

ESE 531: Digital Signal Processing

Lec 4: January 24, 2017

Discrete Time Fourier Transform



Lecture Outline

- ❑ Difference Equations
- ❑ Eigenfunctions
- ❑ Discrete Time Fourier Transform
 - Definition
 - Properties
- ❑ Frequency Response of LTI Systems

LTI Systems

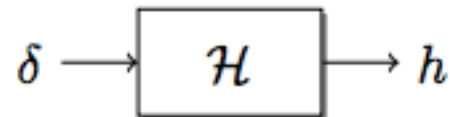
Difference Equations

LTI Systems

DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

A block diagram representing an LTI system. An input signal x enters a rectangular block labeled h . The output of the block is a signal y .

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$



Difference Equations

□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



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$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m y[n-m]$$

Difference Equations

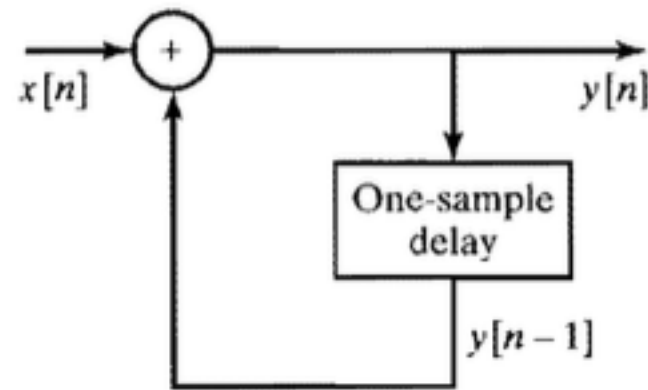
□ Accumulator example

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$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m y[n-m]$$



Example: Difference Equation

- ❑ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- ❑ Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Eigenfunctions



Eigenfunction

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

Eigenvalue (frequency response)

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
 - Re and Im
 - Mag and Phase



CT vs DT Frequency Response

□ $H(e^{j(\omega + 2\pi)n})?$

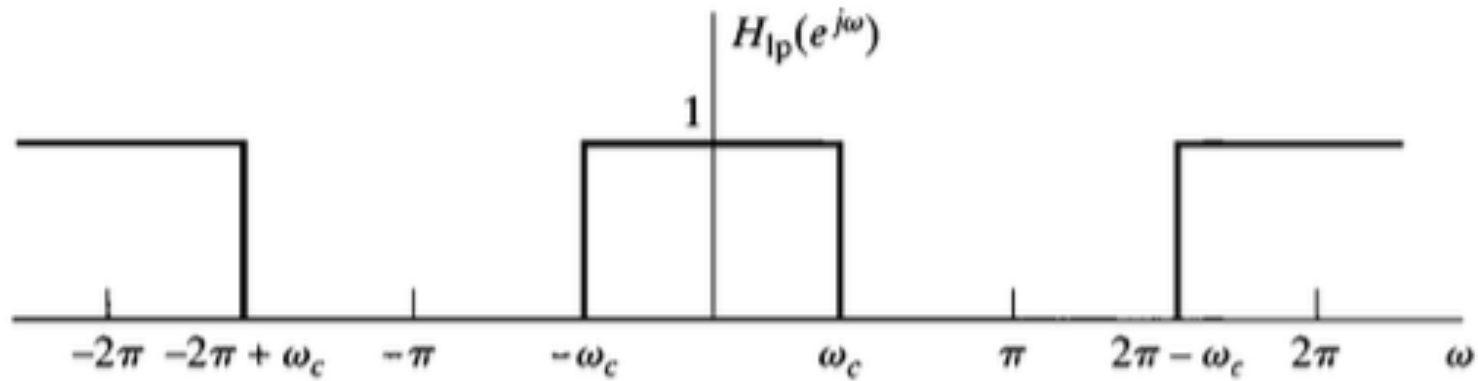


CT vs DT Frequency Response

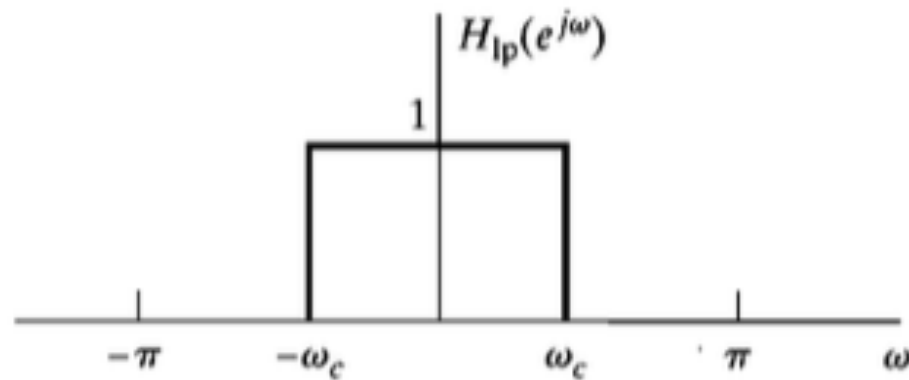
□ $H(e^{j(\omega+2\pi)n})$?

