the inputs have been reordered so that the outputs come out in their proper sequence.
- We can define a *butterfly operation*, e.g., the computation of X[0] and X[4] from G[0] and H[0]:



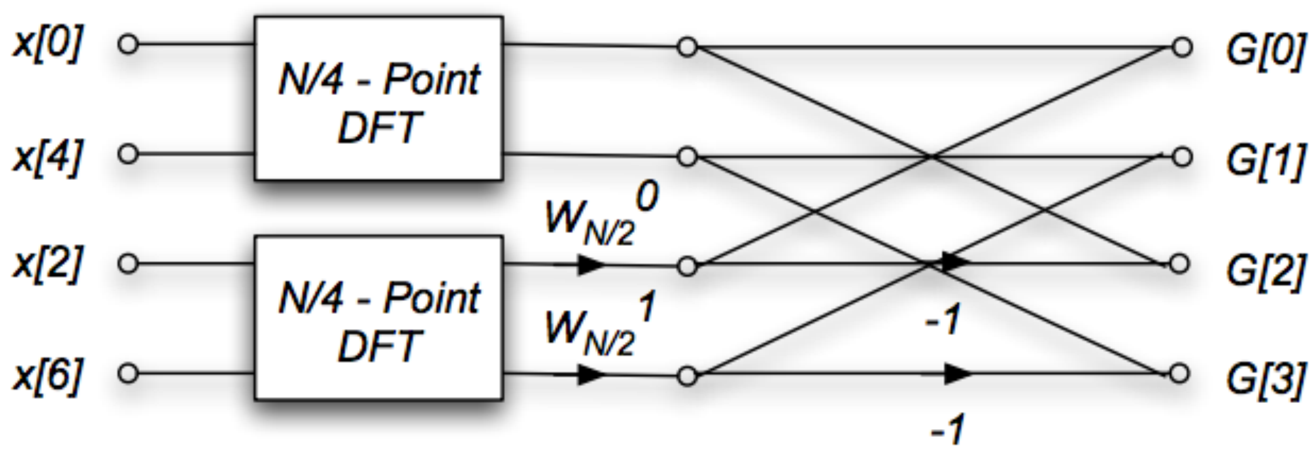
Decimation-in-Time FFT

- Still O(N^2) operations…. What should we do?



Decimation-in-Time FFT

- We can use the same approach for each of the N/2 point DFT's. For the N = 8 case, the N/2 DFTs look like



*Note that the inputs have been reordered again.

Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the N/4 = 2 sample DFT's with a single butterfly. The coefficient is

$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$

Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the N/4 = 2 sample DFT's with a single butterfly. The coefficient is

