ESE 531: Digital Signal Processing

Lec 7: February 2nd, 2017

Sampling and Reconstruction

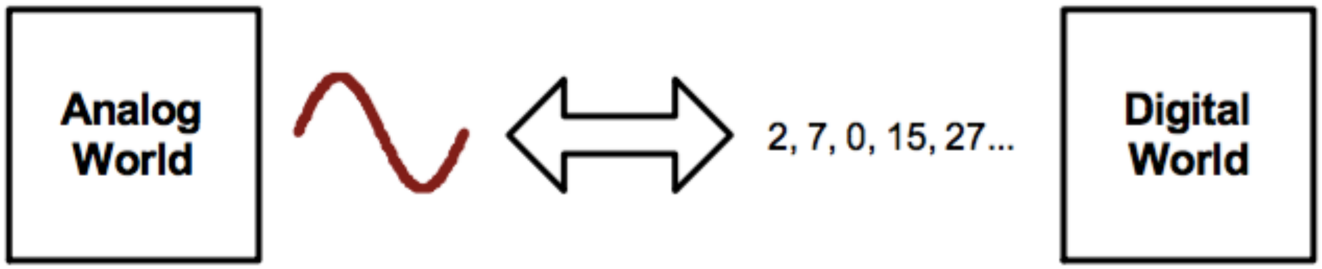
Lecture Outline

- Sampling
- Frequency Response of Sampled Signal
- Reconstruction

Video Example

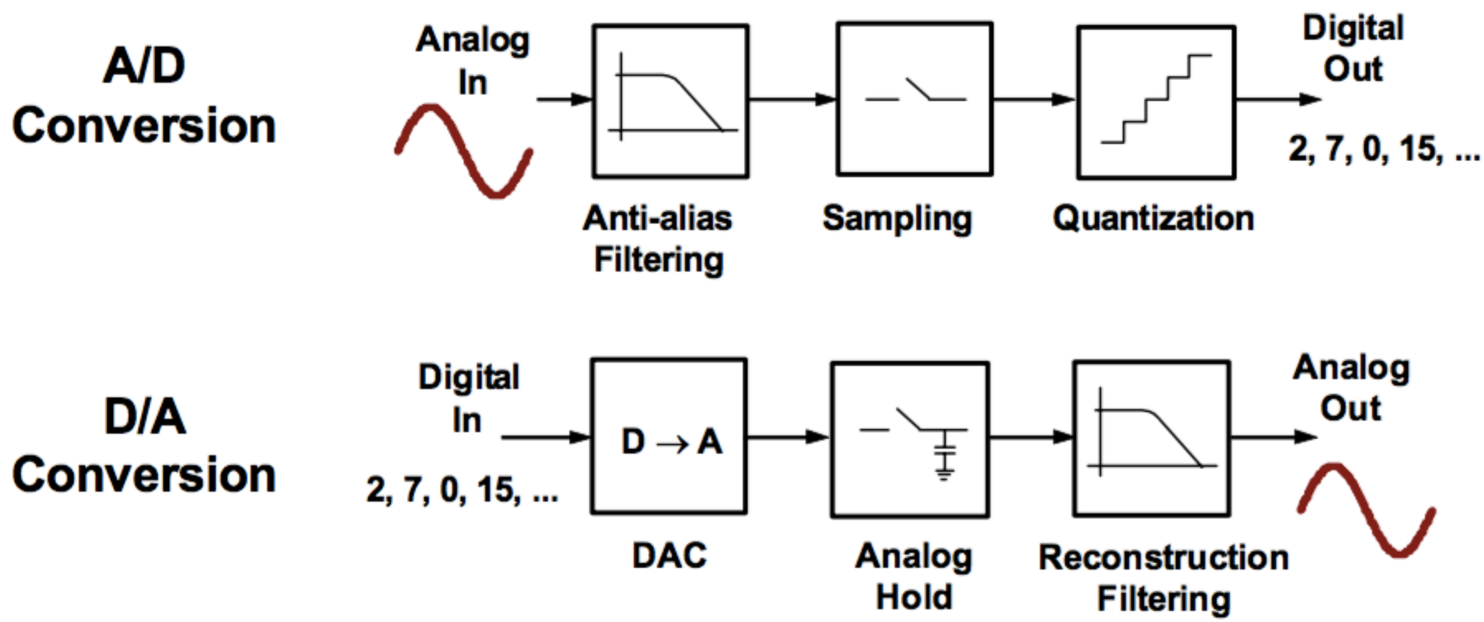
- https://www.youtube.com/watch?v=ByTsISFXUoY

The Data Conversion Problem



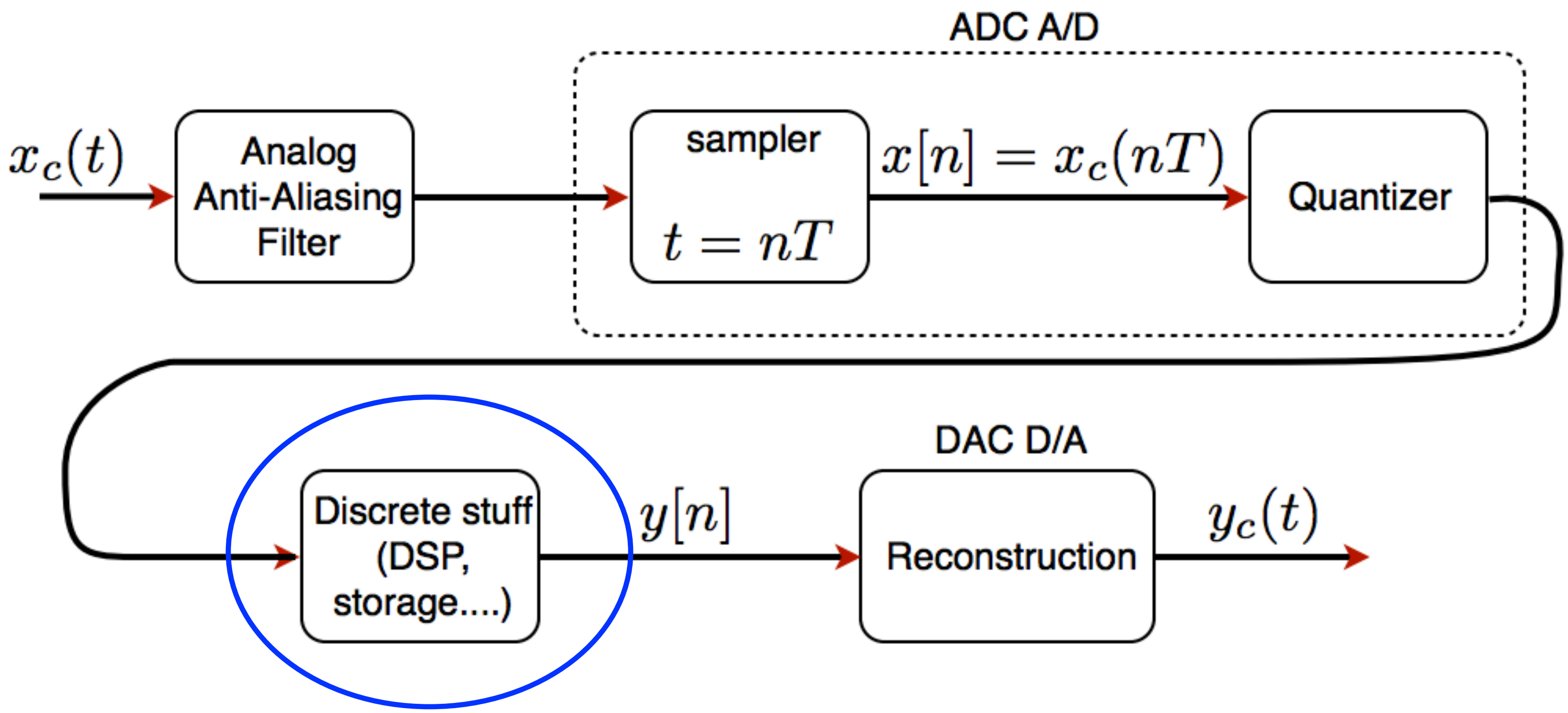
- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

