ESE 531: Digital Signal Processing

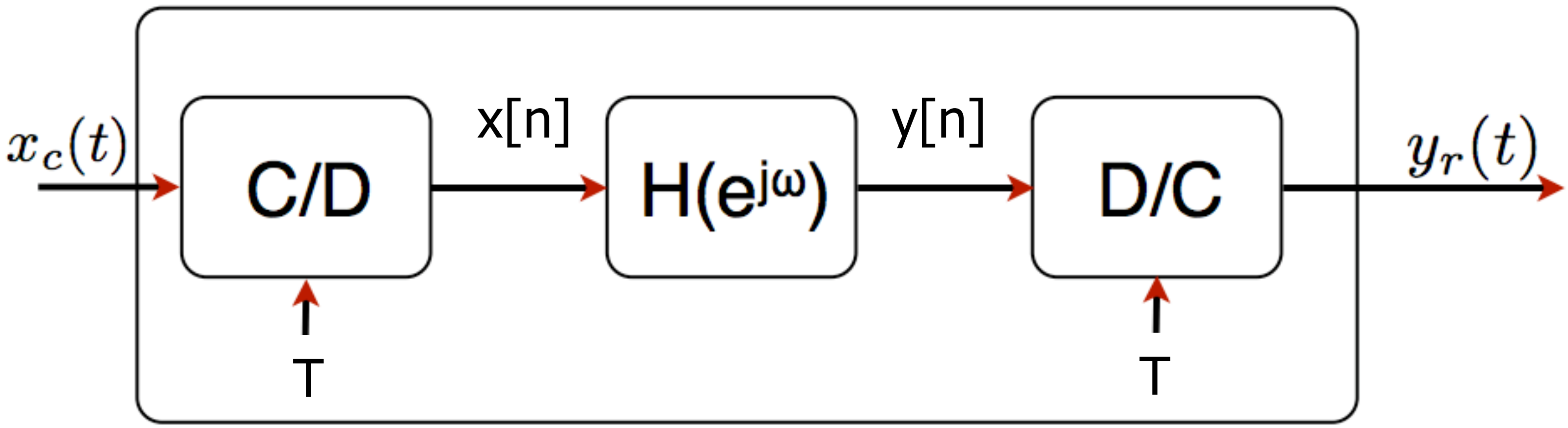
Lec 9: February 9th, 2017

CT Processing, Downsampling/Upsampling

Lecture Outline

- DT processing of CT signals
  - Impulse Invariance
- CT processing of DT signals (why??)
- Downsampling
- Upsampling

Discrete-Time Processing of Continuous Time

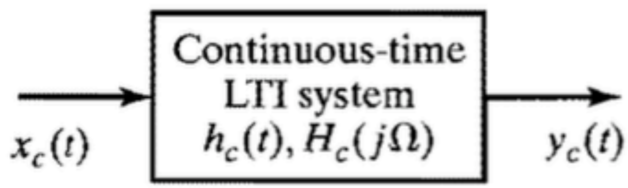


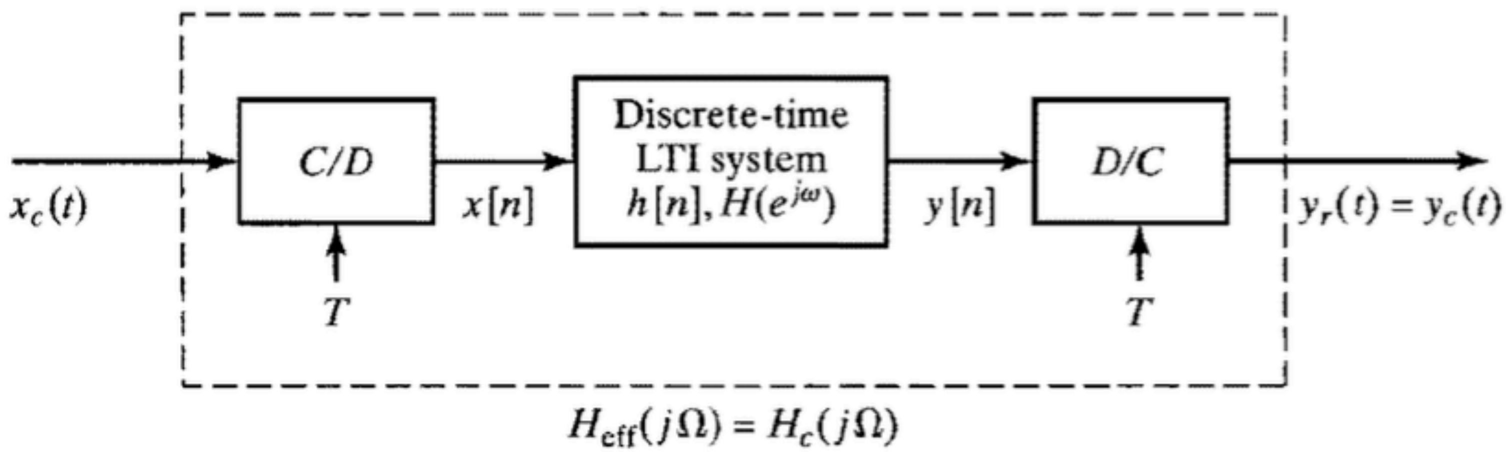
- If $x_c(t)$ is bandlimited by π/T
  - I.e $X_c(j\Omega) = 0$ for $|\Omega| > \pi/T$
- then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \pi/T \\ 0 & else \end{cases}$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time





Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \qquad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