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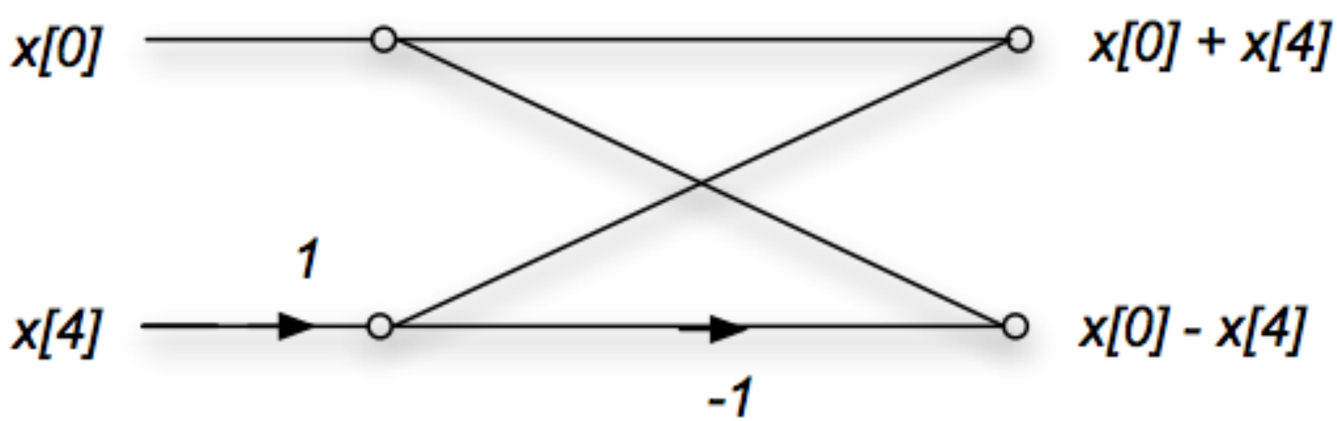
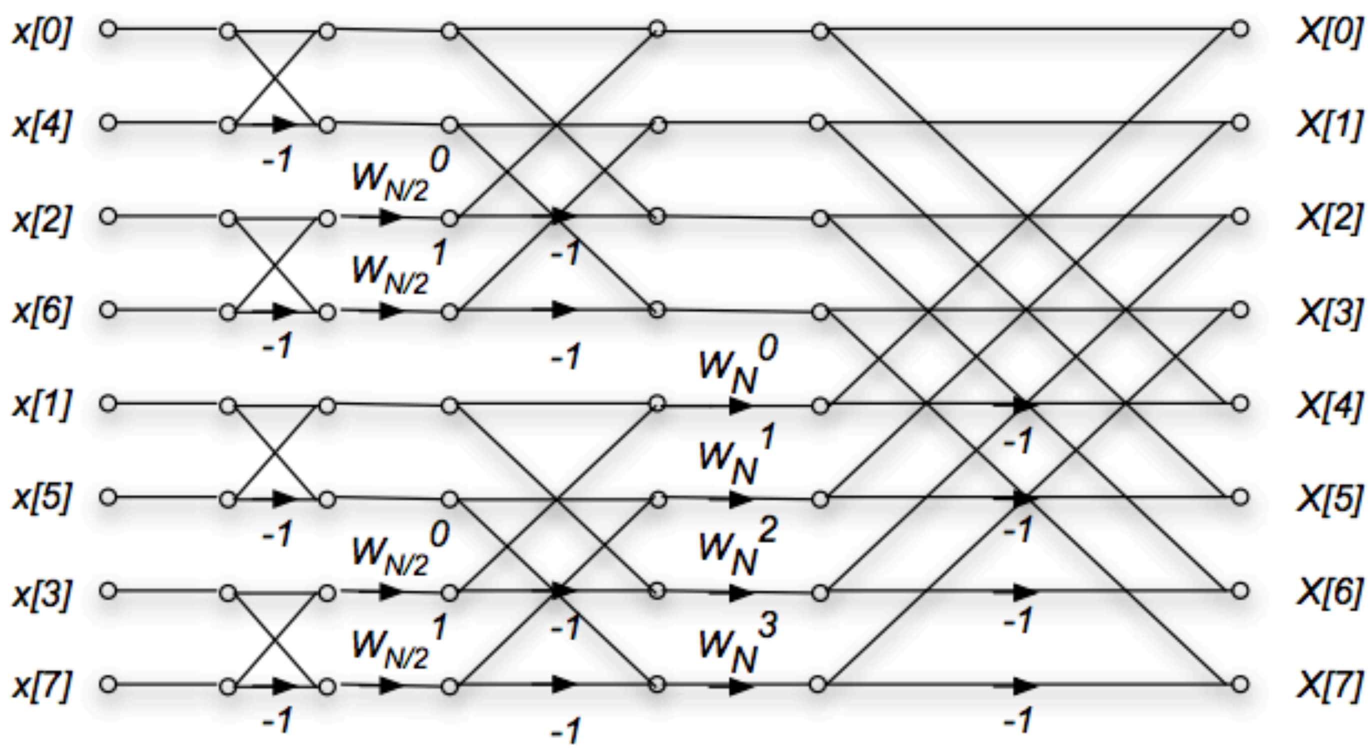
The diagram of this stage is then



Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:



This the decimation-in-time FFT algorithm.

Decimation-in-Time FFT

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2)\log_2 N$, or $O(N \log_2 N)$

Decimation-in-Time FFT

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2) \log_2 N$, or $O(N \log_2 N)$

- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even. Zero low-order bit (LSB) first
  - Next stage repeats with next zero-lower bit first.
  - Net effect is reversing the bit order of indexes

Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary	Bit-Reversed Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Decimation-in-Frequency FFT

The DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

If we only look at the even samples of $X[k]$, we can write $k = 2r$,

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}$$

We split this into two sums, one over the first $N/2$ samples, and the second of the last $N/2$ samples.