Overview

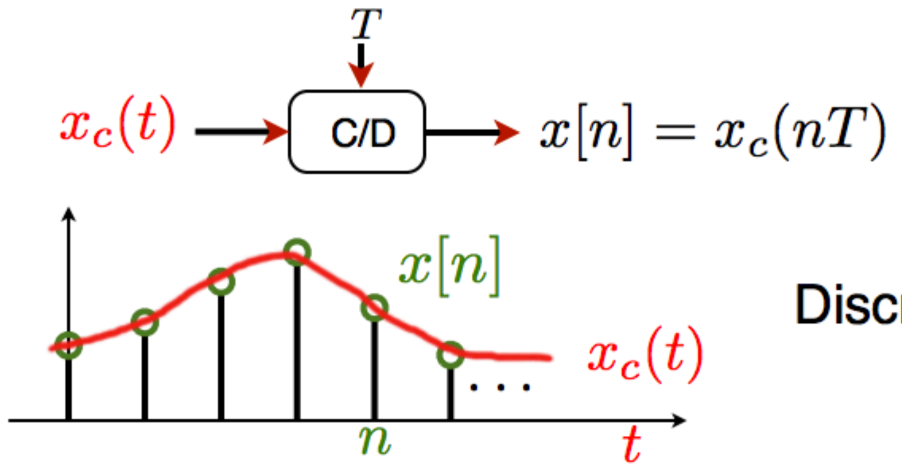


- We'll first look at these building blocks from a functional, "black box" perspective
  - Later refine and look at implementation

DSP System


ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
Discrete stuff (DSP, storage....)
$y[n]$
DAC D/A
Reconstruction
$y_c(t)$

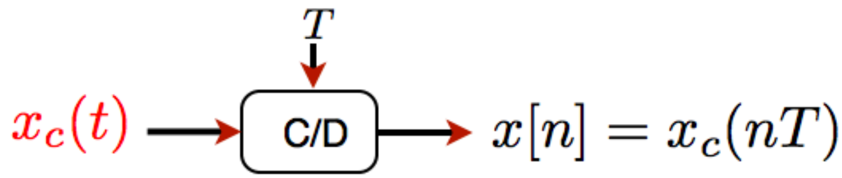
Ideal Sampling Model

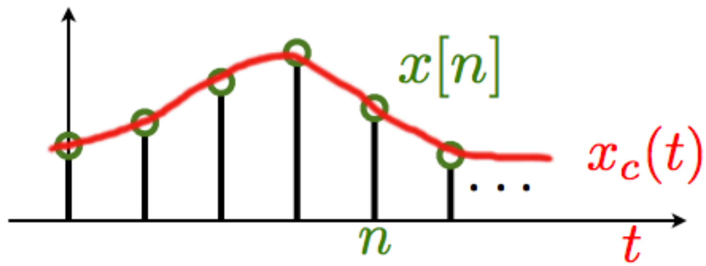


Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2π/T

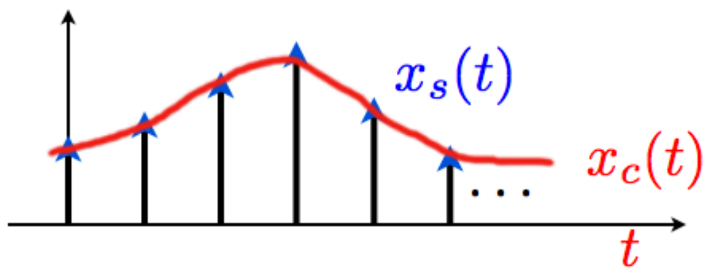
Ideal Sampling Model





Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, $\delta(t)$
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT)$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad\qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x[n]$:D.T

$$X(e^{j\omega}) = \sum_{n} x[n]e^{-j\omega n}$$

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

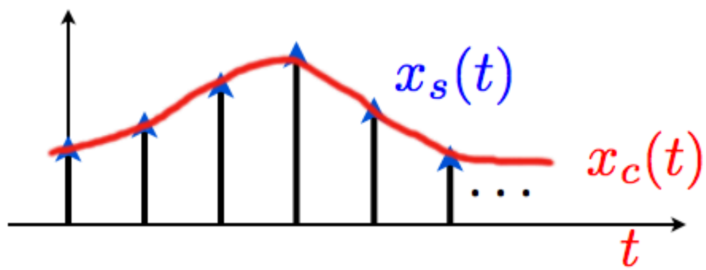
$x_s(t)$:C.T
$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$x[n]$:D.T
$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})\big|_{\omega=\Omega T}$$

Frequency Domain Analysis



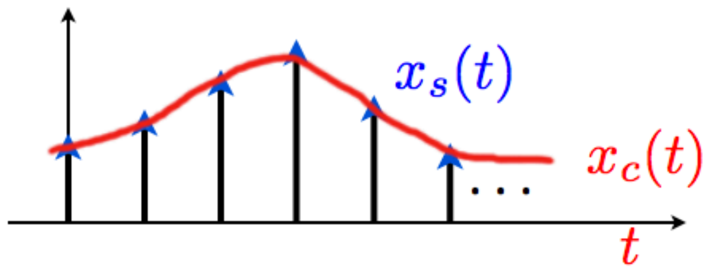
Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Frequency Domain Analysis



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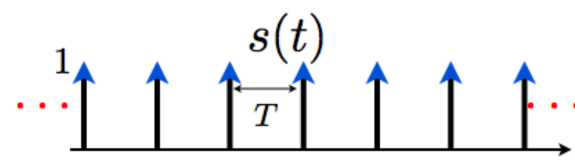
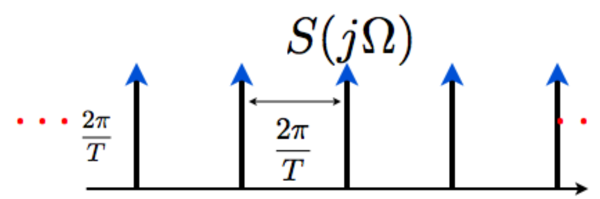
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) \quad \leftrightarrow \quad S(j\Omega)$$

$$S(j\Omega) \quad = \quad \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