$$\begin{aligned} H(e^{j(\omega+2\pi)n}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-j(\omega+2\pi)k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} e^{-j2\pi k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \\ &= H(e^{j\omega n}) \end{aligned}$$

Periodicity of Low Pass Freq Response



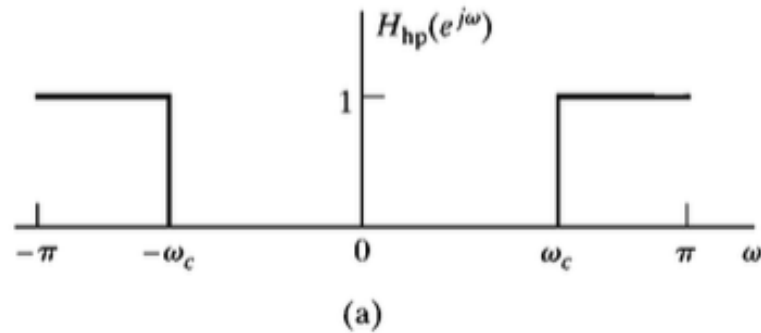
(a)



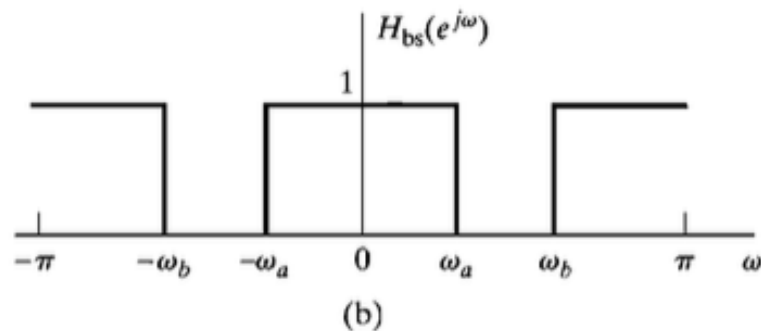
(b)

Other Filters

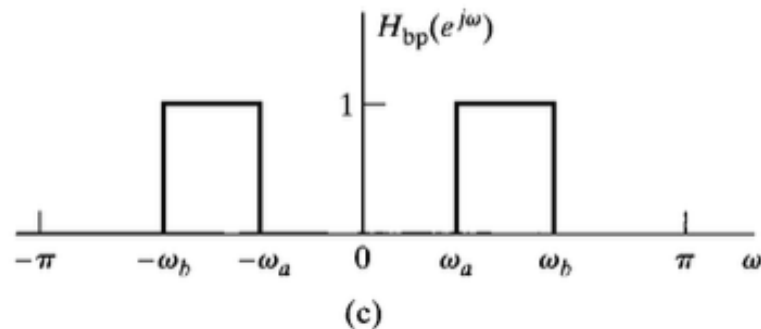
High-pass



Band-stop



Band-pass



Discrete-Time Fourier Transform (DTFT)



DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$



DTFT Definition

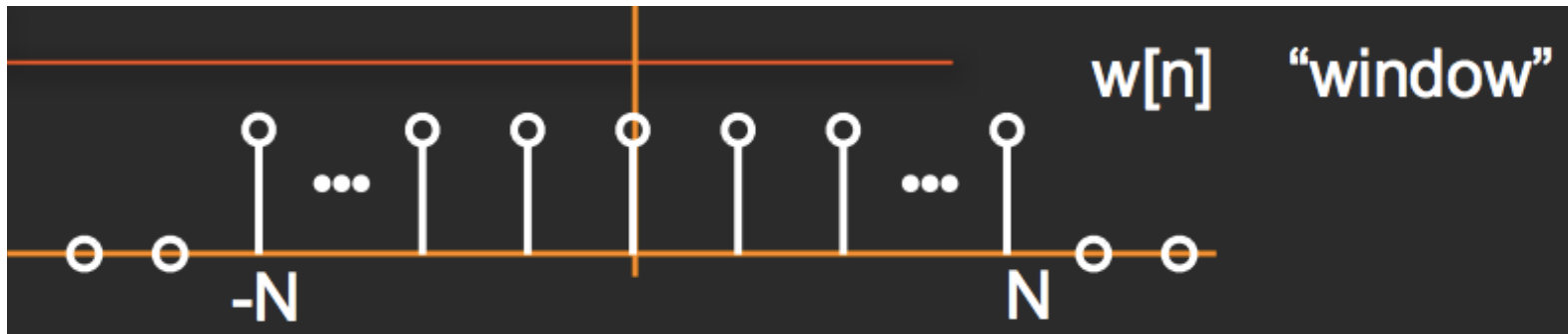
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k}$$
$$x[n] = \int_{-0.5}^{0.5} X(f) e^{j2\pi f n} df$$

Example: Window DTFT



$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k} \\ &= \sum_{k=-N}^N e^{-j\omega k} \end{aligned}$$



Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \dots + p^M = \frac{1 - p^{M+1}}{1 - p}$

$$\begin{aligned} &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \\ &= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}} \end{aligned}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \\ &= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)} \end{aligned}$$