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n + N/2] W_N^{2r(n+N/2)}$$

Decimation-in-Frequency FFT

But $W_N^{2r(n+N/2)} = W_N^{2rn} W_N^N = W_N^{2rn} = W_{N/2}^{rn}$.
We can then write

$$
\begin{aligned}
X[2r] &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)} \\
&= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2rn} \\
&= \sum_{n=0}^{(N/2)-1} \left(x[n] + x[n+N/2]\right) W_{N/2}^{rn}
\end{aligned}
$$

This is the $N/2$-length DFT of first and second half of $x[n]$ summed.

Decimation-in-Frequency FFT

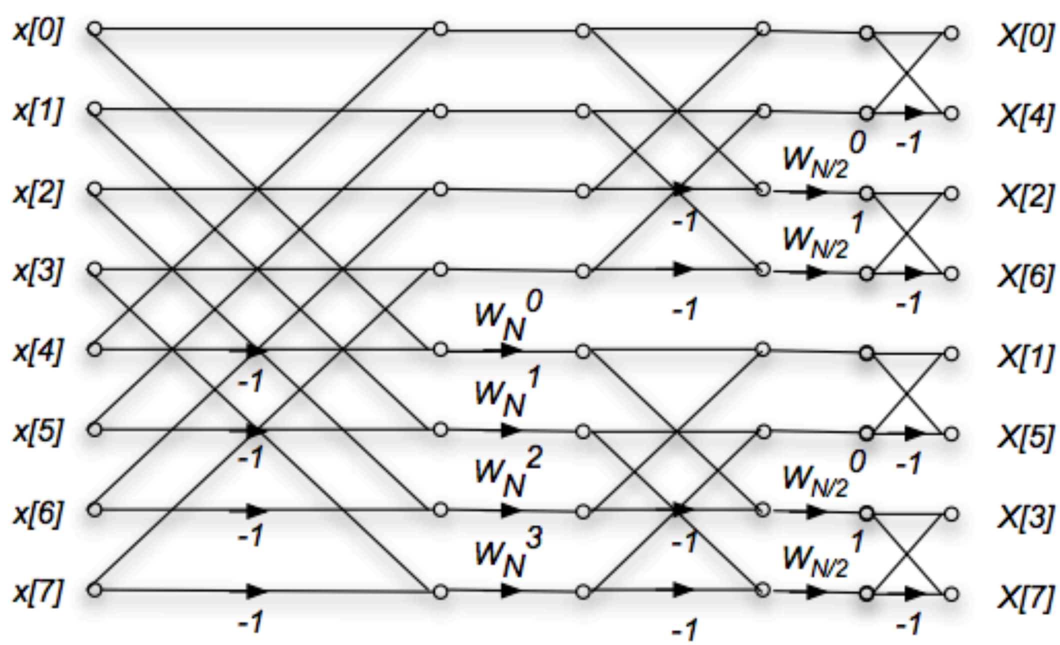
$$\begin{aligned} X[2r] &= \mathrm{DFT}_{\frac{N}{2}}\{(x[n] + x[n+N/2])\} \\ X[2r+1] &= \mathrm{DFT}_{\frac{N}{2}}\{(x[n] - x[n+N/2])\, W_N^n\} \end{aligned}$$

(By a similar argument that gives the odd samples)

Continue the same approach is applied for the $N/2$ DFTs, and the $N/4$ DFT's until we reach simple butterflies.