$\frac{2\pi}{T} = \Omega_s$





Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

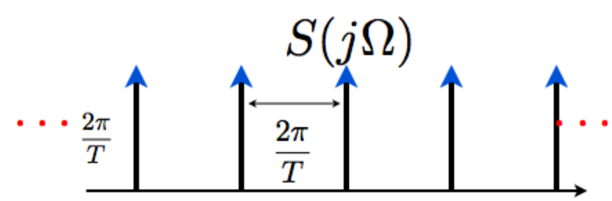
Frequency Domain Analysis

$$x_s(t) = s(t)\cdot x_c(t)$$

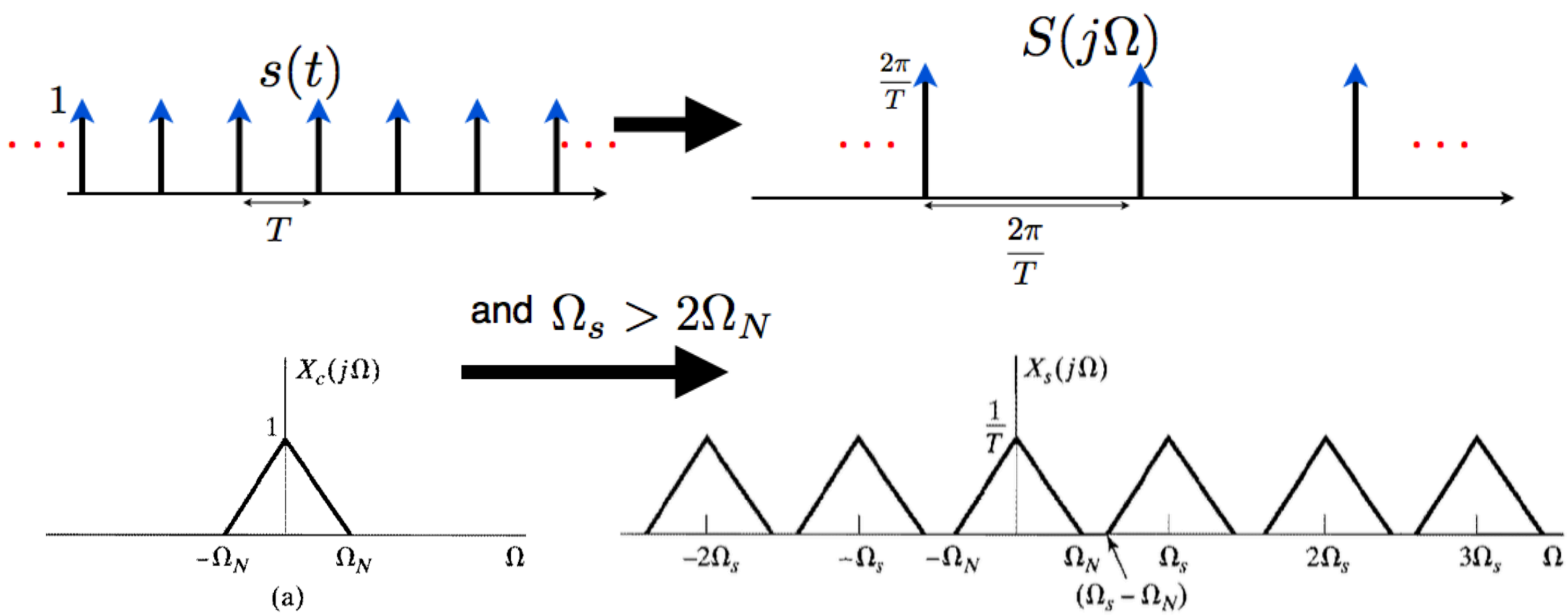
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$

$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - \frac{2\pi}{T}k)$$

$\frac{2\pi}{T} = \Omega_s$



Frequency Domain Analysis



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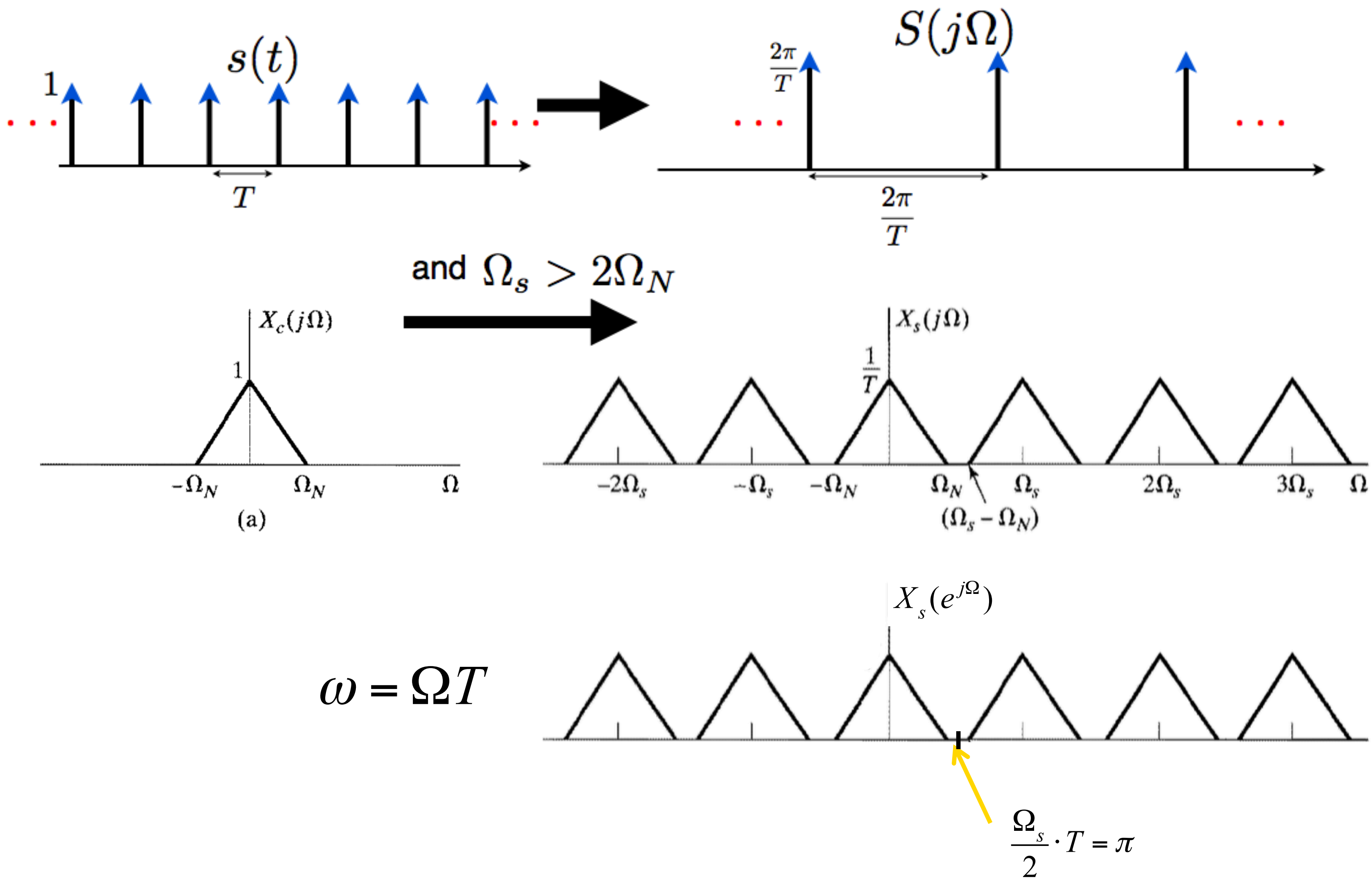

