

ESE 531: Digital Signal Processing

Lec 5: January 26, 2017
z-Transform



Lecture Outline

- ❑ z-Transform
 - Regions of convergence (ROC) & properties
 - z-Transform properties
- ❑ Inverse z-transform

z-Transform



z-Transform

- ❑ The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- ❑ Very useful for designing and analyzing signal processing systems
- ❑ Properties are very similar to the DTFT with a few caveats



Reminder: DTFT Definition

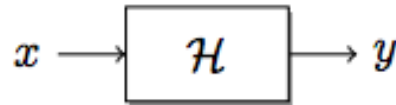
- ❑ The core “basis functions” of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- ❑ The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$Y(\omega) = X(\omega)H(\omega)$$

Reminder: Complex Exponentials

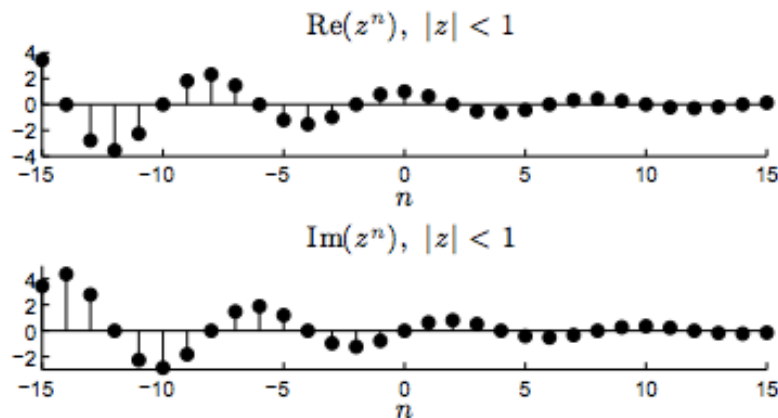
- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z| e^{j\omega}$, $z \in \mathbb{C}$
 - $|z|$ = magnitude of z
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**
- Now we have
$$z^n = (|z|e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$
 - $|z|^n$ is a **real exponential** (a^n with $a = |z|$)
 - $e^{j\omega n}$ is a **complex sinusoid**

Reminder: Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

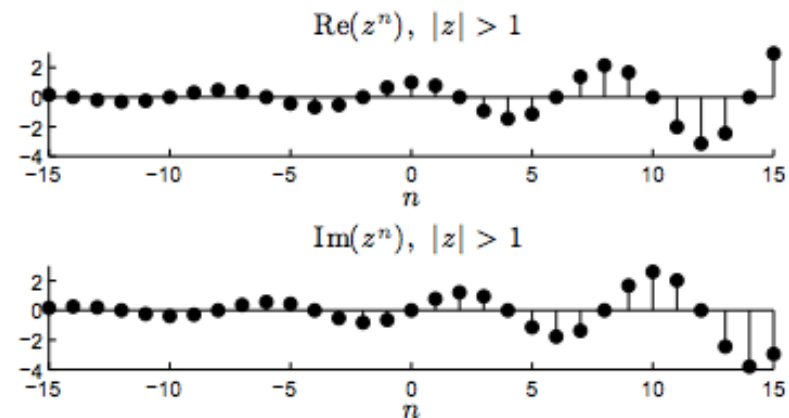
- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