Decimation-in-Frequency FFT

The diagram for and 8-point decimation-in-frequency DFT is as follows

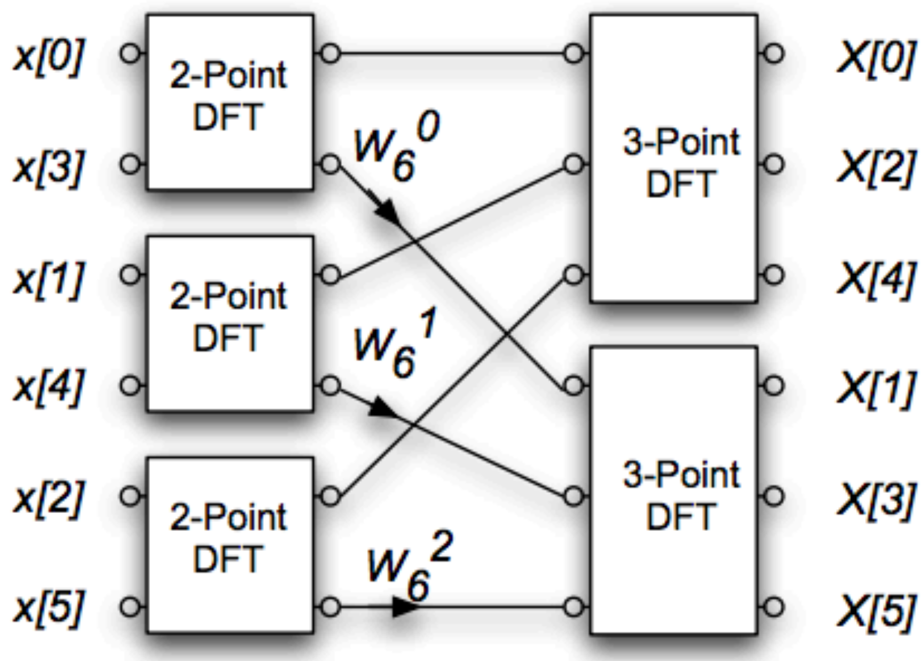


This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Non-Power-of-2 FFTs

A similar argument applies for any length DFT, where the length N is a composite number.

For example, if $N = 6$, a decimation-in-time FFT could compute three 2-point DFT's followed by two 3-point DFT's



Non-Power-of-2 FFTs

Good component DFT's are available for lengths up to 20 or so. Many of these exploit the structure for that specific length. For example, a factor of

$$W_N^{N/4} = e^{-j\frac{2\pi}{N}(N/4)} = e^{-j\frac{\pi}{2}} = -j \qquad \textit{Why?}$$

just swaps the real and imaginary components of a complex number, and doesn't actually require any multiplies.
Hence a DFT of length 4 doesn't require any complex multiplies.
Half of the multiplies of an 8-point DFT also don't require multiplication.
Composite length FFT's can be very efficient for any length that factors into terms of this order.