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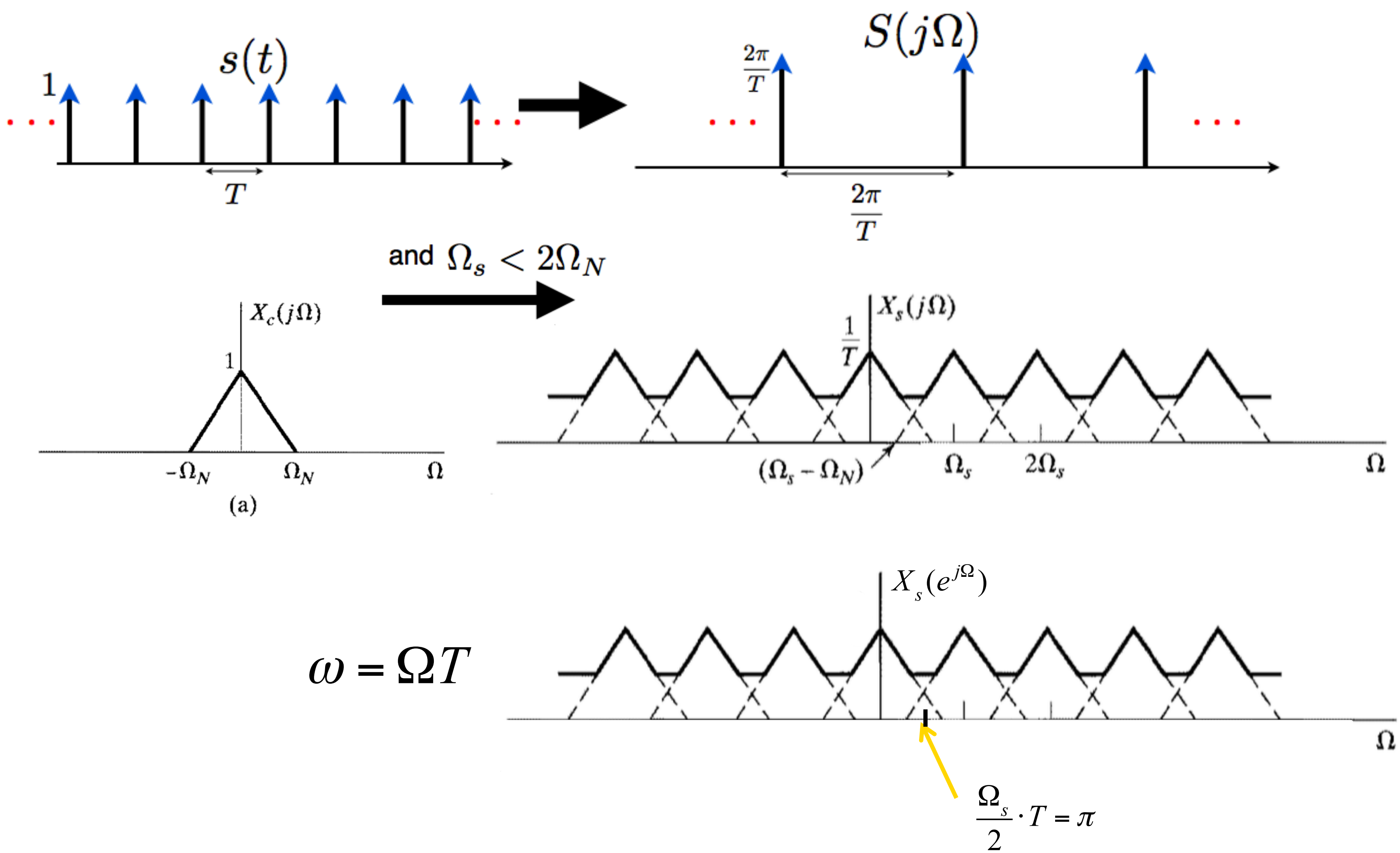
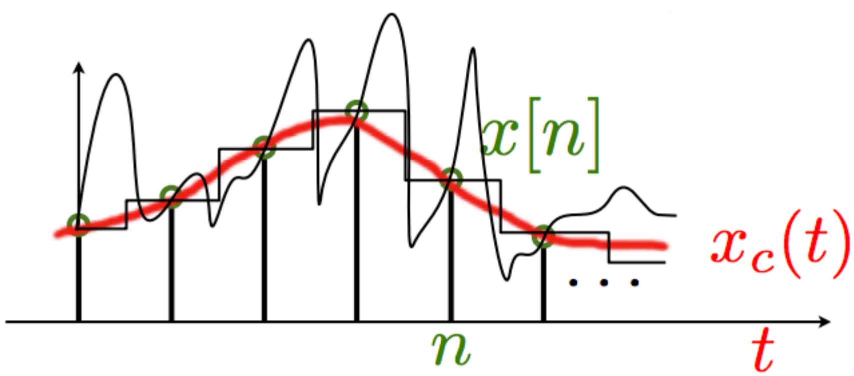


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

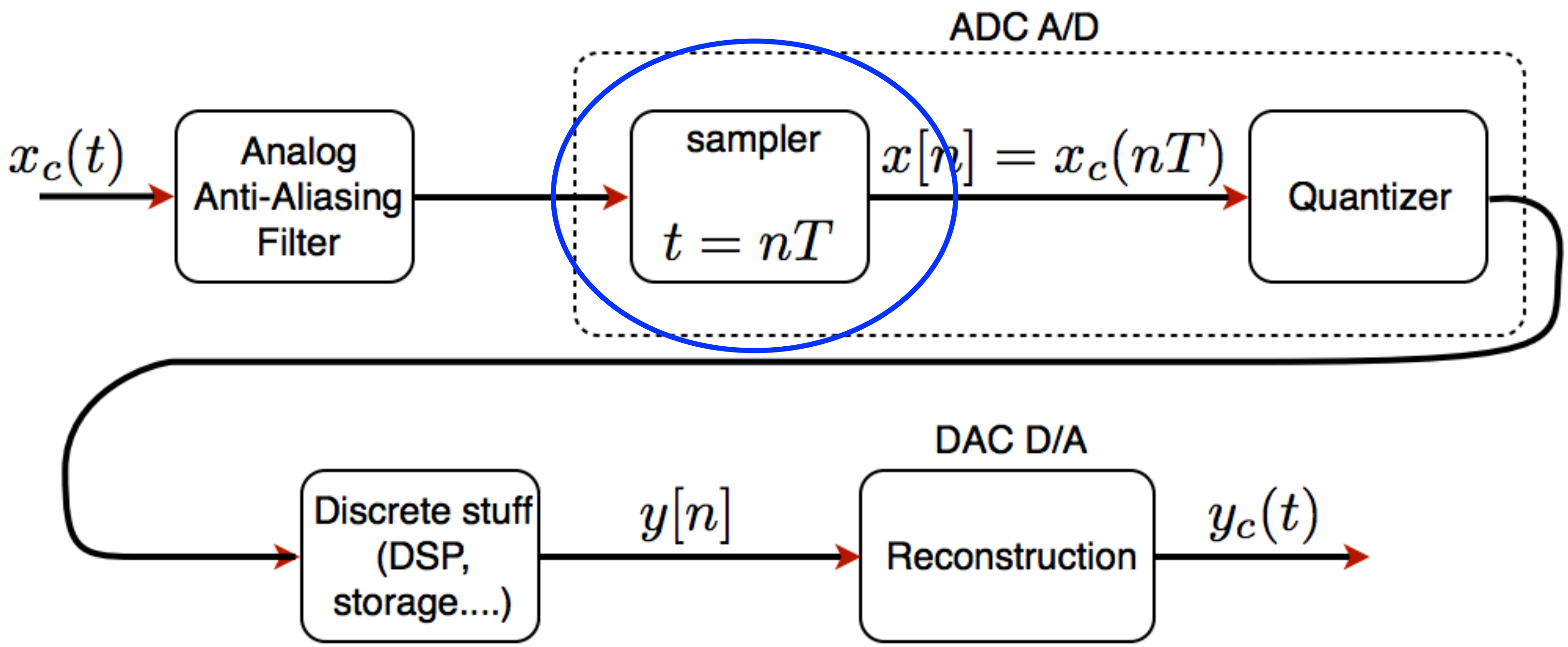
$$X_c(j\Omega) = 0 \ \ \forall \ \ |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness



Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

DSP System



ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
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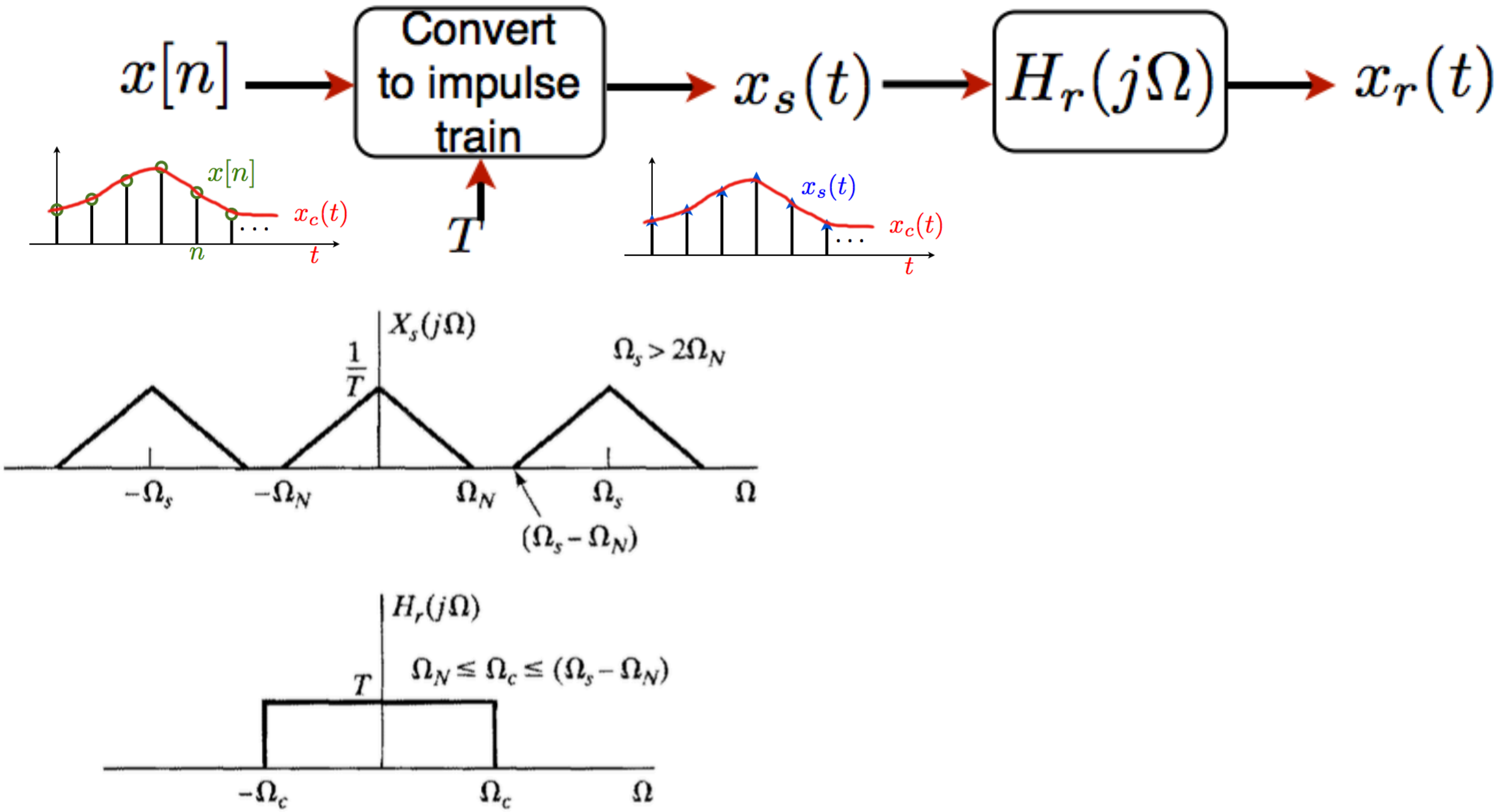
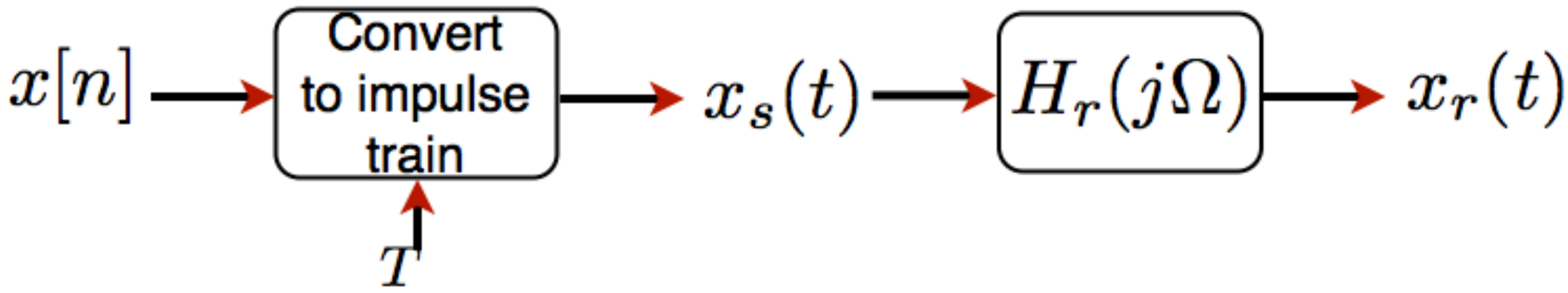
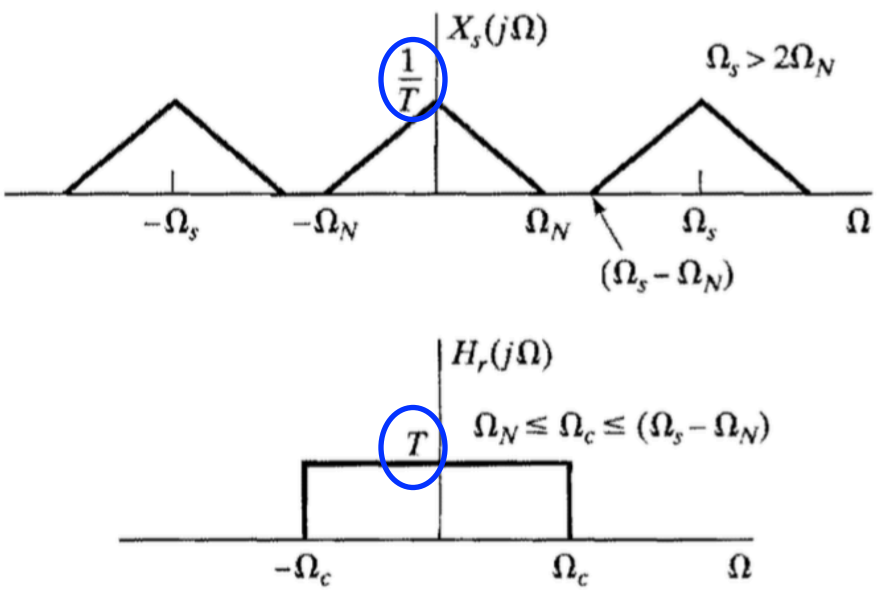
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