$$|z| < 1$$



Bounded

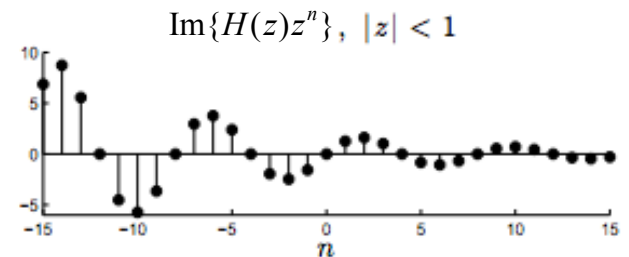
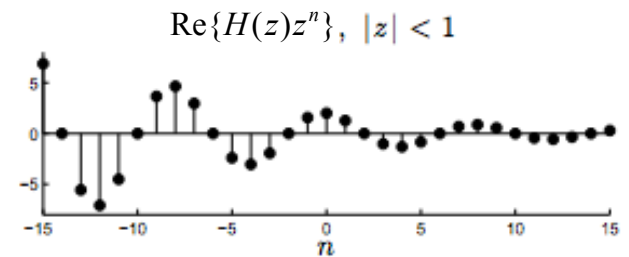
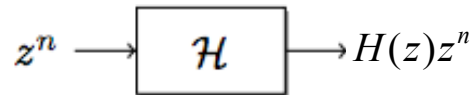
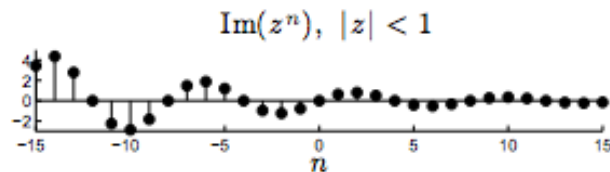
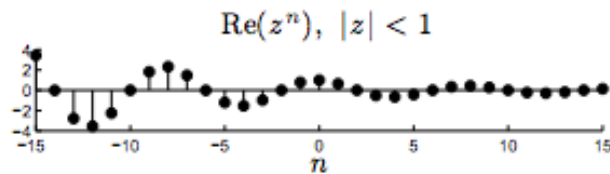
$$|z| > 1$$



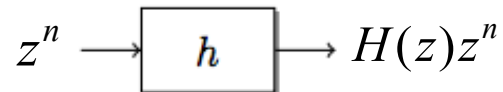
Unbounded

Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials z^n , $z \in \mathbb{C}$



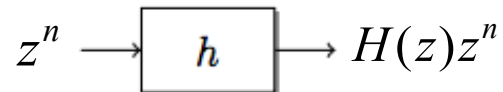
Complex Exponentials as Eigenfunctions



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m} h[m] = \sum_{m=-\infty}^{\infty} z^n z^{-m} h[m] \\ &= \left(\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right) z^n \\ &= H(z) z^n \end{aligned}$$

Complex Exponentials as Eigenfunctions



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m} h[m] = \sum_{m=-\infty}^{\infty} z^n z^{-m} h[m] \\ &= \left(\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right) z^n \\ &= H(z) z^n \end{aligned}$$

$$\sum_{n=-\infty}^{\infty} h[n] z^{-n} = H(z)$$



z-Transform

- Define the **forward z-transform** of $x[n]$ as

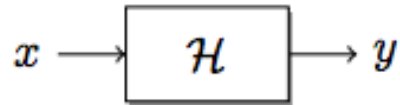
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The core “basis functions” of the z-transform are the complex exponentials z^n with arbitrary $z \in \mathbb{C}$; these are the eigenfunctions of LTI systems for infinite-length signals
- **Notation abuse alert:** We use $X(\bullet)$ to represent both the DTFT $X(\omega)$ and the z-transform $X(z)$; they are, in fact, intimately related

$$X_{\text{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



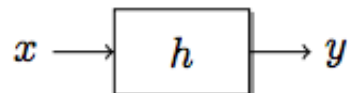
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$Y(z) = X(z) H(z)$$

Proof



- Compute the z -transform of the result of the convolution of x and h
(Note: we are a little cavalier about exchanging the infinite sums below)

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m] h[n-m] \right) z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m] z^{-n} \right) \quad (\text{let } r = n - m) \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r] z^{-r-m} \right) = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) \left(\sum_{r=-\infty}^{\infty} h[r] z^{-r} \right) \\ &= X(z) H(z) \quad \checkmark \end{aligned}$$

