## ESE 531: Digital Signal Processing

Lec 4: January 24, 2017

Discrete Time Fourier Transform



## Lecture Outline

- Difference Equations
- Eigenfunctions
- Discrete Time Fourier Transform
  - Definition
  - Properties
- Frequency Response of LTI Systems



## LTI Systems

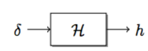
Difference Equations



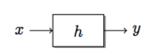
## LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response




- Then the output for an arbitrary input is a sum of weighted, delay impulse responses




$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$



## Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



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$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m y[n-m]}$$

## Difference Equations

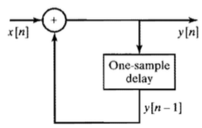
- Accumulator example

$$y[n]=\sum_{k=-\infty}^{n}x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1}x[k]$$

$$y[n]=x[n]+y[n-1]$$

$$\boxed{y[n]-y[n-1]=x[n]}$$




$$\boxed{\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m y[n-m]}$$



## Example: Difference Equation

- Moving Average System

$$y[n]=\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}x[n-k]$$

- Let $M_1=0$ (i.e. system is causal)

$$y[n]=\frac{1}{M_2+1}\sum_{k=0}^{M_2}x[n-k]$$



## Eigenfunctions



## Eigenfunction

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty}e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$



## Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty}x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty}e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$

$$\boxed{H(e^{j\omega})=\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}}$$

- Describes the change in amplitude and phase of signal at frequency $\omega$
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase



## CT vs DT Frequency Response

- $H(e^{j(\omega+2\pi)n})$?

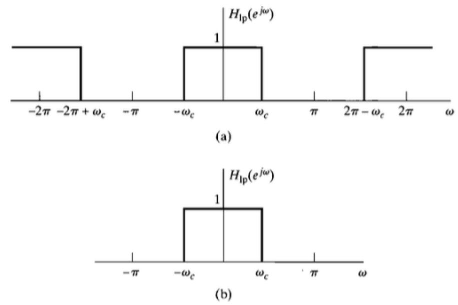
## CT vs DT Frequency Response

- $H(e^{j(\omega+2\pi)n})$?

$$H(e^{j(\omega+2\pi)n}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k}$$
$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k}$$
$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$= H(e^{j\omega n})$$



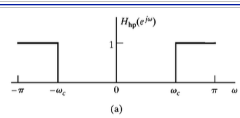
## Periodicity of Low Pass Freq Response



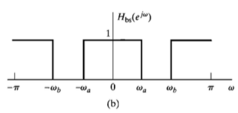



## Other Filters

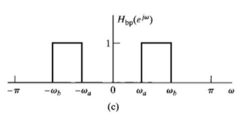
High-pass



Band-stop



Band-pass





## Discrete-Time Fourier Transform (DTFT)



## DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$



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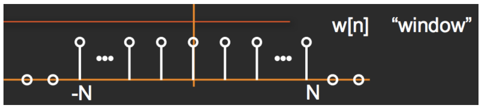
Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi fn}df$$

## Example: Window DTFT




$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k}$$
$$= \sum_{k=-N}^{N} e^{-j\omega k}$$



## Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$
$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$
$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$



## Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$
$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$
$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$
$$= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$



## Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$
$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}}$$
$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$
$$= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

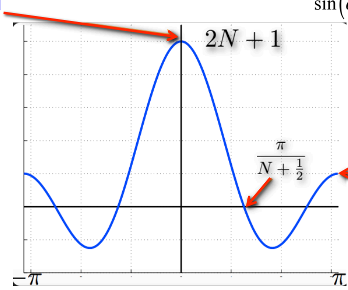
Periodic sinc



## Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σx[n]

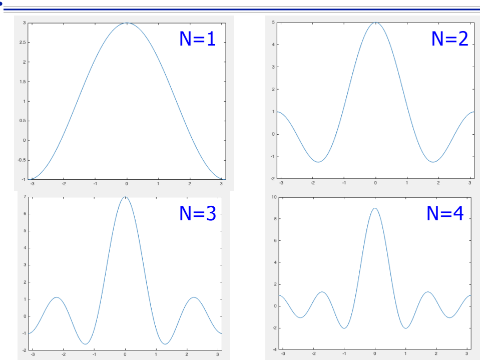



=1 why?