Impulse Invariance

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$$\boxed{h[n] = Th_c(nT)}$$

Impulse Invariance

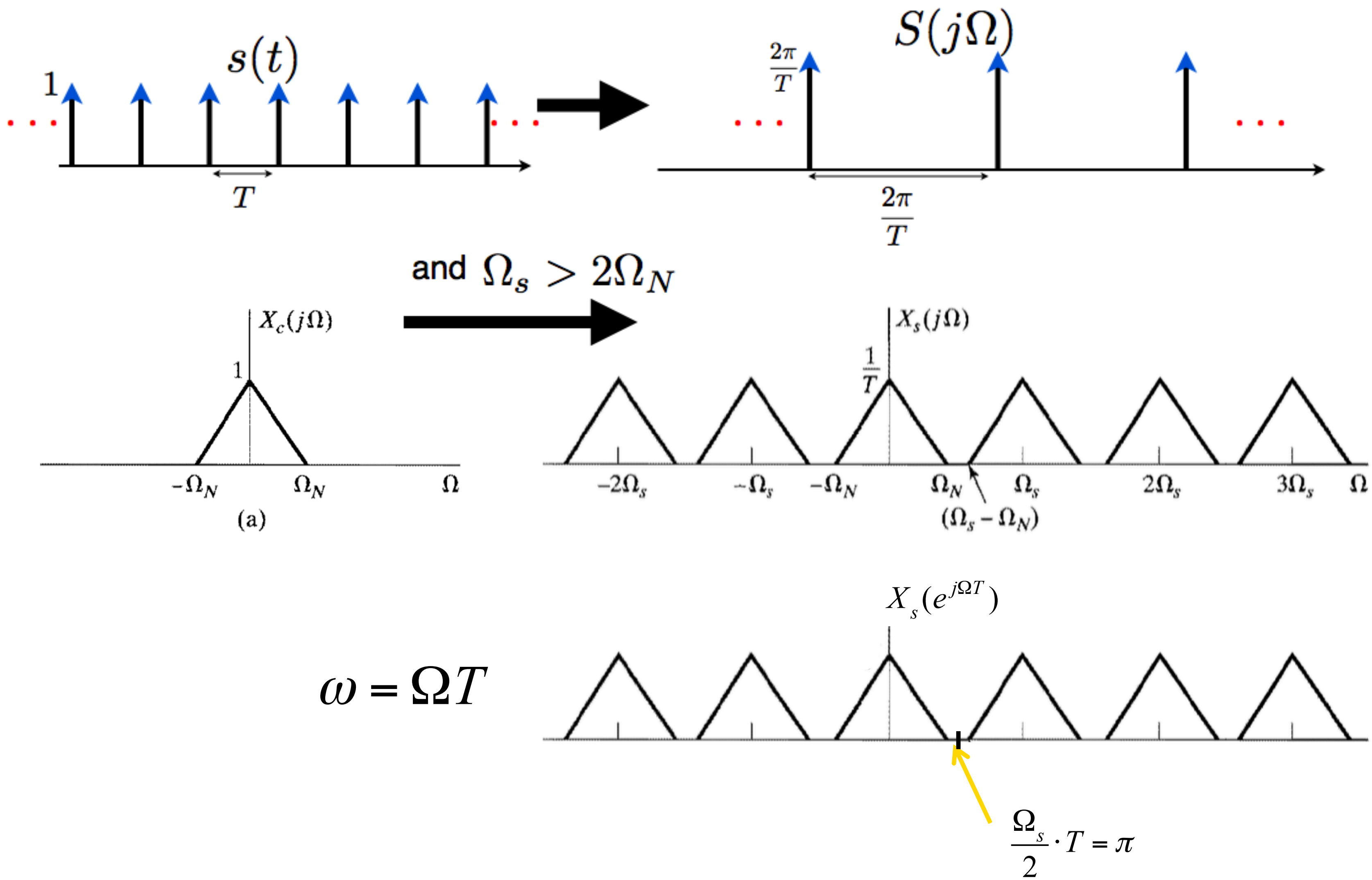
- Let,

$$h[n] = h_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

Frequency Domain Analysis



Impulse Invariance

- Let,

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

$$h[n] = h_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi$$

Impulse Invariance

- Let,

$$h[n] = T h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

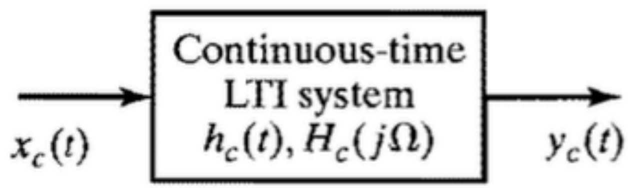
- If sampling at Nyquist Rate then

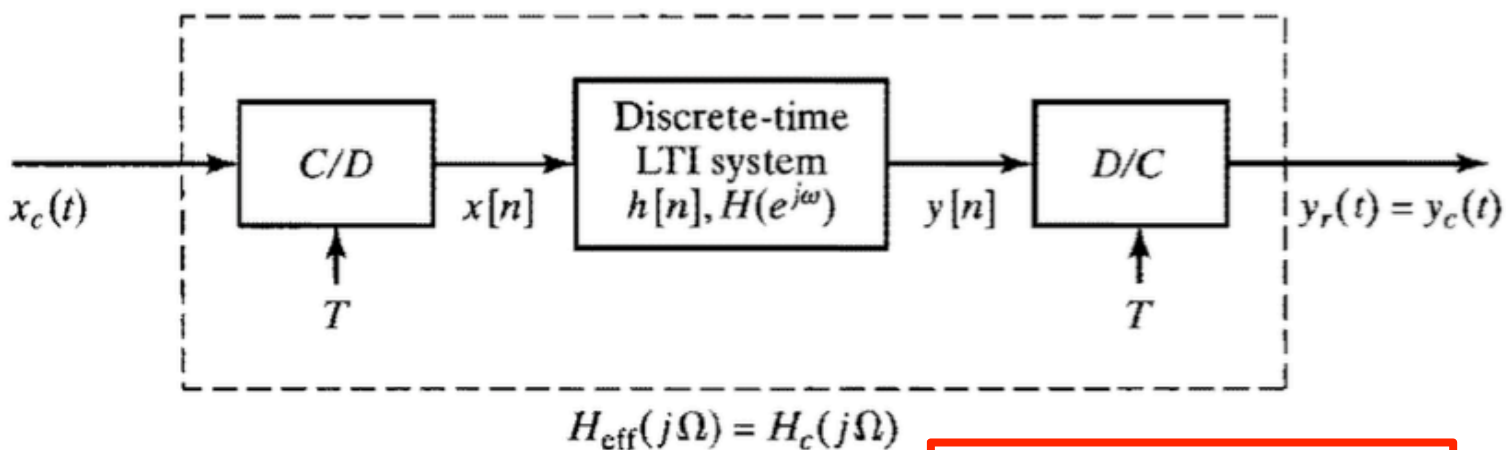
$$H(e^{j\omega}) = \frac{T}{T} \sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = \frac{T}{T} H_c\left(j\frac{\omega}{T} \right), \qquad |\omega| < \pi$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time

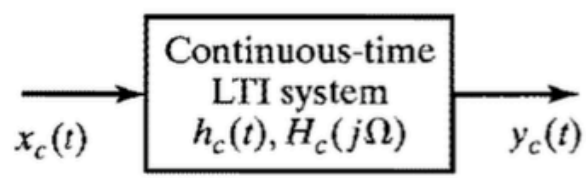


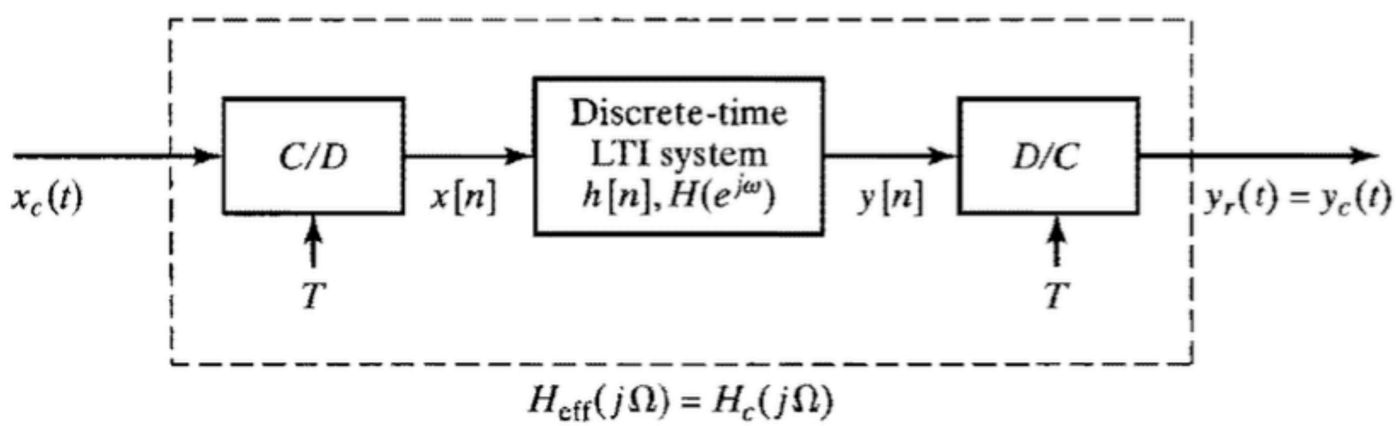


$$h[n] = Th_c(nT)$$

Example: DT Lowpass Filter

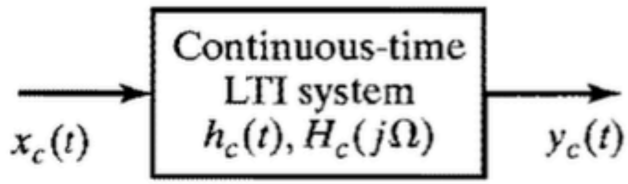
- We wish to implement a lowpass filter with cutoff frequency Ω_c on continuous time signal in discrete time with the following system