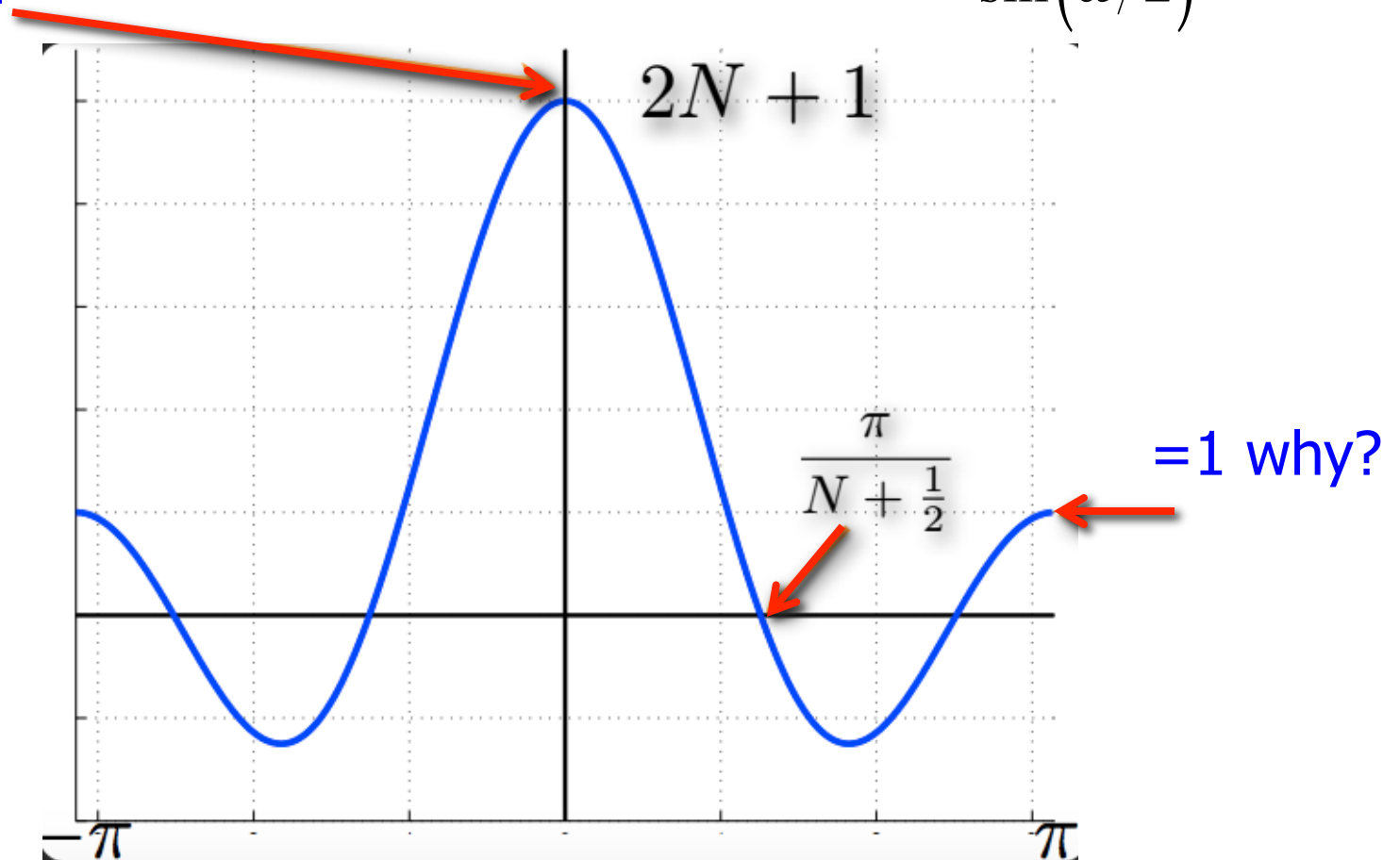
Periodic sinc

Example: Window DTFT

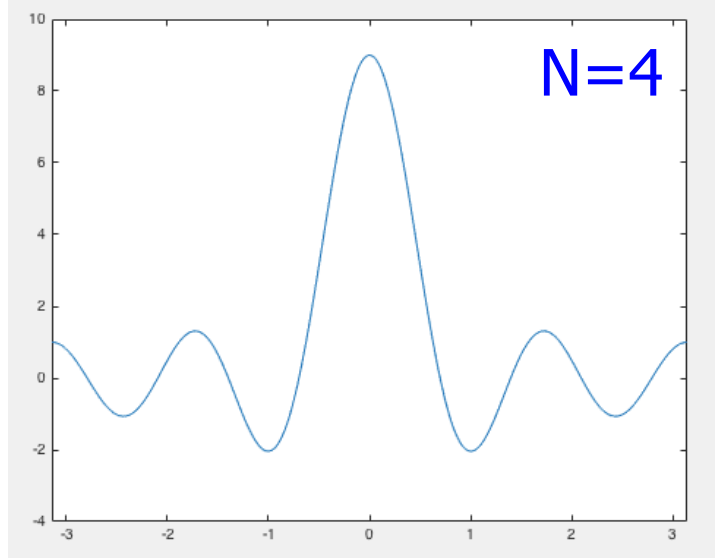
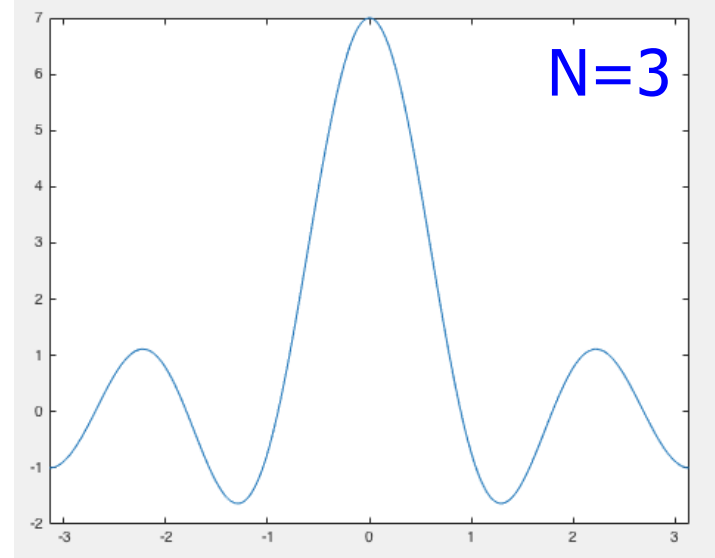
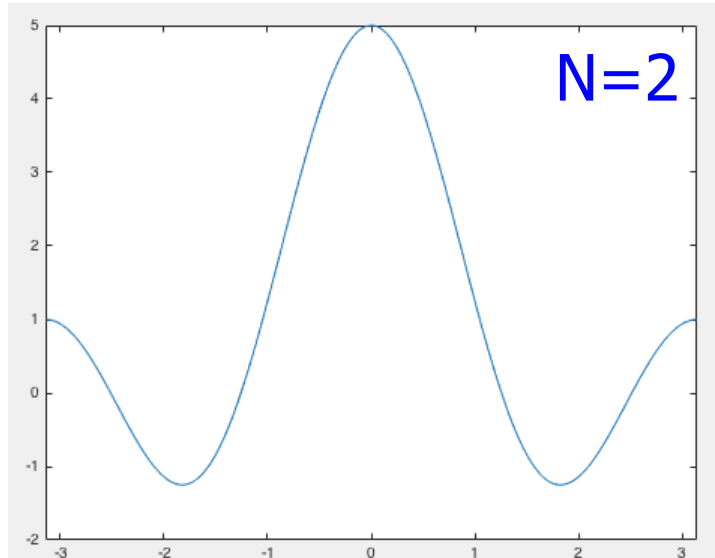
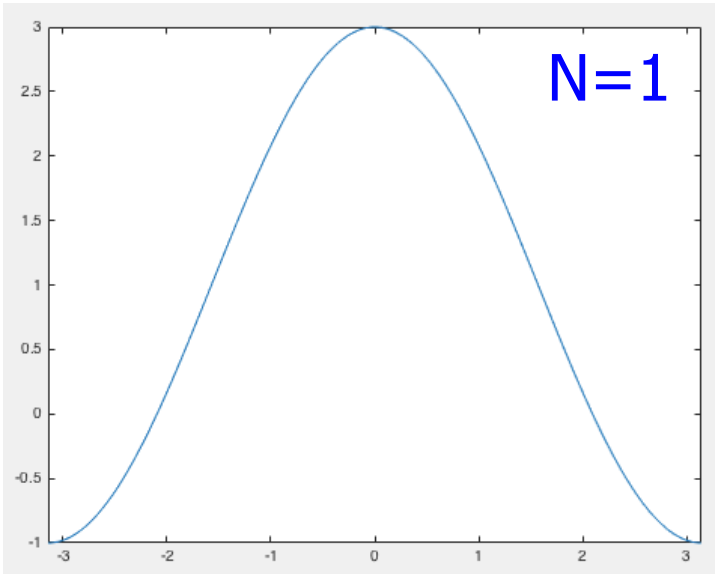
$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Also, $\sum x[n]$

Plot for $N=2$



Periodic Sinc



Properties of the DTFT

□ Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

□ Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

□ Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega}) \quad \text{If } x[n] \text{ real}$$

$$\operatorname{Re}\{X(e^{-j\omega})\} = \operatorname{Re}\{X(e^{j\omega})\}$$

$$\operatorname{Im}\{X(e^{-j\omega})\} = -\operatorname{Im}\{X(e^{j\omega})\}$$



Properties of the DTFT

□ Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega})$$

If $x[n]$ real

$$x[-n] \leftrightarrow X(e^{-j\omega}) \quad x[-n] \leftrightarrow X^*(e^{-j\omega})$$

□ Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$



Properties of the DTFT

□ Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$$

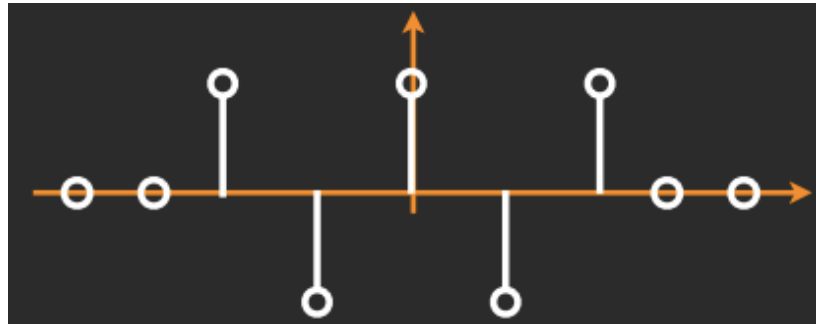
□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Example: Windowed $\cos(\pi n)$

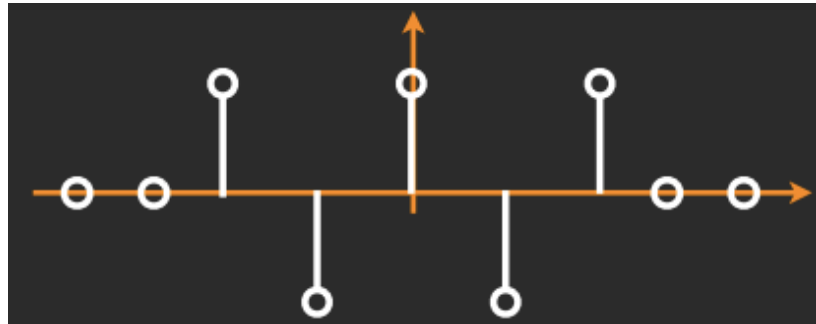
□ What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

Example: Windowed $\cos(\pi n)$

□ What is DTFT of:



Properties of the DTFT

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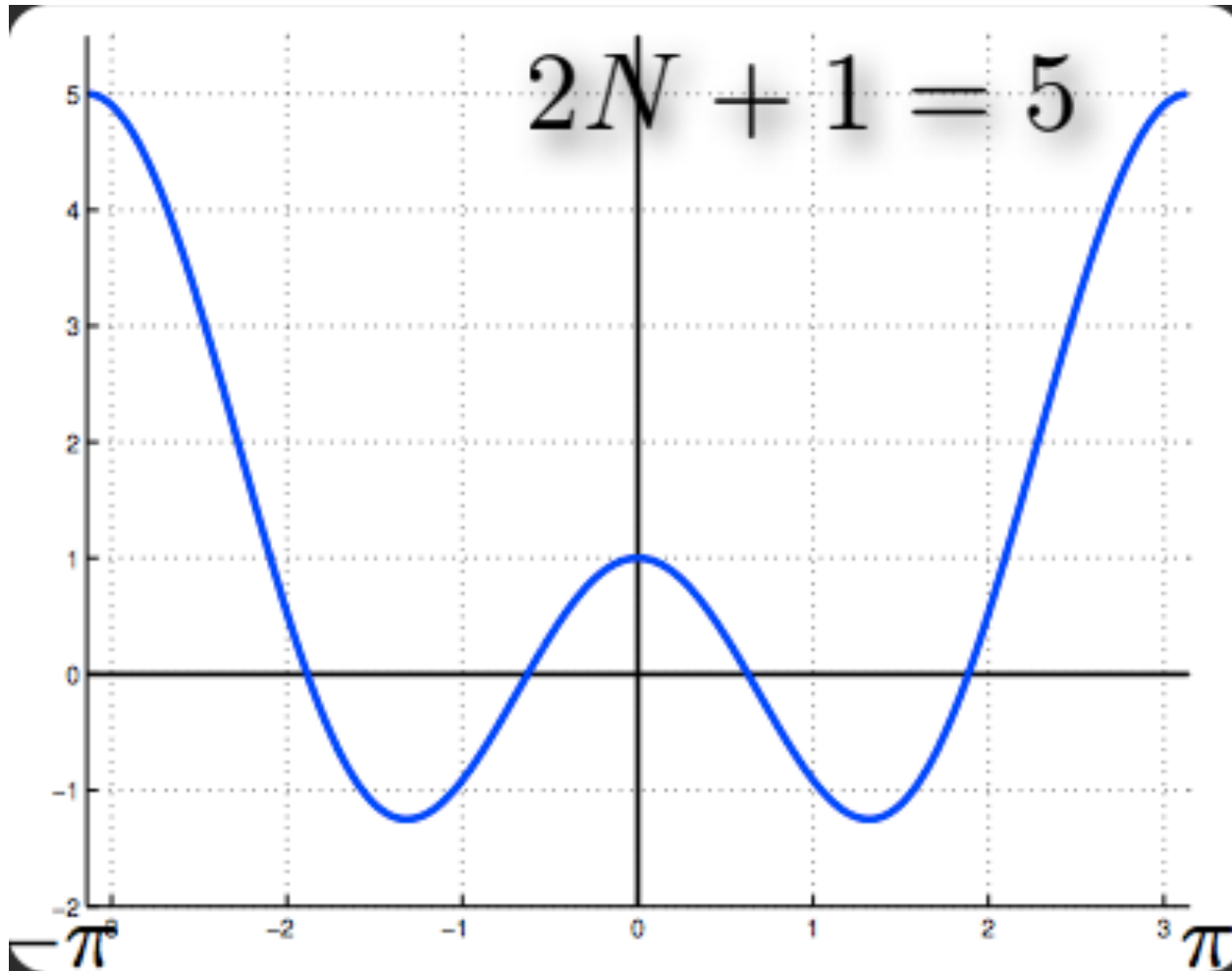
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$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$