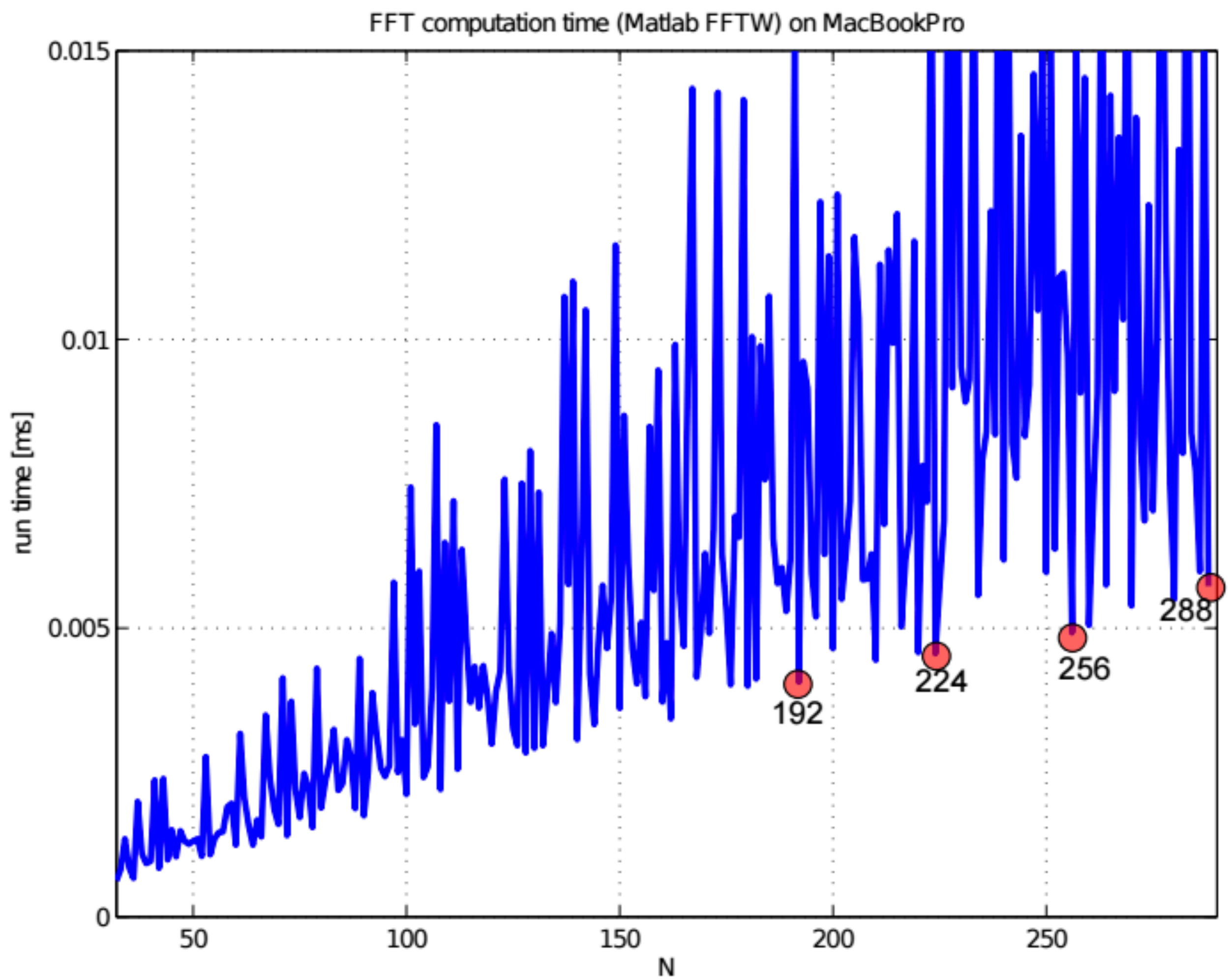
Non-Power-of-2 FFTs

- For example N = 693 factors into
  - N = (7)(9)(11)
- each of which can be implemented efficiently. We would perform
  - 9 x 11 DFT's of length 7
  - 7 x 11 DFT's of length 9, and
  - 7 x 9 DFT's of length 11

Non-Power-of-2 FFTs

- Historically, the power-of-two FFTs were much faster (better written and implemented).
- For non-power-of-two length, it was faster to zero pad to power of two.
- Recently this has changed. The free FFTW package implements very efficient algorithms for almost any filter length. Matlab has used FFTW since version 6

FFT Computation Time



FFT as Matrix Operation

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- W_N is fully populated → N^2 entries

FFT as Matrix Operation

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- W_N is fully populated → N^2 entries
- FFT is a decomposition of W_N into a more sparse form:

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} W_{N/2} & 0 \\ 0 & W_{N/2} \end{bmatrix} \begin{bmatrix} \text{Even-Odd Perm.} \\ \text{Matrix} \end{bmatrix}$$

- $I_{N/2}$ is an identity matrix. $D_{N/2}$ is a diagonal matrix with entries $1, W_N, \cdots, W_N^{N/2-1}$