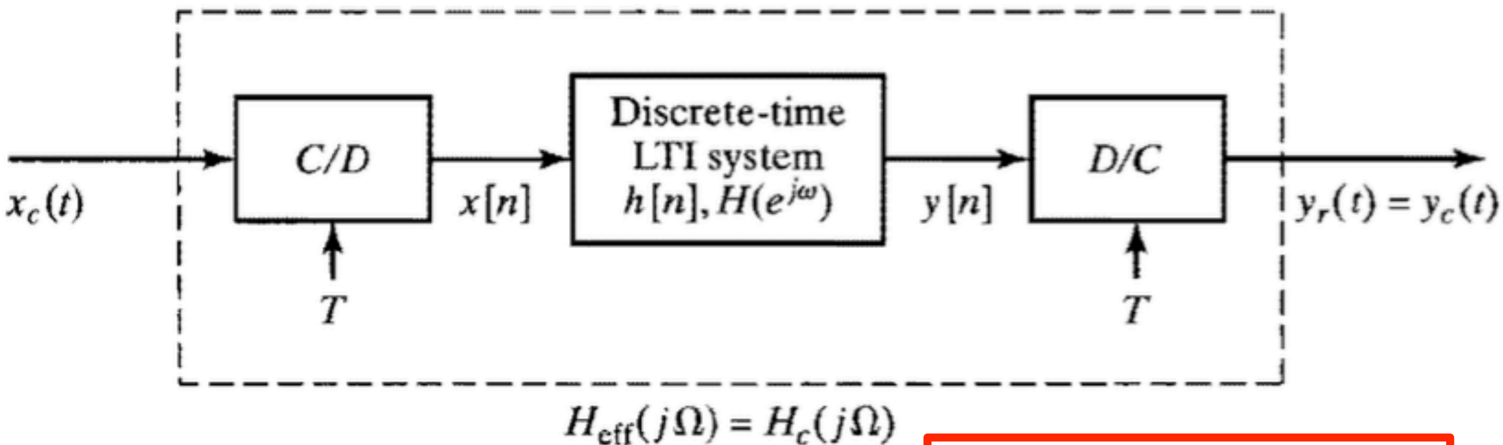




Impulse Invariance

- Want to implement continuous-time system in discrete-time

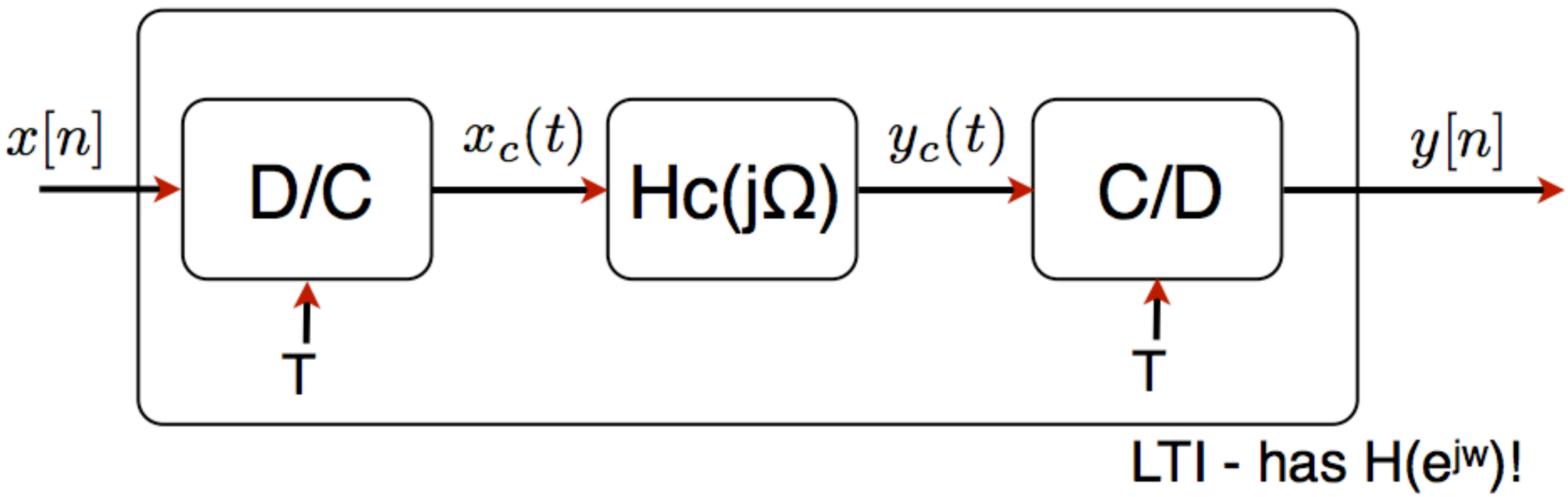




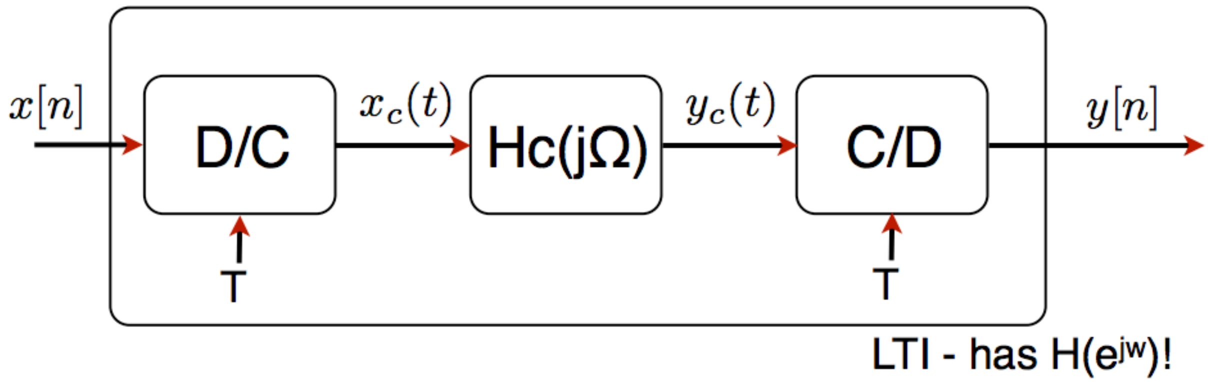
$$h[n] = Th_c(nT)$$

Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time

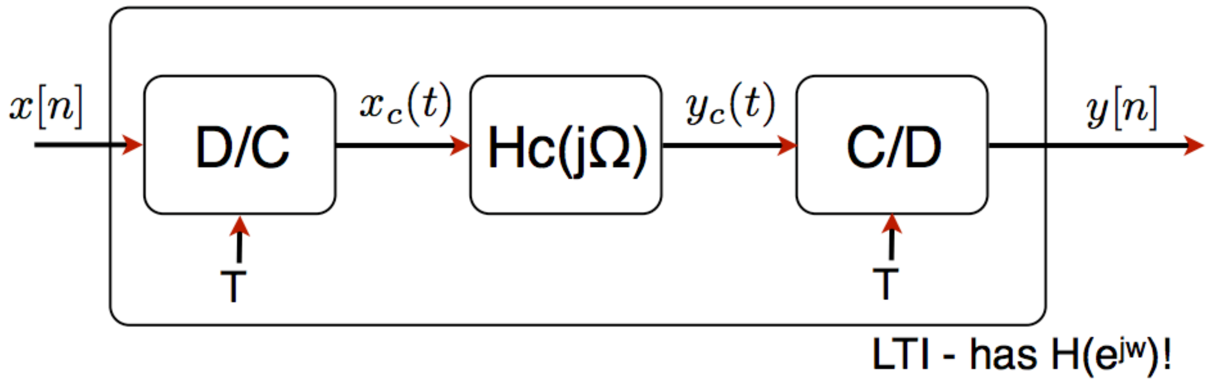


Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Continuous-Time Processing of Discrete-Time