Example: Windowed $\cos(\pi n)$



Frequency Response of LTI Systems

LTI Systems

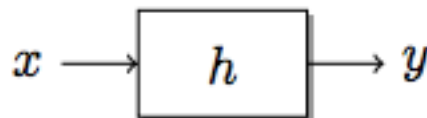
DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be characterized by its impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$

LTI System Frequency Response

- Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

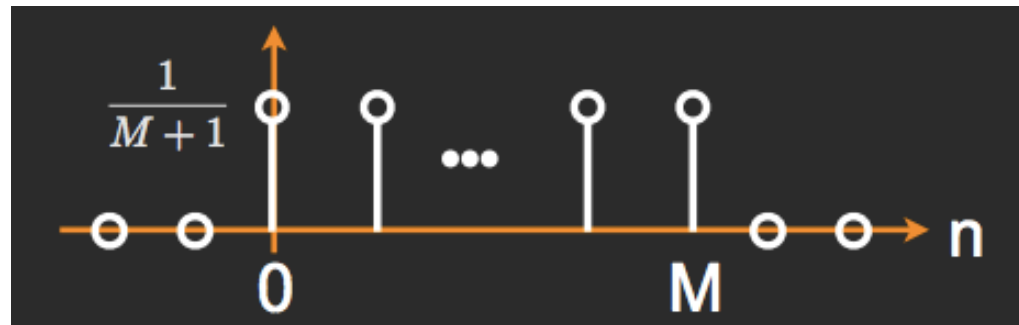
Example: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