Plot for N=2



## Periodic Sinc



## Properties of the DTFT

- Linearity:

$$ax_1[n]+bx_2[n] \leftrightarrow aX_1(e^{j\omega})+bX_2(e^{j\omega})$$

- Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega}) \qquad \text{If x[n] real}$$

$$\mathrm{Re}\{X(e^{-j\omega})\} = \mathrm{Re}\{X(e^{j\omega})\}$$

$$\mathrm{Im}\{X(e^{-j\omega})\} = -\mathrm{Im}\{X(e^{j\omega})\}$$



## Properties of the DTFT

- Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega}) \qquad \text{If x[n] real}$$

$$x[-n] \leftrightarrow X(e^{-j\omega}) \qquad x[-n] \leftrightarrow X^*(e^{-j\omega})$$

- Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n-n_d] \leftrightarrow e^{-j\omega n_d}X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$



## Properties of the DTFT

- Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Convolution in Time:

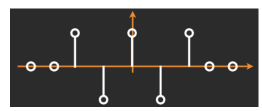
$$y[n] = x[n]*h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$



## Example: Windowed cos(π n)

- What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$



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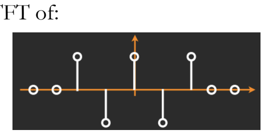

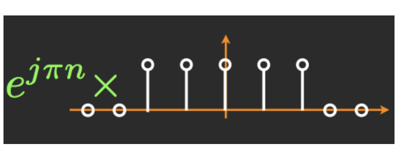




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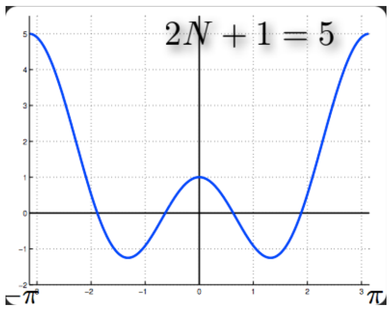
$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

## Example: Windowed cos($\pi$ n)






## Frequency Response of LTI Systems



## LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response
  $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

$x \rightarrow$ [ $h$ ] $\rightarrow y$

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

$$y[n] = x[n] * h[n]$$



## LTI System Frequency Response

- Fourier Transform of impulse response

$x[n]=e^{j\omega n} \rightarrow$ [ LTI System ] $\rightarrow y[n]=H(e^{j\omega})e^{j\omega n}$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

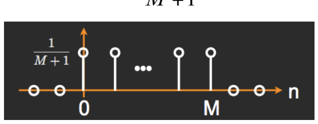


## Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M]+\ldots+x[n]}{M+1}$$

Impulse response



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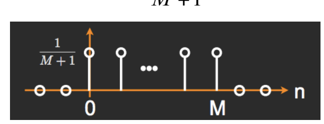
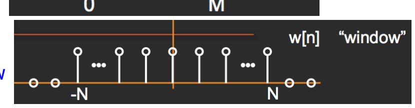



Scaled &Time Shifted window



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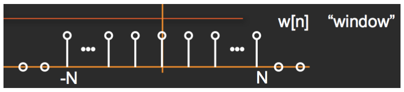
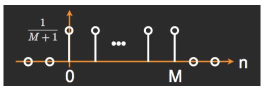


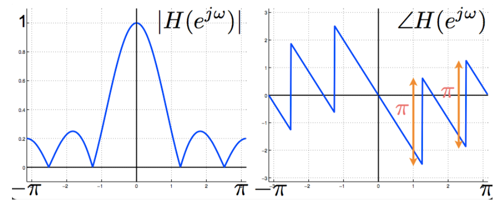

$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$




$$\frac{1}{M+1} w[n-M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$



## Example: Moving Average

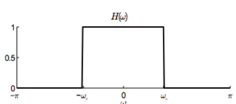





## Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$





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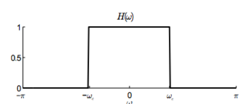

- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \frac{e^{j\omega n}}{2\pi j n}\Big|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$



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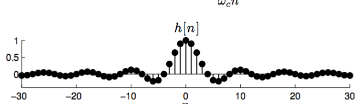



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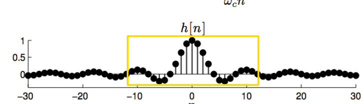
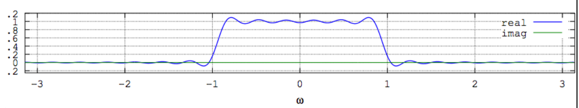


Truncation and shift $\quad h_{LP}[n] = w_N[n-N] \cdot h[n-N]$

## Example: Practical LP Filter




- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of main lobes



## Big Ideas

- Difference Equations
  - Help see implementation… more next time with z-transform
- Discrete Time Fourier Transform
  - Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)
  - Continuous in frequency over $2\pi$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

- Frequency Response of LTI Systems

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

  - Frequency response of impulse response
  - Describes scaling and phase shifting of a pure frequency

$x[n]=e^{j\omega n}$ → LTI System → $y[n]=H(e^{j\omega n})e^{j\omega n}$



## Admin

- HW 1 out now
  - Due 1/27 at midnight
  - Submit in Canvas
- Updated office hours
  - See Piazza and course website for info