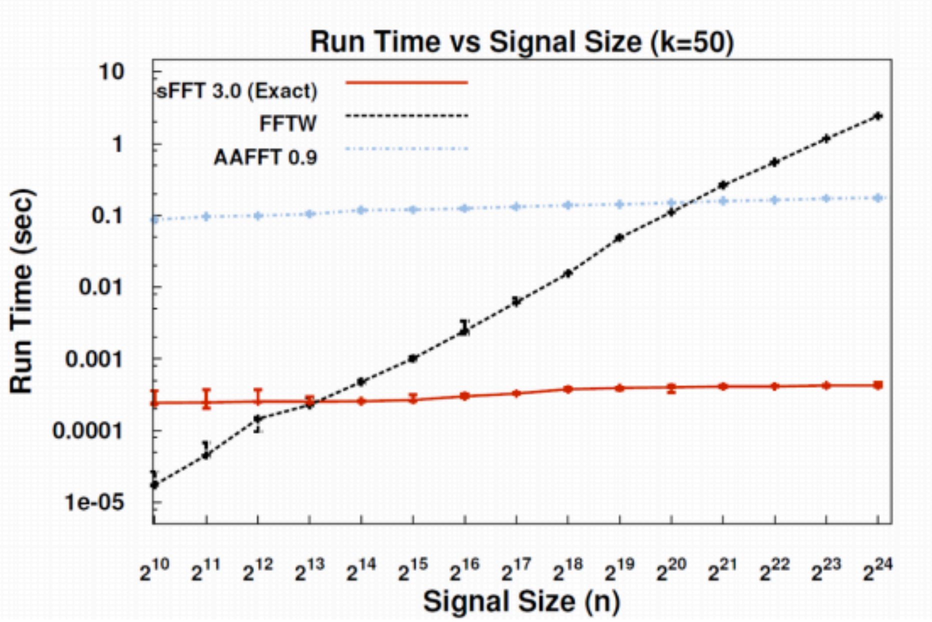
FFT as Matrix Operation

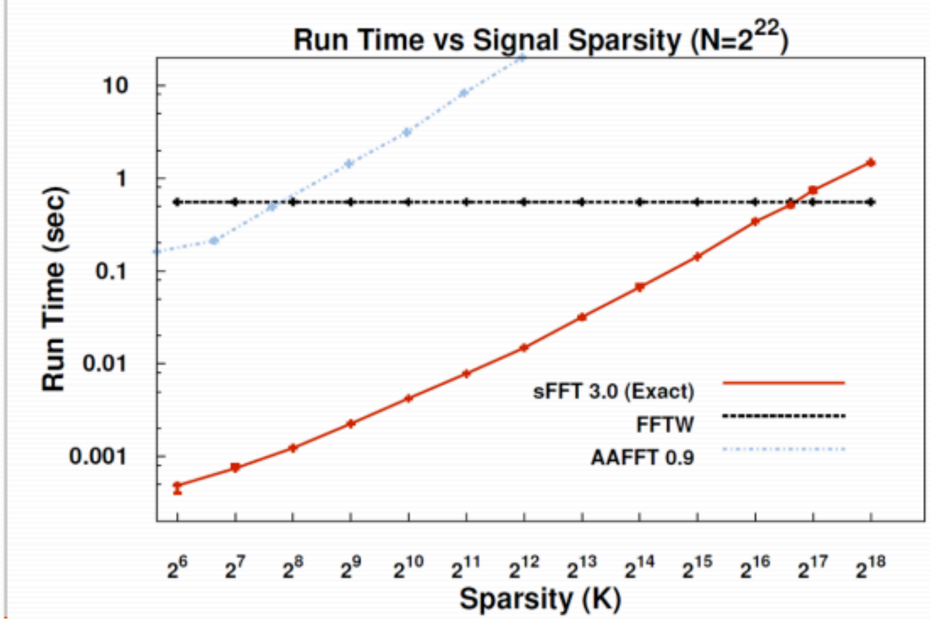
Example: $N = 4$

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & W_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -W_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beyond NlogN

- What if the signal x[n] has a k sparse frequency
  - A. Gilbert et. al, "Near-optimal sparse Fourier representations via sampling
  - H. Hassanieh et. al, "Nearly Optimal Sparse Fourier Transform"
  - Others…
  - O(K Log N) instead of O(N Log N)





From: http://groups.csail.mit.edu/netmit/sFFT/paper.html

Big Ideas

- Fast Fourier Transform
  - Enable computation of an N-point DFT (or DFT^{-1}) with the order of just $N \cdot \log_2 N$ complex multiplications.
  - Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
    - Decimation-in-time algorithms
    - Decimation-in-frequency
  - Historically, power-of-2 DFTs had highest efficiency
  - Modern computing has led to non-power-of-2 FFTs with high efficiency
  - Sparsity leads to reduce computation on order $K \cdot \log N$

Admin

- Project
  - Due 4/25