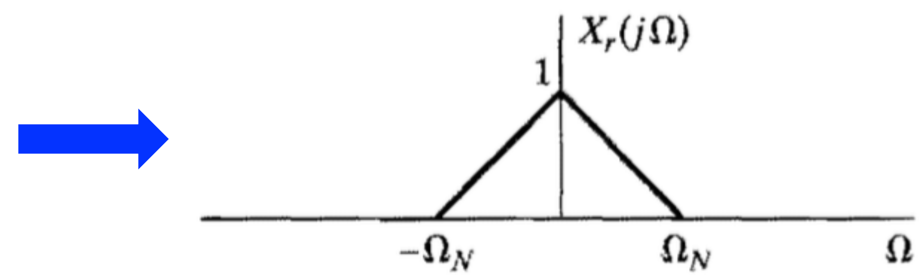
Reconstruction in Frequency Domain



Reconstruction in Frequency Domain







Example: Cosine Input

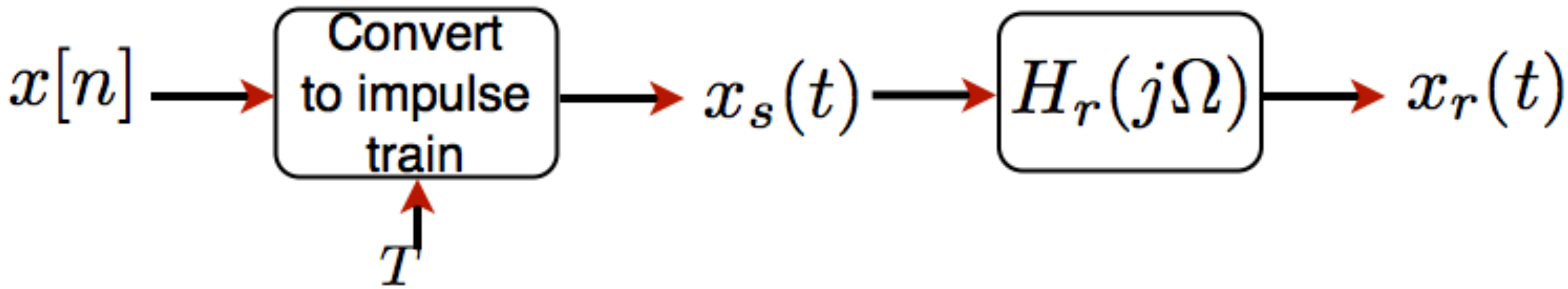
- Sample the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

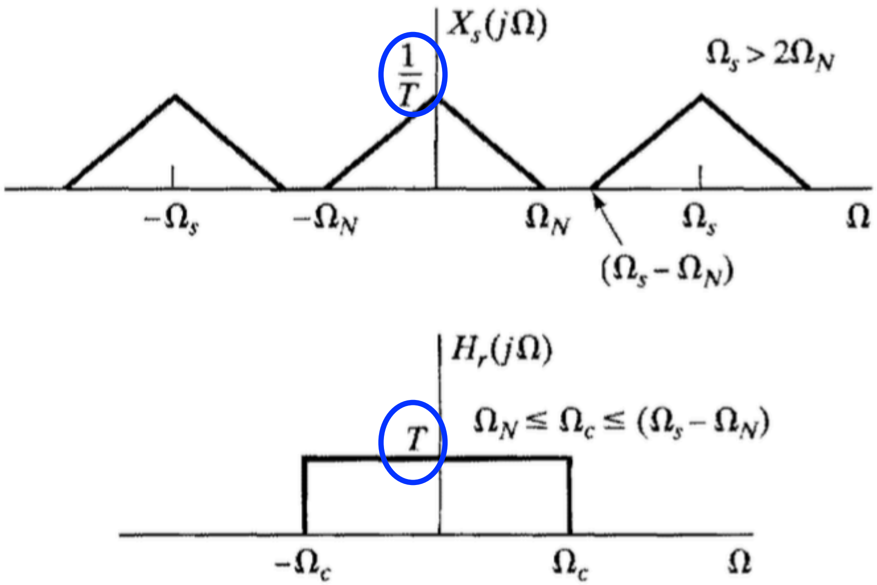
$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

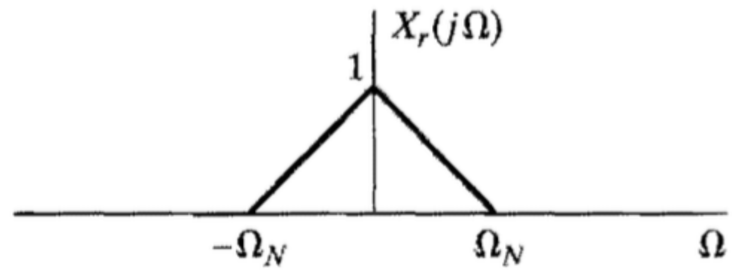
Example: Cosine Input

- Sample the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

Reconstruction in Frequency Domain





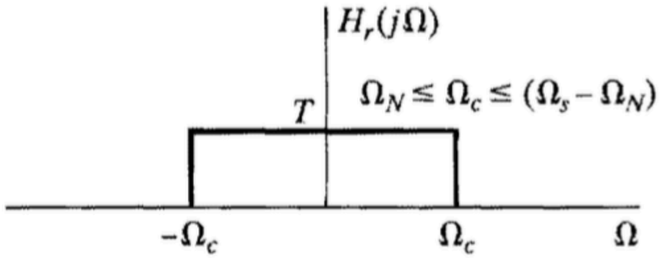


$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

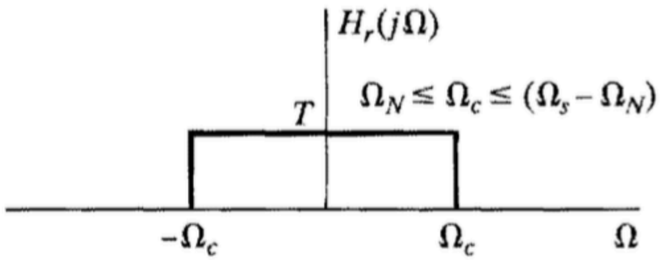
Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

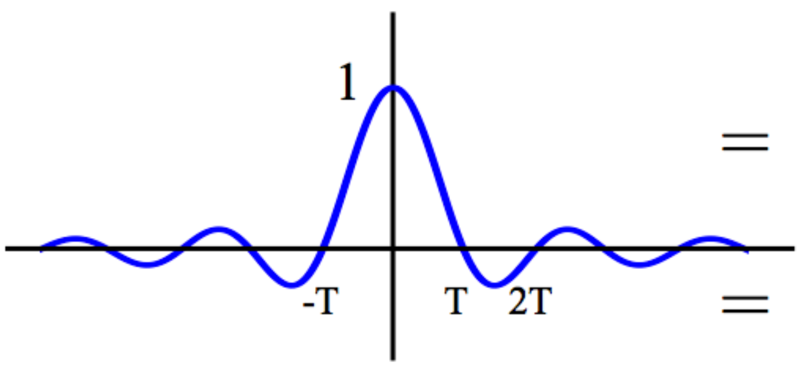


$$h_r(t) = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain



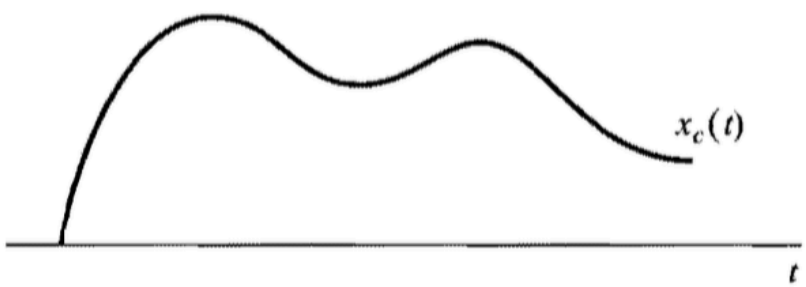
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} s^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\
&= \text{sinc}(\frac{t}{T})
\end{aligned}
$$