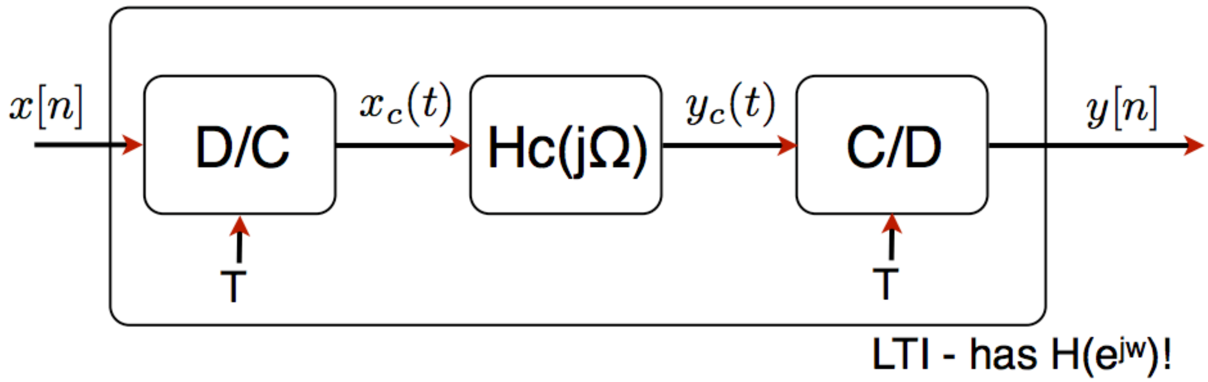


$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Also bandlimited

$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

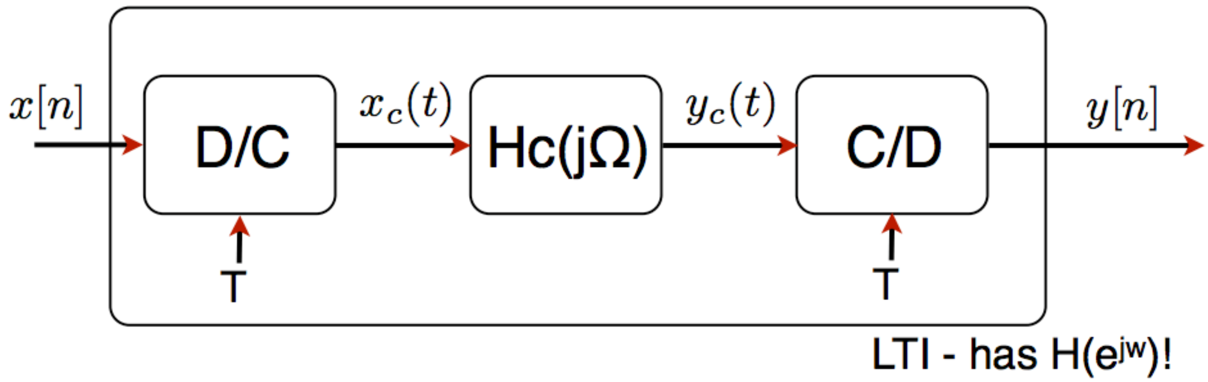
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_c\left[j\left(\Omega - k\Omega_s\right)\right]\Bigg|_{\Omega=\omega/T} = \frac{1}{T} Y_c(j\Omega)\Bigg|_{\Omega=\omega/T}$$

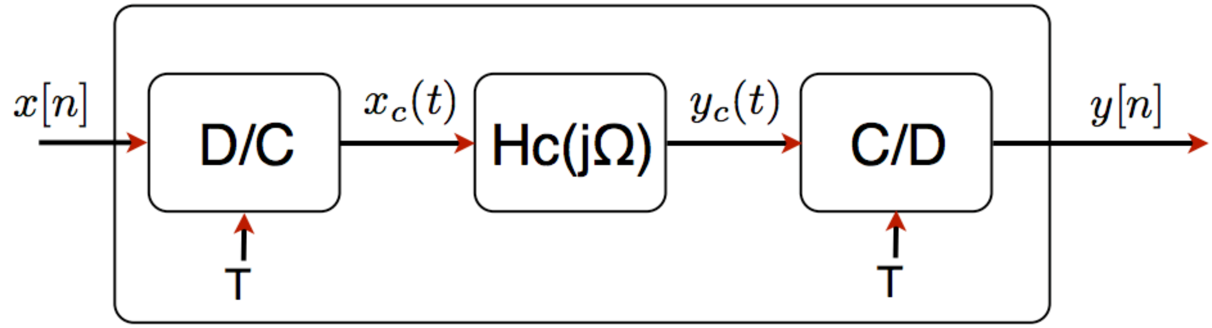
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} Y_c(j\Omega)\Big|_{\Omega=\omega/T}$$

Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

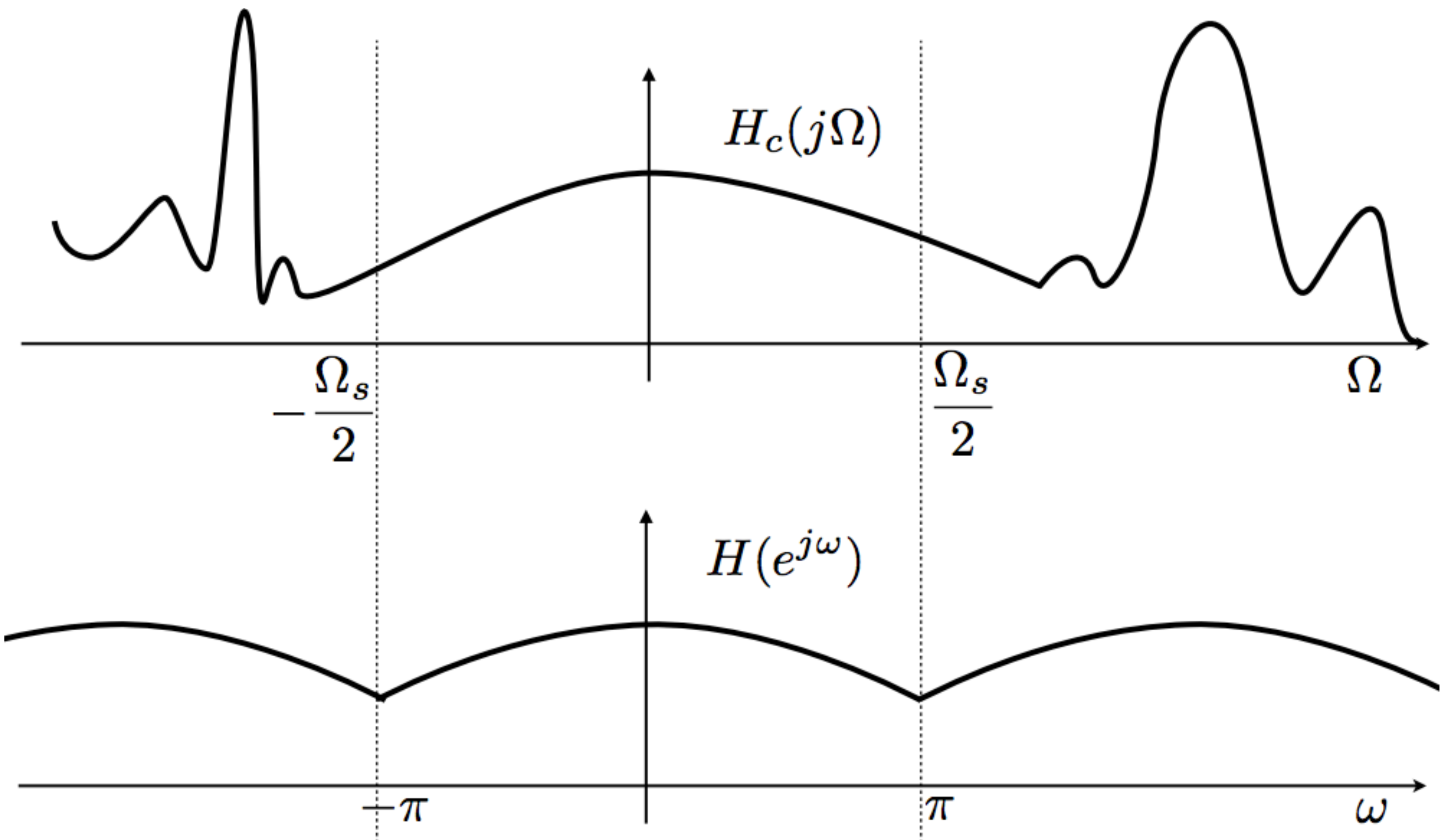
$$Y(e^{j\omega}) = \frac{1}{T} Y_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$Y(e^{j\omega}) = \frac{1}{T} H_c(j\Omega)\Big|_{\Omega=\omega/T} X_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T} H_c(j\Omega)\Big|_{\Omega=\omega/T} (TX(e^{j\omega})) \qquad |\omega| < \pi$$

$H(e^{j\omega})$

Example



$H_c(j\Omega)$
$-\frac{\Omega_s}{2}$
$\frac{\Omega_s}{2}$
Ω
$H(e^{j\omega})$
$-\pi$
π
ω

Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

$$\delta[n] \leftrightarrow 1$$

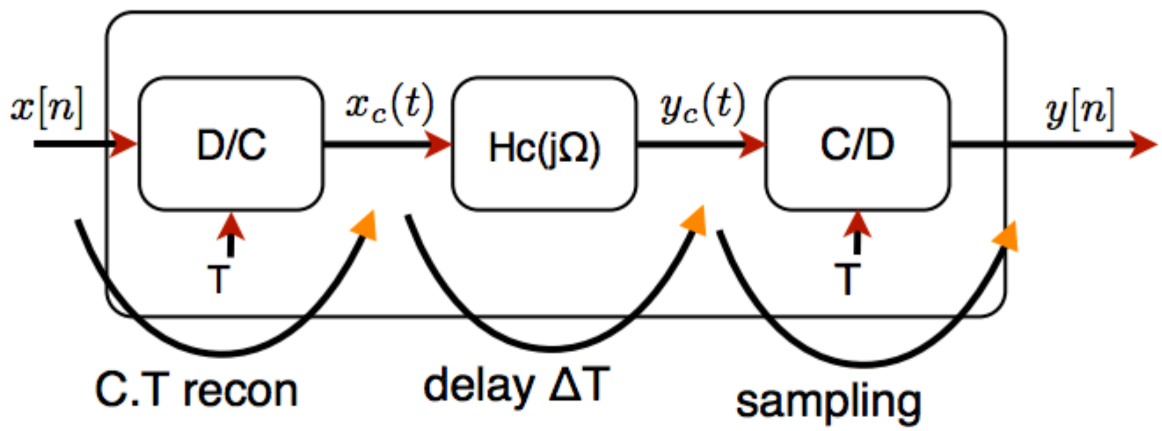
$$\delta[n - n_d] \leftrightarrow e^{-j\omega n_d}$$

Example: Non-integer Delay

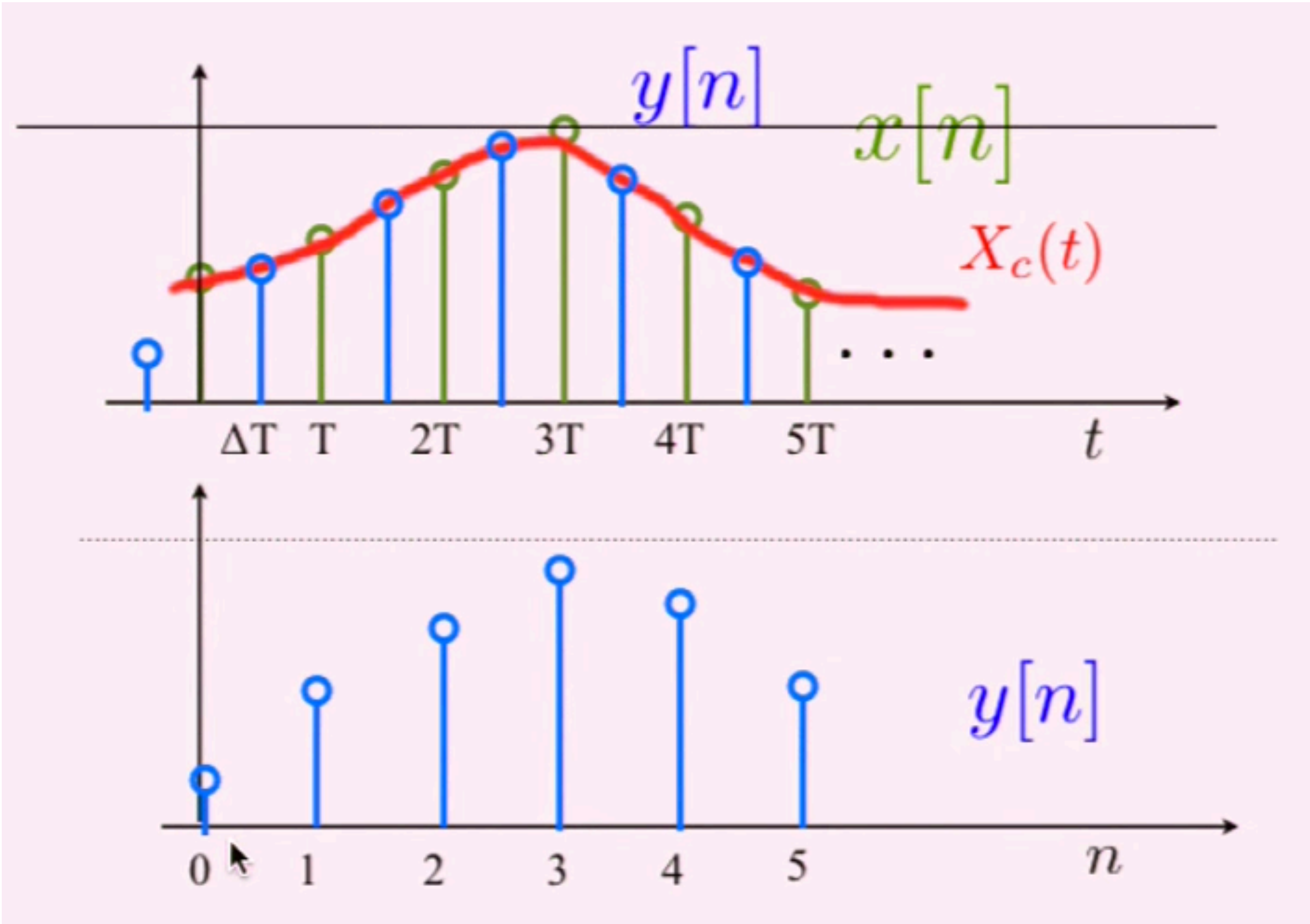
- What is the time domain operation when Δ is non-integer? I.e Δ=1/2

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ <u>delay of ΔT in time</u>

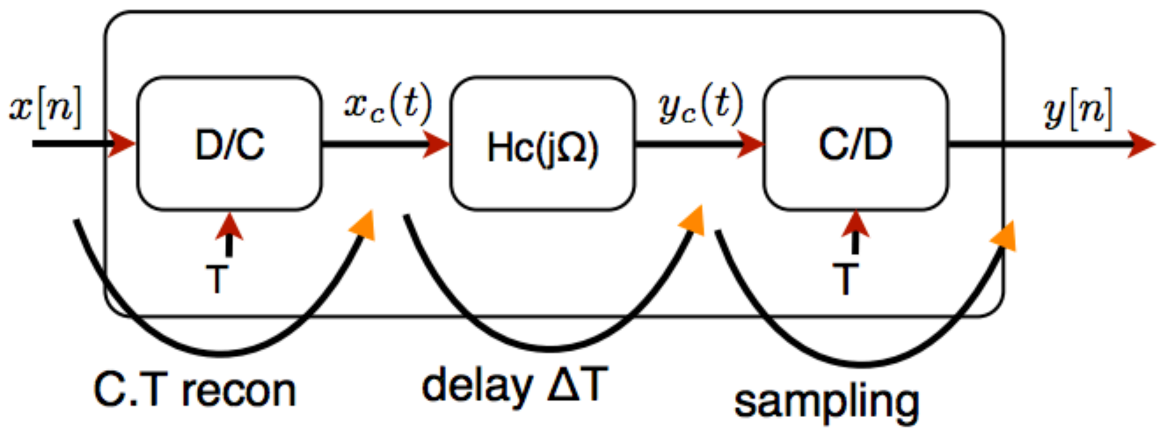


Example: Non-integer Delay



Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$y_c(t) = x_c(t - T\Delta)$$

Example: Non-integer Delay

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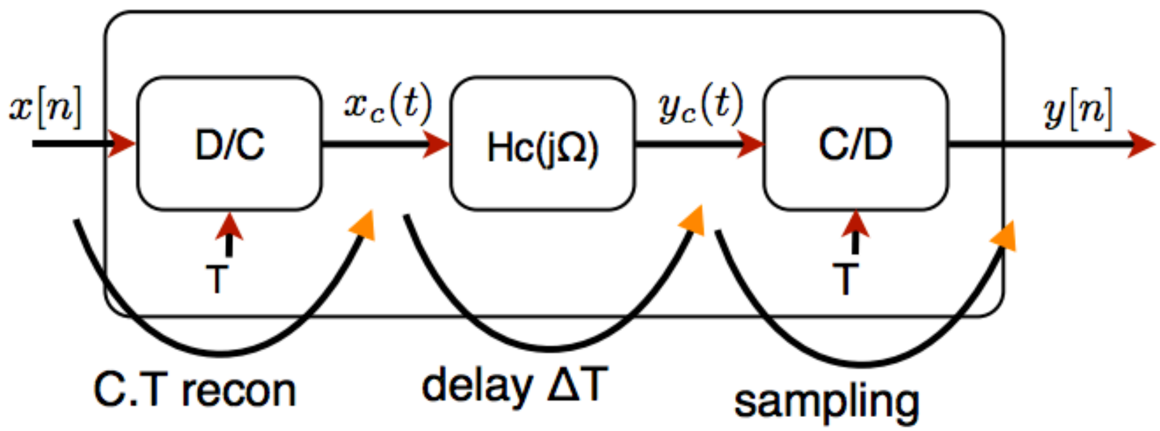

$$y_c(t) = x_c(t - T\Delta)$$

$$y[n] = y_c(nT) = x_c(nt - T\Delta)$$

$$= \sum_k x[k] \mathrm{sinc}\left(\frac{t - kT - T\Delta}{T}\right)\Bigg|_{t=nT}$$

$$= \sum_k x[k] \mathrm{sinc}\left(n - k - \Delta\right)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$h[n] = \text{sinc}(n - \Delta)$$