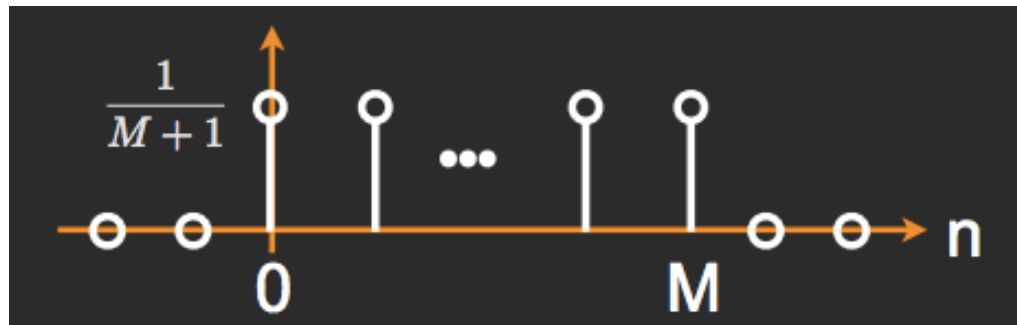
Example: Moving Average

□ Moving Average Filter

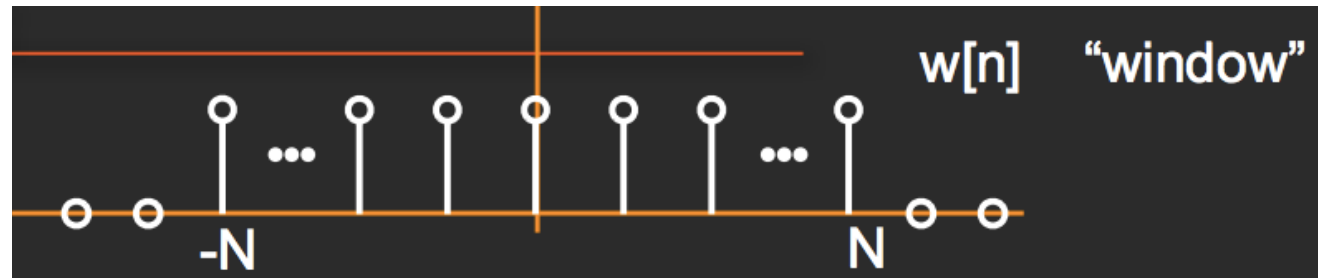
- Causal: $M_1=0$, $M_2=M$

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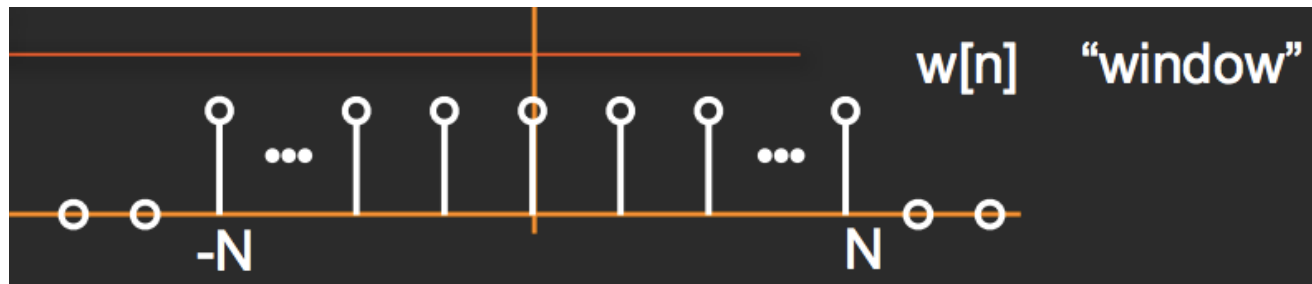
Impulse
response



Scaled & Time
Shifted window



Example: Moving Average

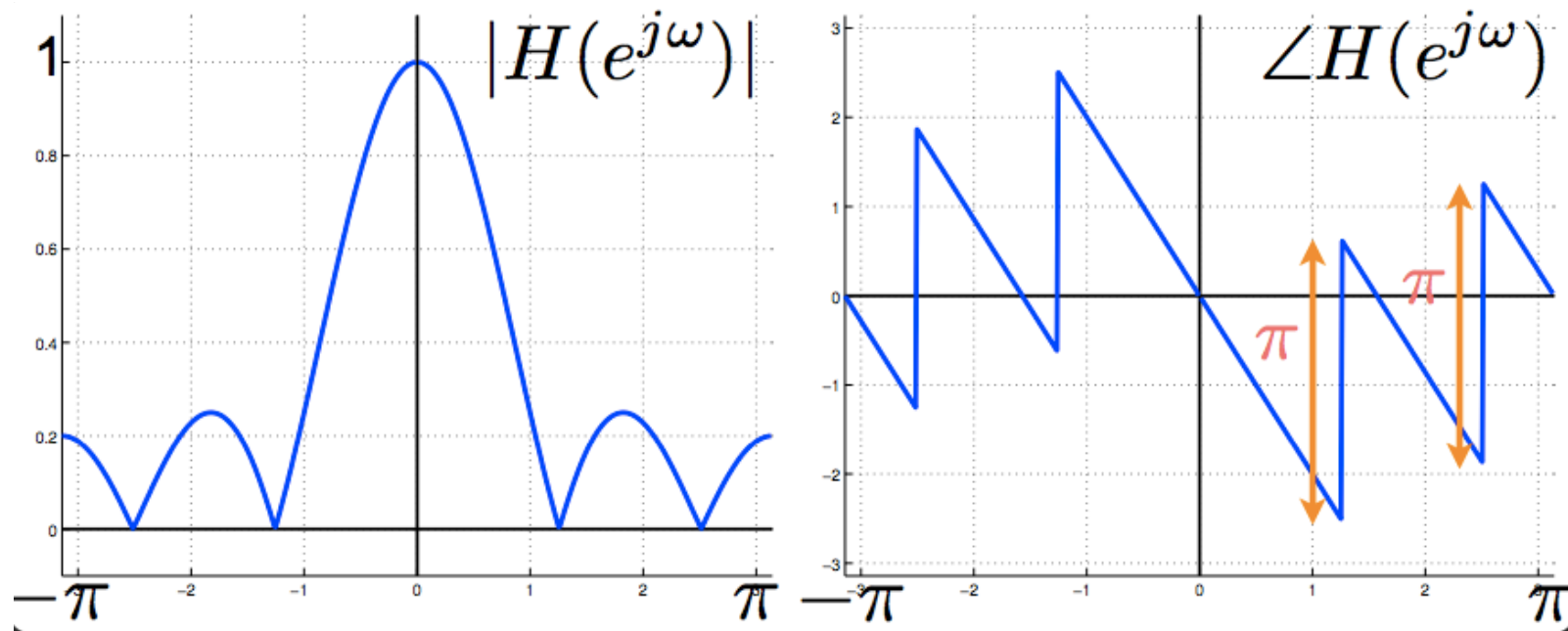


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

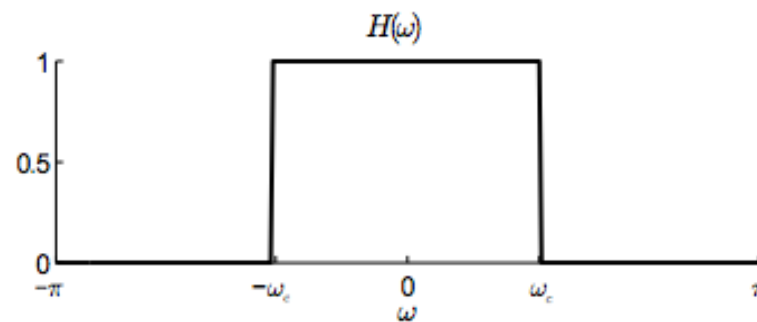
Example: Moving Average



Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

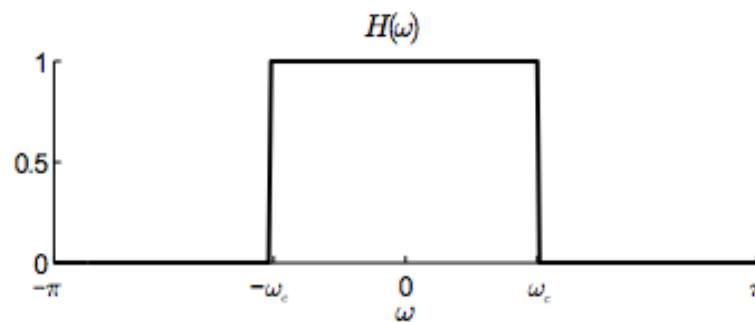
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