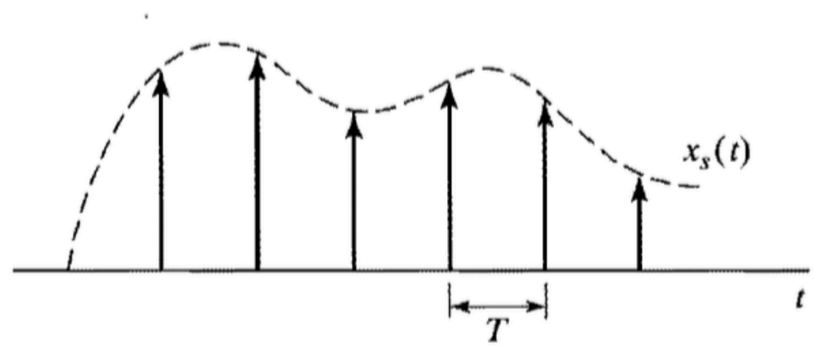


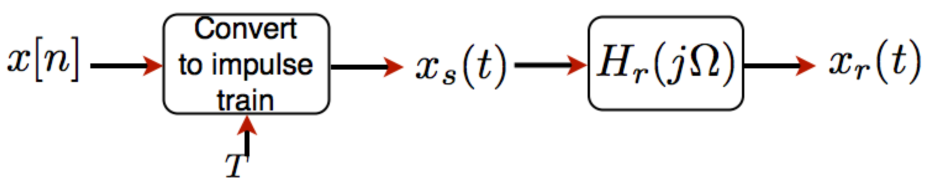
Reconstruction in Time Domain

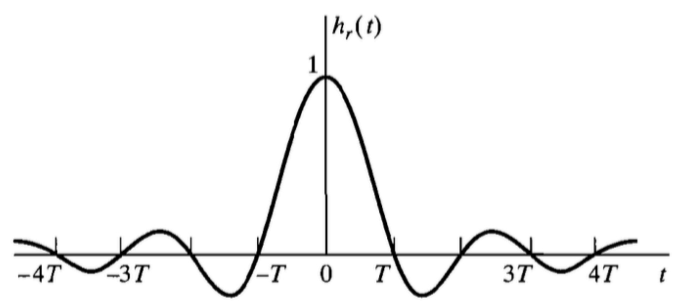
$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

$$= \sum_n x[n] h_r(t - nT)$$









Reconstruction in Time Domain

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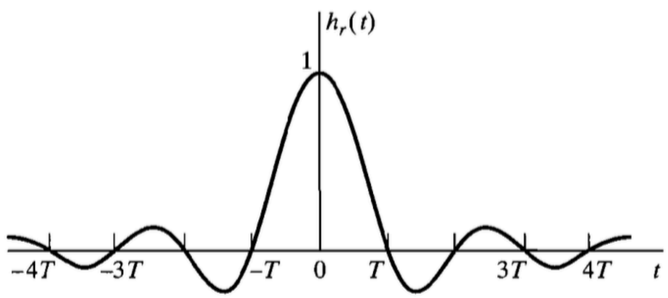

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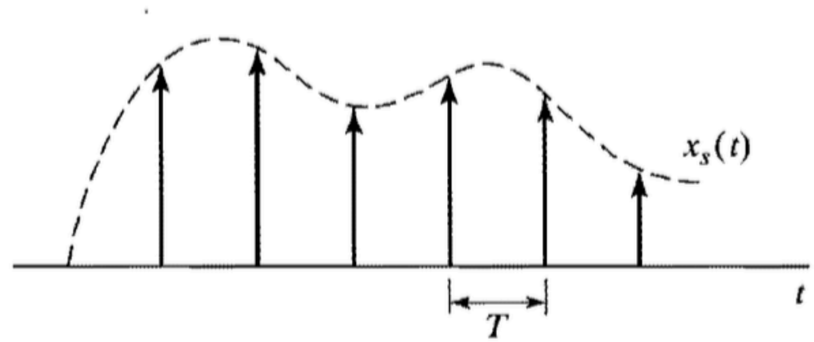

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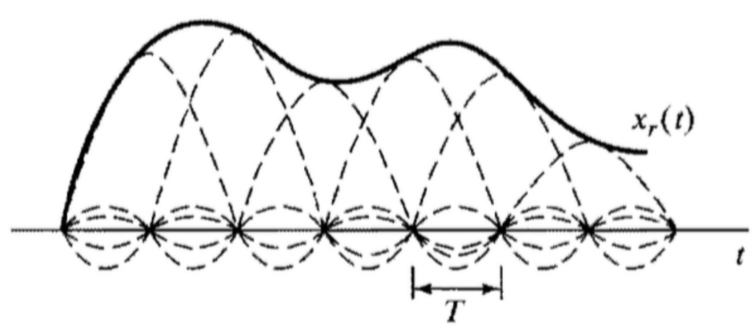
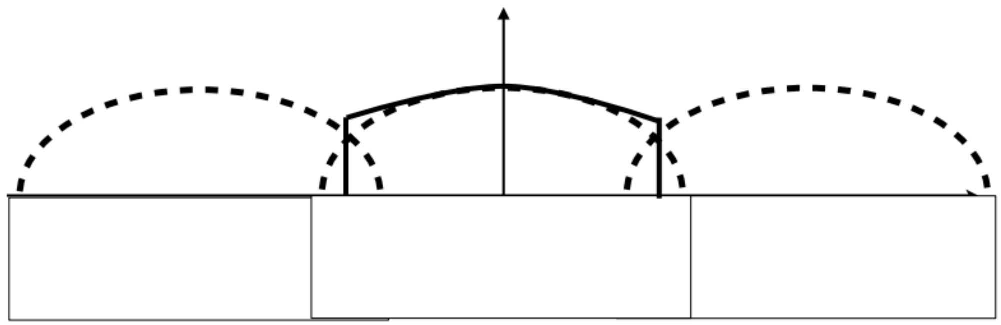


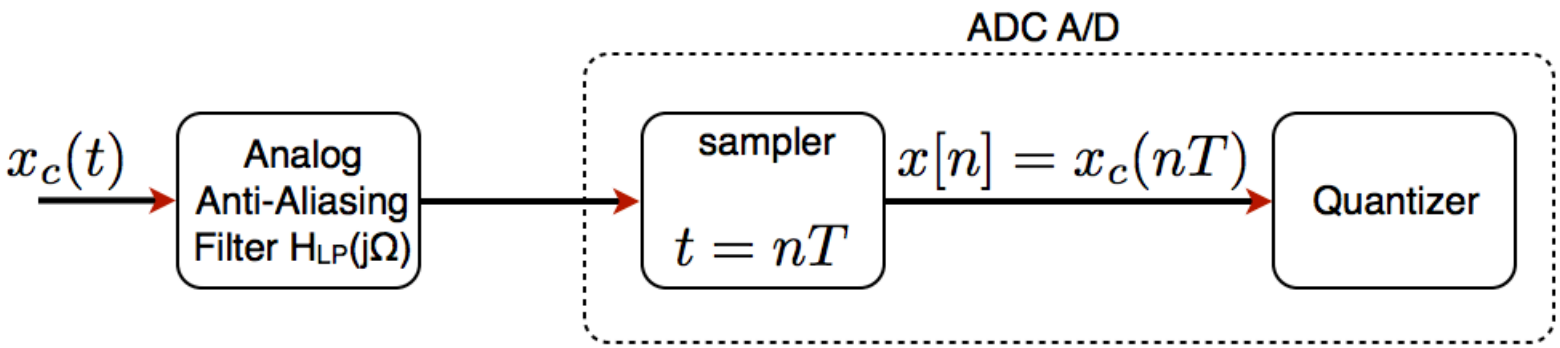
The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Aliasing