Example: Non-integer Delay

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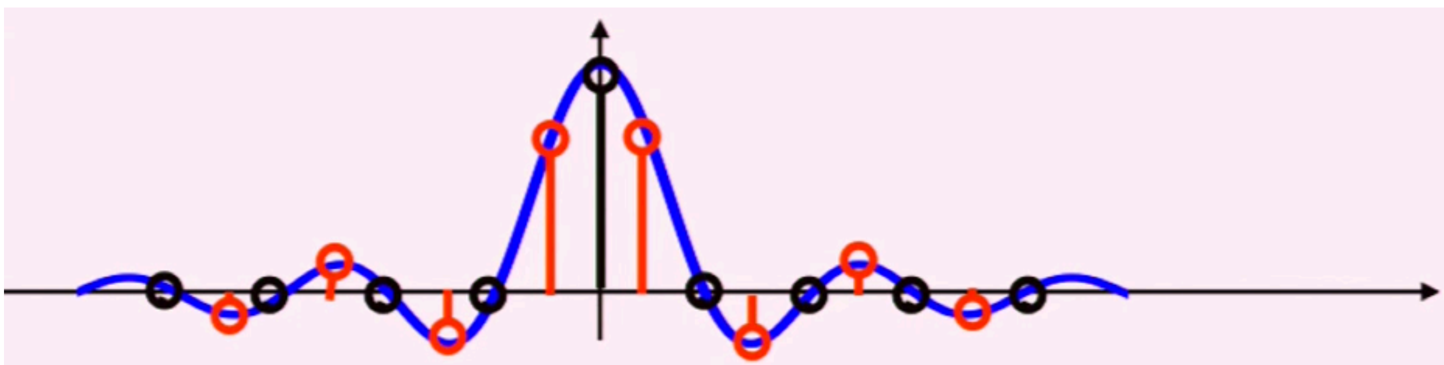


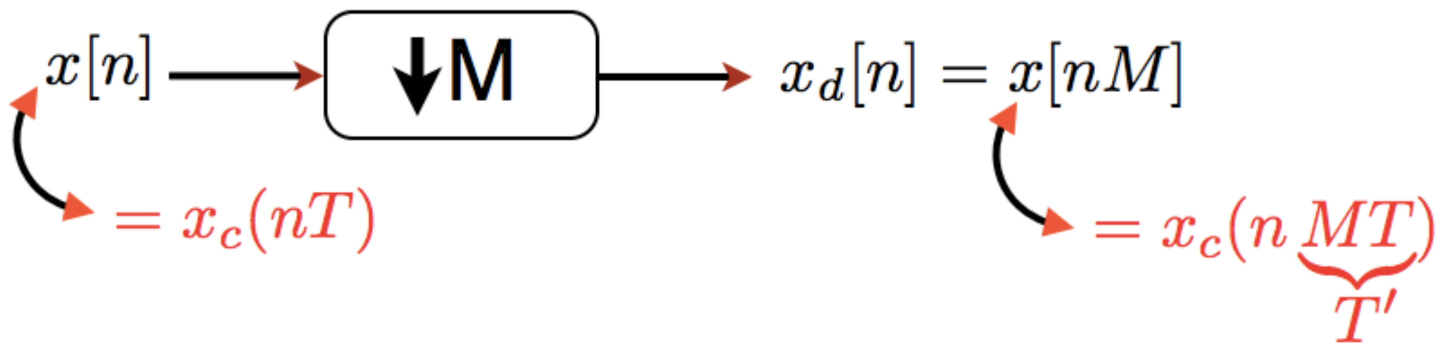
Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects

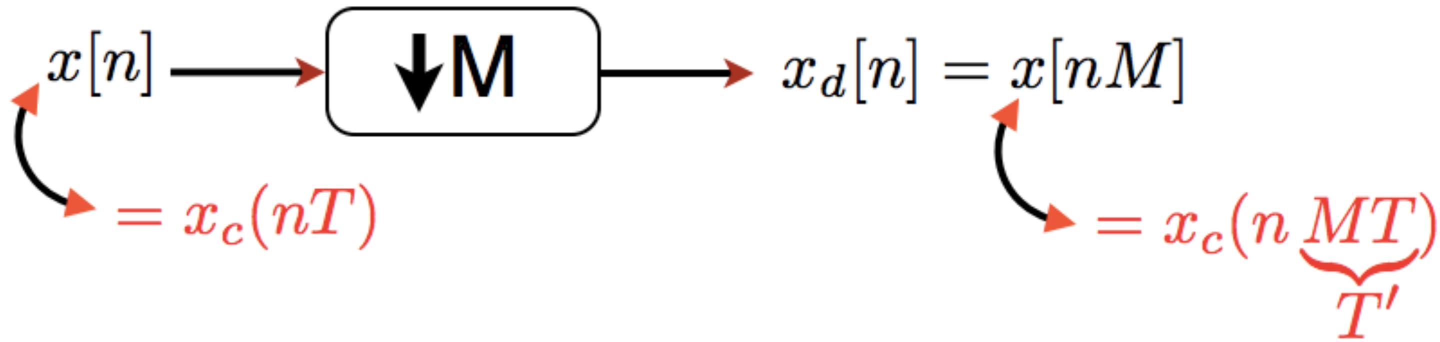
- If your discrete time signal is finely sample almost like a CT signal
  - Downsampling is just like sampling (C/D conversion)

Downsampling

- Definition: Reducing the sampling rate by an integer number



Downsampling



The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Downsampling

The DTFT:

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$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$
- Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)

Downsampling

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- k=rM+i
  - i = 0, 1, …, M-1
  - r = -∞, ..., ∞

Downsampling

$$
\begin{aligned}
X_d(e^{j\omega}) &= \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi}{MT}k\right)\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi}{MT}i-\frac{2\pi}{T}r\right)\right)
\end{aligned}
$$

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi}{MT}k\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1}\underbrace{\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT}-\frac{2\pi}{MT}i-\frac{2\pi}{T}r\right)\right)}_{X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi}{M}i\right)}\right)}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}-\underbrace{\frac{2\pi}{T}}k\right)\right)$$

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

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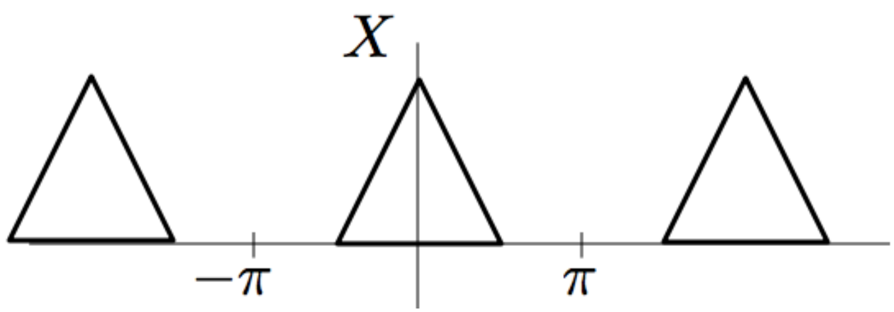
$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}} - \underbrace{\frac{2\pi}{T}} k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M (↑ $\frac{\omega}{M}$) — replicate (↑ $\frac{2\pi}{M}i$)

Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



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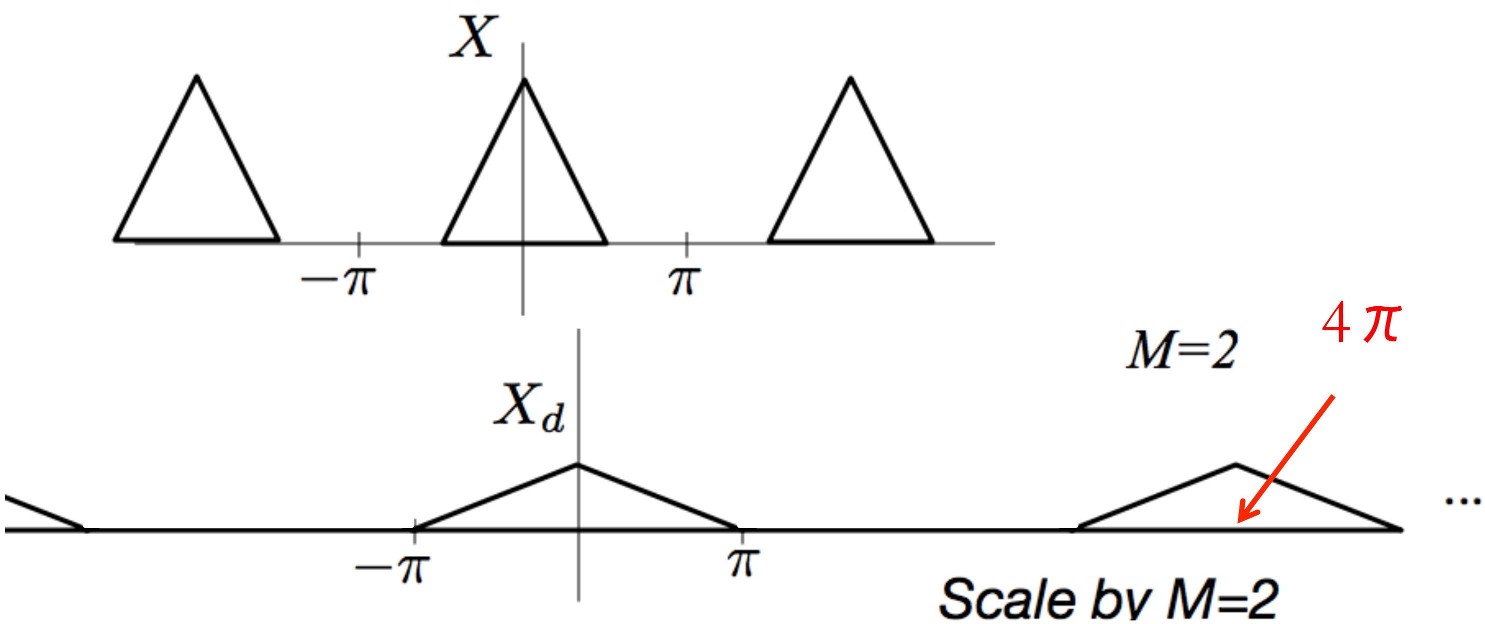

Example

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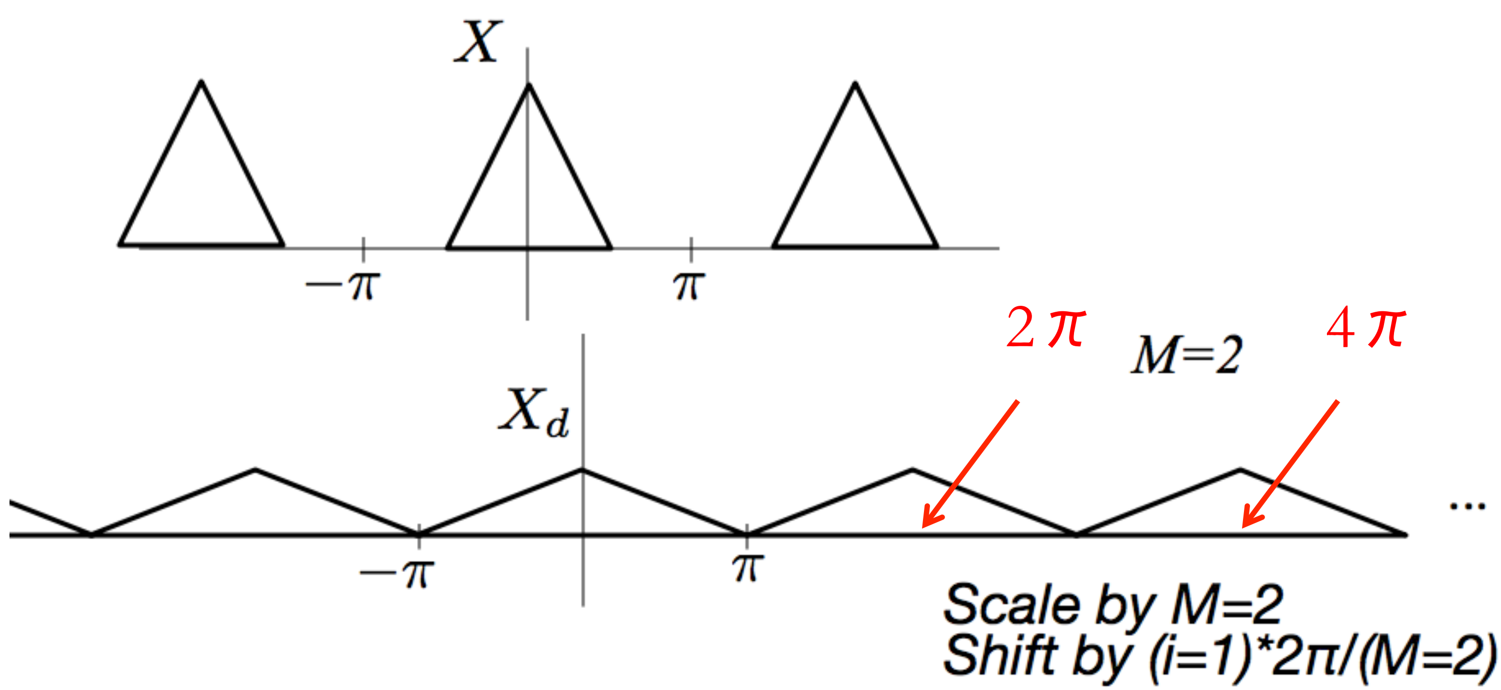

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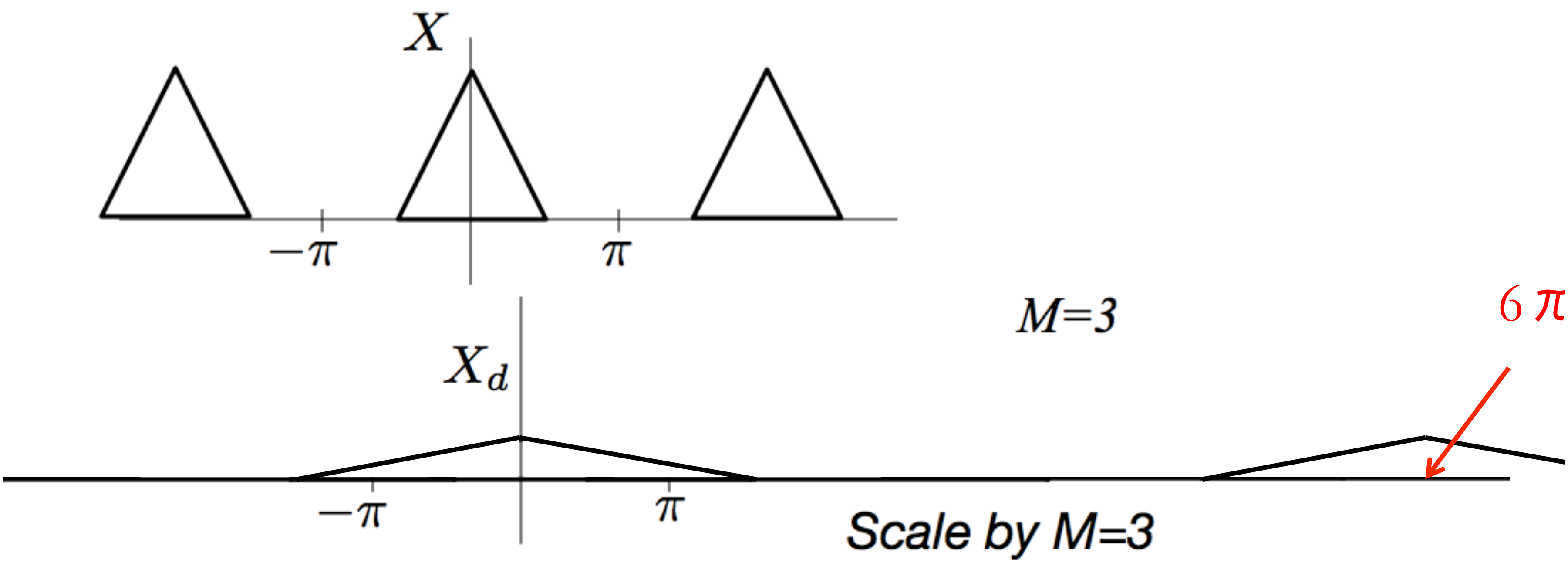