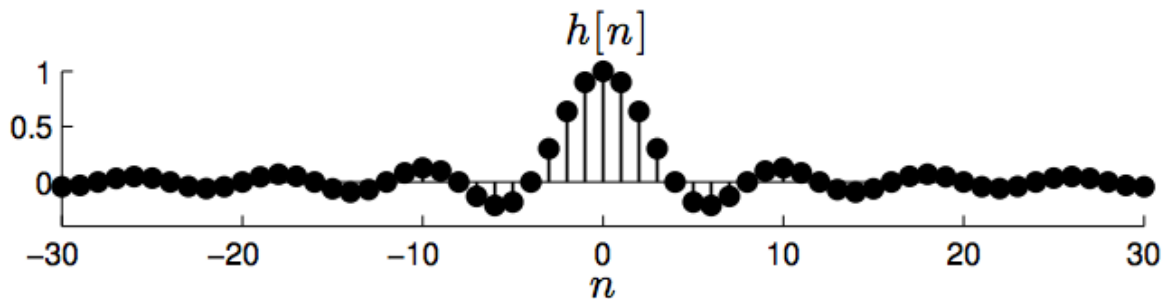
$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Example: Ideal Low-Pass Filter

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$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$

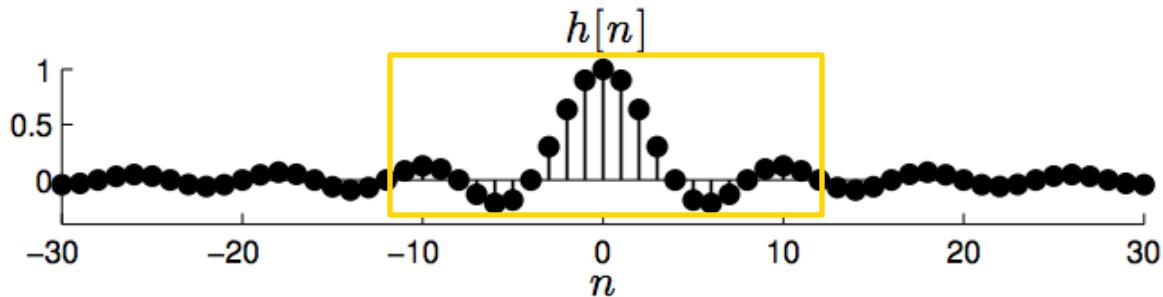


Example: Ideal Low-Pass Filter

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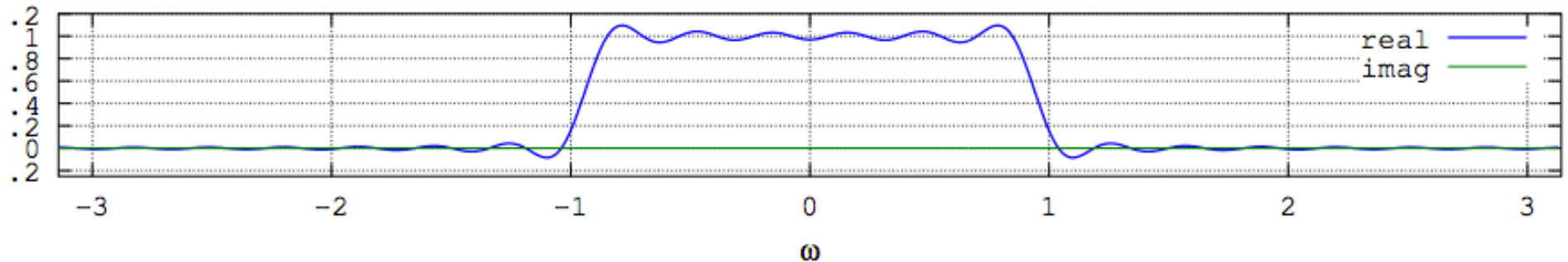
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncation
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of main lobes

Big Ideas

□ Difference Equations

- Help see implementation... more next time with z-transform

□ Discrete Time Fourier Transform

- Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)
- Continuous in frequency over 2π

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

□ Frequency Response of LTI Systems

- Frequency response of impulse response
- Describes scaling and phase shifting of a pure frequency

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$





Admin

- ❑ HW 1 out now
 - Due 1/27 at midnight
 - Submit in Canvas
- ❑ Updated office hours
 - See Piazza and course website for info