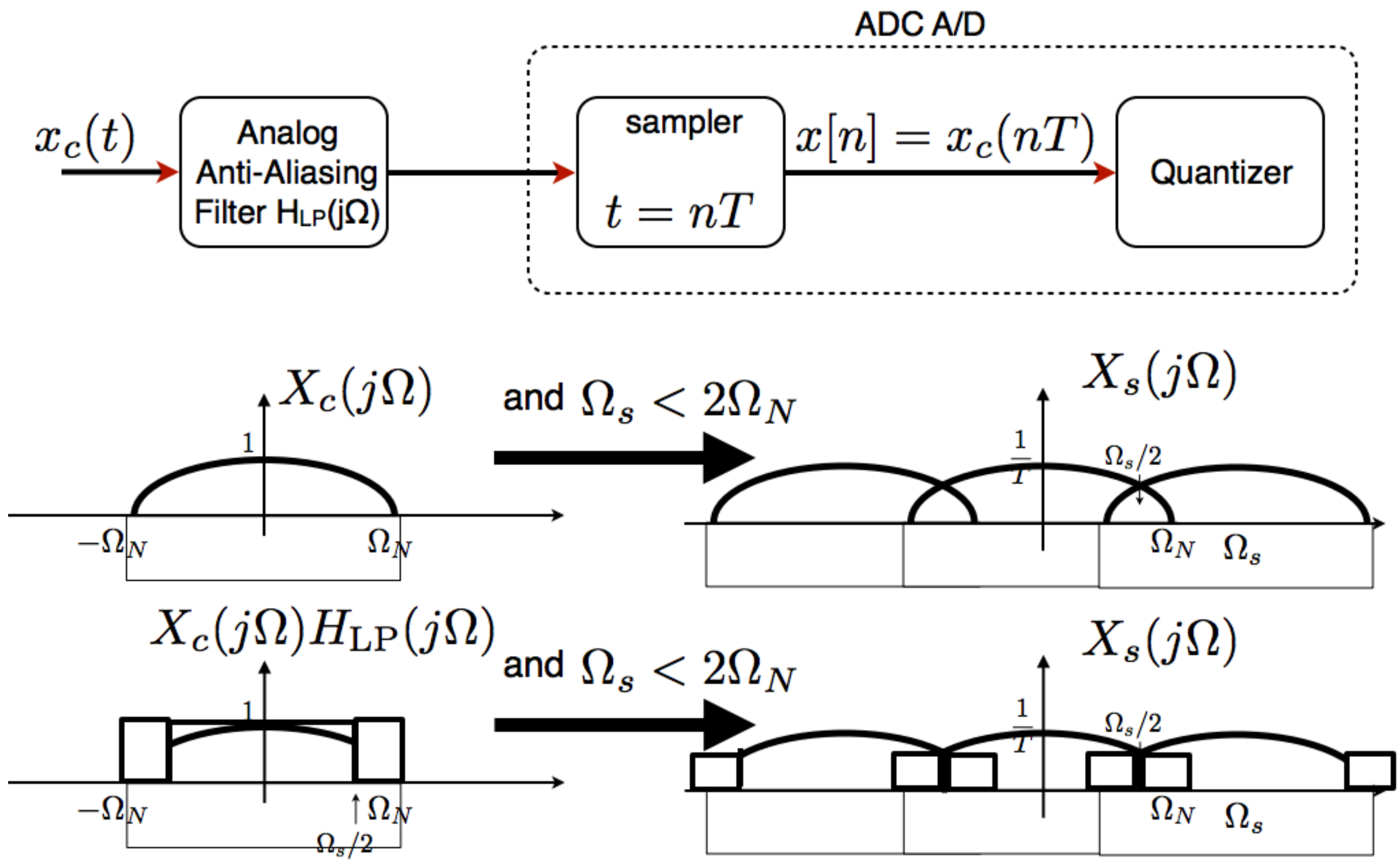
- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$
- 

$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

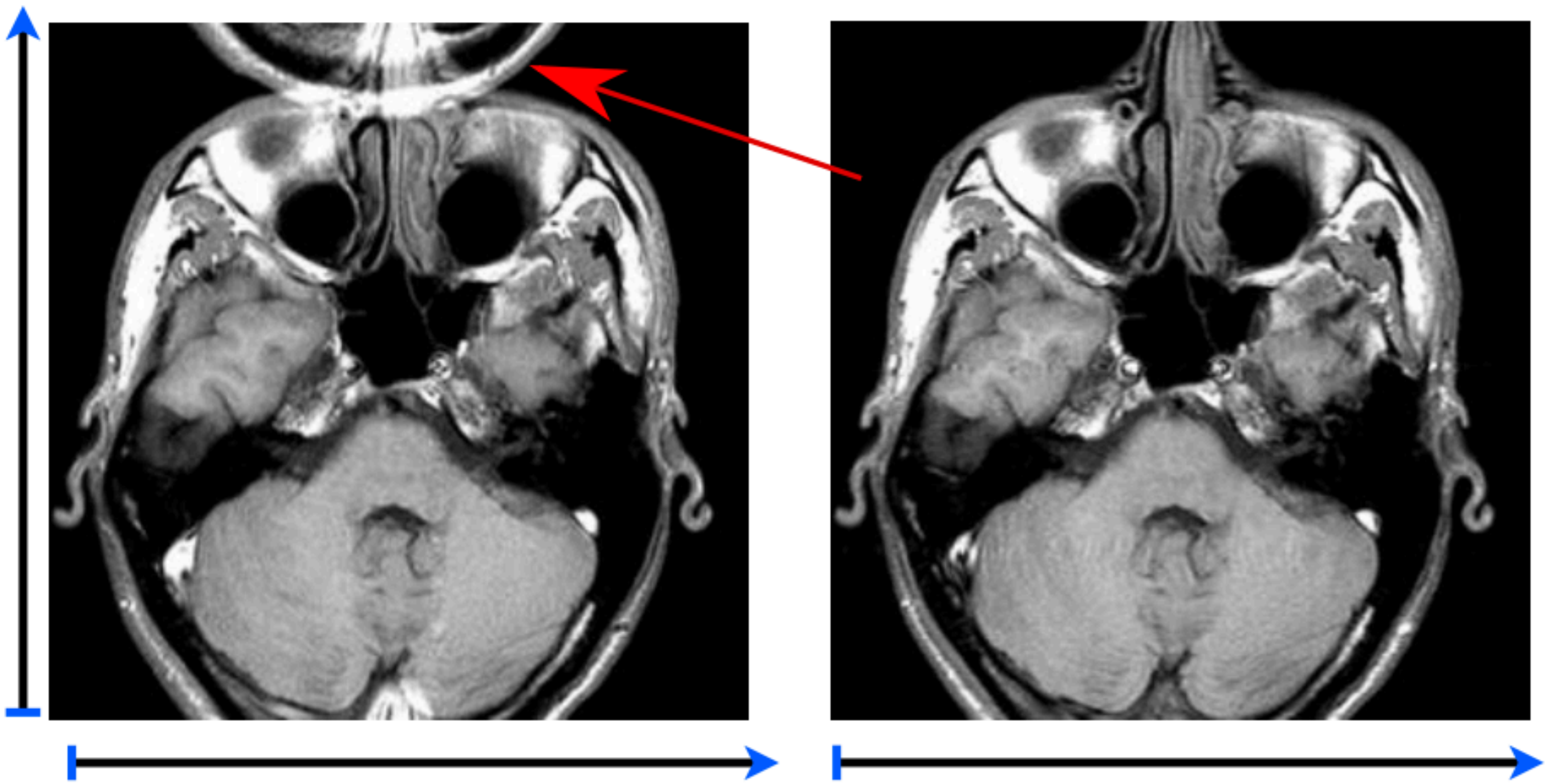
Anti-Aliasing Filter



Anti-Aliasing Filter



MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

Admin

- HW 2 due tomorrow at midnight
- HW 3 posted after class
  - Due 2/10 at midnight
- Ahead of schedule
  - Watch course calendar online for changes