Reminder: Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope

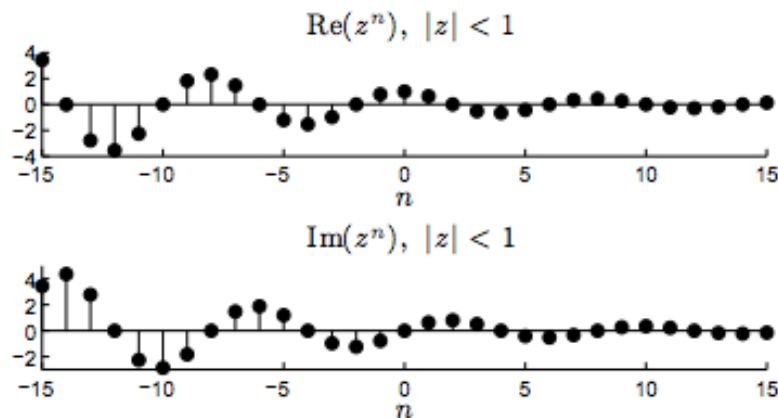
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- $e^{j\omega n}$ is a **complex sinusoid**

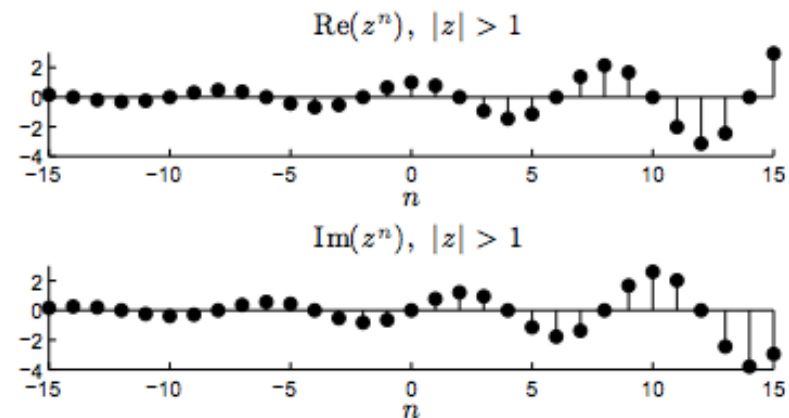
What are we missing?

$$|z| < 1$$



Bounded

$$|z| > 1$$



Unbounded

Region of Convergence (ROC)

Region of Convergence (ROC)

DEFINITION

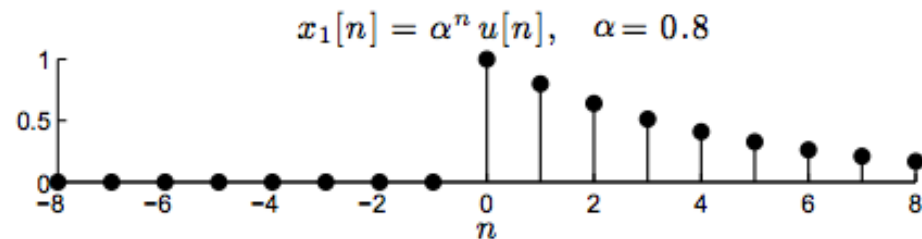
Given a time signal $x[n]$, the **region of convergence** (ROC) of its z -transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n] z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence

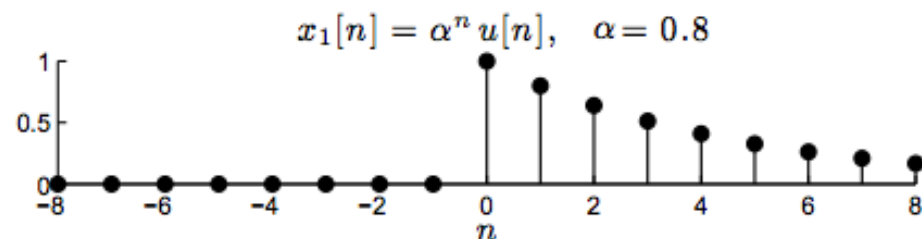
- Example for $\alpha = 0.8$



ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence

- Example for $\alpha = 0.8$



- The **forward** z -transform of $x_1[n]$

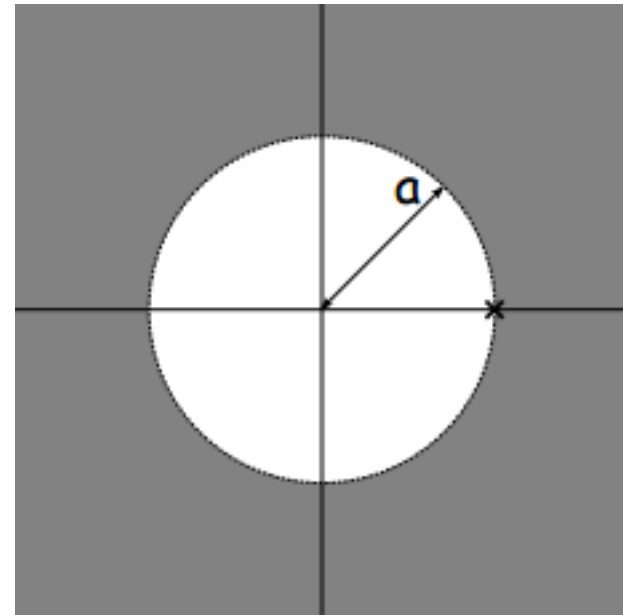
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : |z| > |\alpha|\}$$



- The **forward** z -transform of $x_1[n]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$



ROC Example 1

□ What is the DTFT of $x_1[n] = a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

ROC Example 1

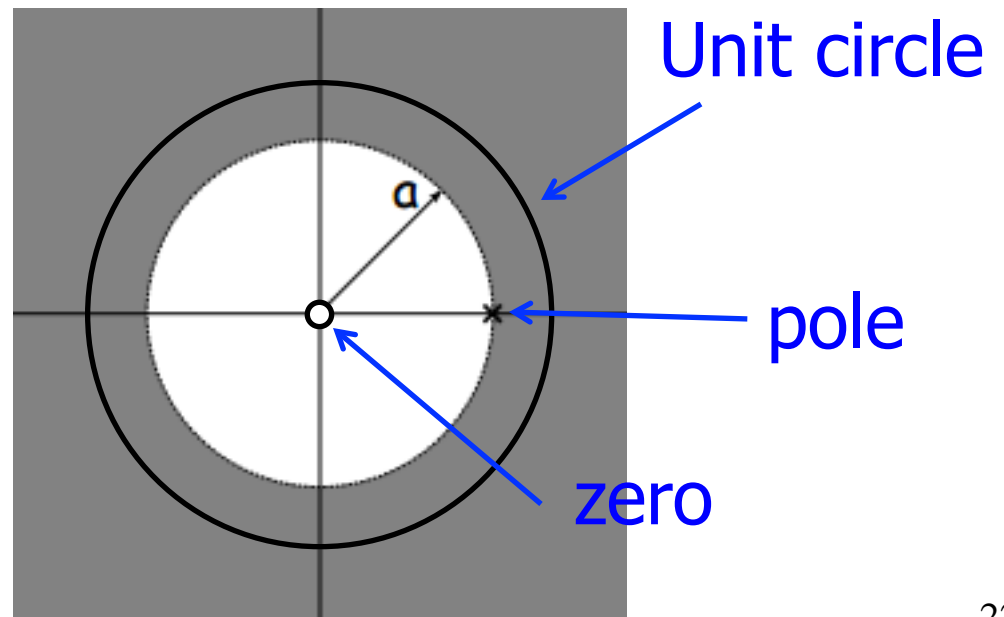
□ What is the DTFT of $x_1[n] = a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

$$X_1(z) = \frac{z}{z - a}$$

$$ROC = \{z : |z| > |a|\}$$





ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$



ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

□ Hint: system is LTI and

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}$$



ROC Example 3

□ What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

Left-sided sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!



ROC Example 4

□ What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

- Hint: system is LTI and

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| < |a|\}$$



ROC Example 5

□ What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence



ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}} \quad \text{Zero cancels pole}$$
$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

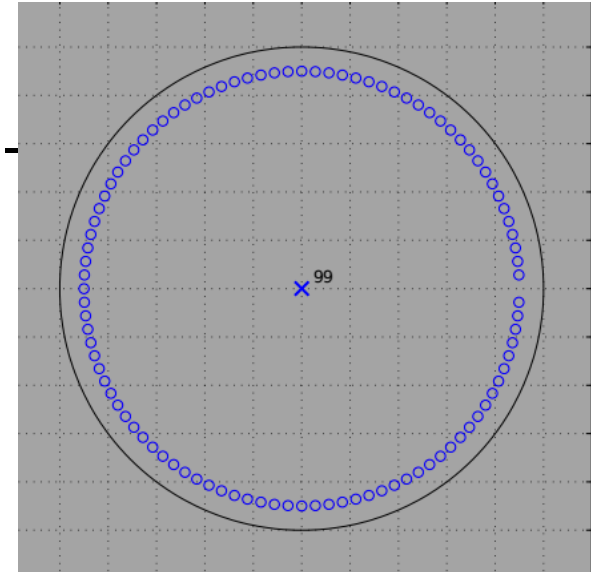
$$x_6[n] = a^n u[n] u[-n + M]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

M=100



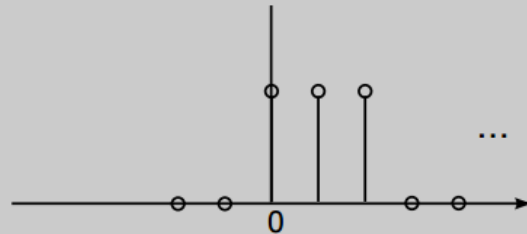


Properties of ROC

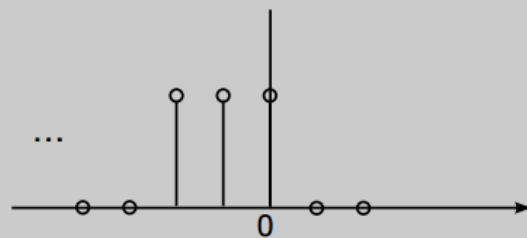
- ❑ A ring or a disk in Z -plane, centered at the origin
- ❑ DTFT converges *if and only if* ROC includes the unit circle
- ❑ ROC can't contain poles

Properties of ROC

- For finite duration sequences, ROC is the entire z -plane, except possibly $z=0$, $z=\infty$



$$X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0$$



$$X(z) = 1 + z^1 + z^2 \quad \text{ROC excludes } z = \infty$$



Properties of ROC

- ❑ For right-sided sequences: ROC extends outward from the outermost pole to infinity
 - Examples 1,2
- ❑ For left-sided: inwards from inner most pole to zero
 - Example 3
- ❑ For two-sided, ROC is a ring - or do not exist
 - Examples 4,5



Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \Leftrightarrow X(z)$$

$$x[n - n_d] \Leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \Leftrightarrow X(z)$$

$$z_0^n x[n] \Leftrightarrow X\left(\frac{z}{z_0}\right)$$

Properties of z-Transform

□ Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

□ Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$\text{ROC}_Y \text{ at least } \text{ROC}_X \wedge \text{ROC}_H$$



Inverse z-Transform

- Recall the **inverse DTFT**

$$x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}$$

- There exists a similar formula for the **inverse z-transform** via a contour integral in the complex z-plane

$$x[n] = \oint_{\mathcal{C}} X(z) z^n \frac{dz}{j2\pi z}$$

- Evaluation of such integrals is fun, but beyond the scope of this course



Inverse z-Transform

- Ways to avoid it:
 - Inspection (known transforms)
 - Properties of the z-transform
 - Power series expansion
 - Partial fraction expansion
 - Residue theorem



Big Ideas

□ z-Transform

- Uses complex exponential eigenfunctions to represent discrete time sequence
 - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
- Draw pole-zero plots
- Must specify region of convergence (ROC)

□ z-Transform properties

- Similar to DTFT

□ Inverse z-transform

- Avoid it!



Admin

- ❑ HW 1 due tomorrow at midnight
 - Submit **single pdf** in Canvas
 - Don't need to submit .m file, just the code
- ❑ HW 2 posted tonight after class
- ❑ Advice
 - Want to try and develop intuition
 - Similar to the circuit intuition we always talk about
 - Practice problems at end of chapter with answers in the text
 - Shlesh will post on Piazza suggested problems each week
 - Matlab resources
 - Mathworks website
 - See piazza for other references