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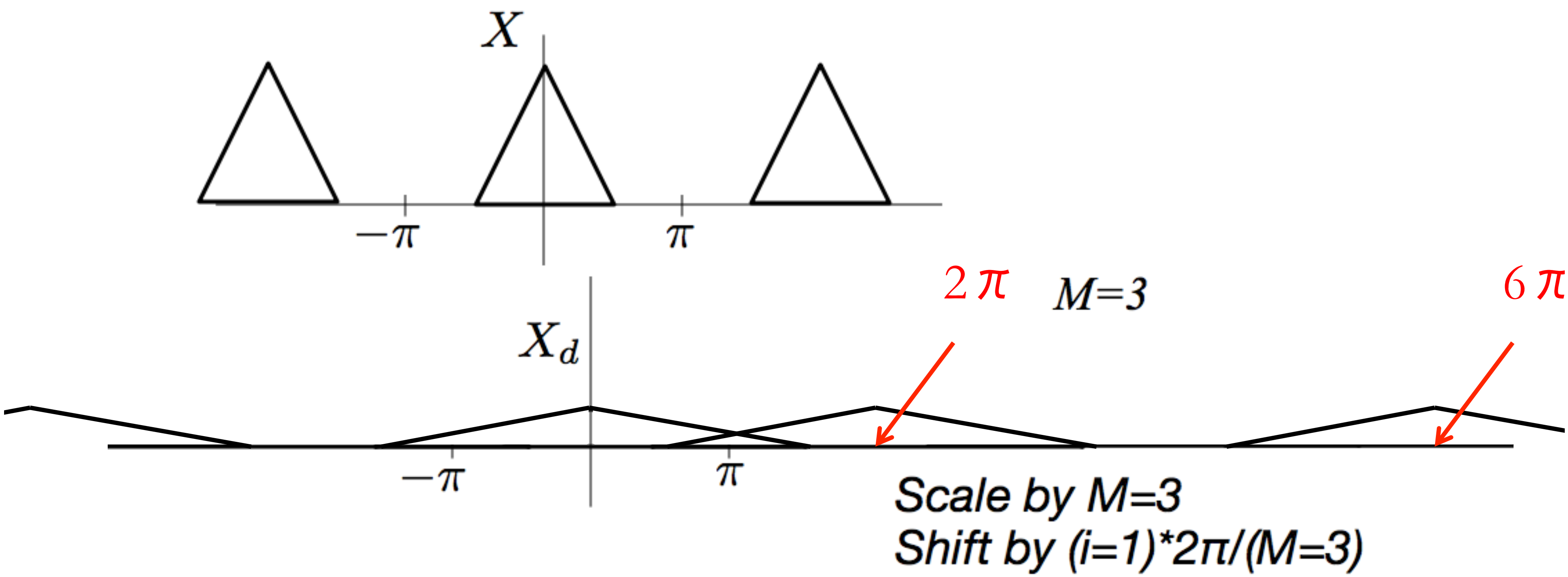
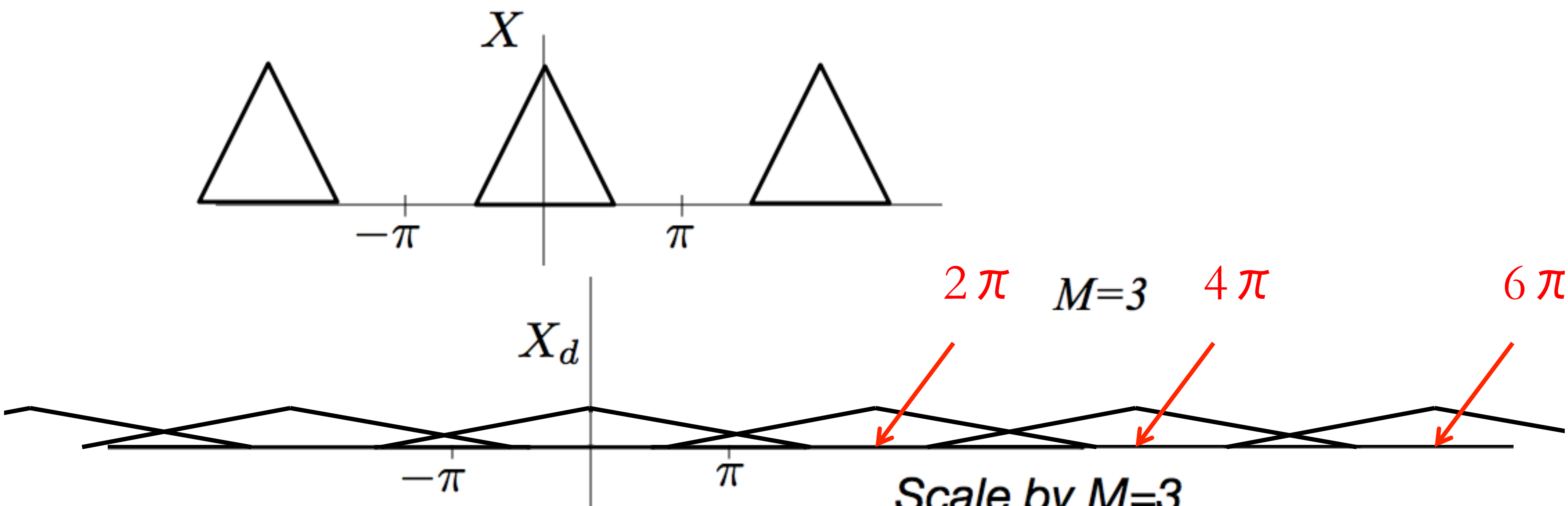


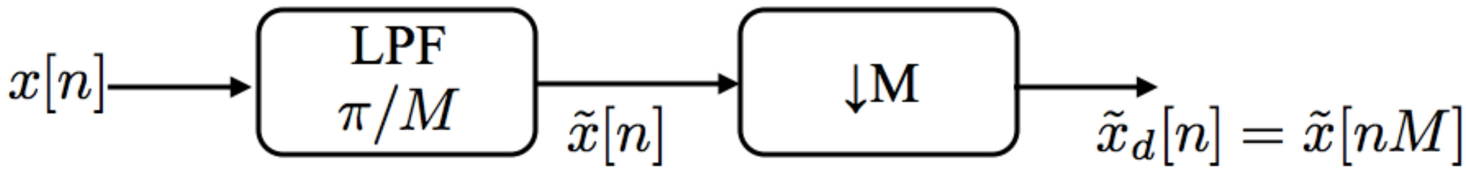
Example

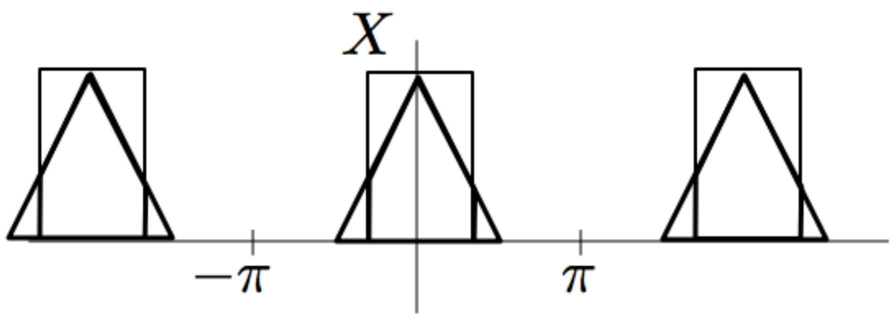
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



Scale by M=3
Shift by (i=1)*2π/(M=3)
Shift by (i=2)*2π/(M=3)

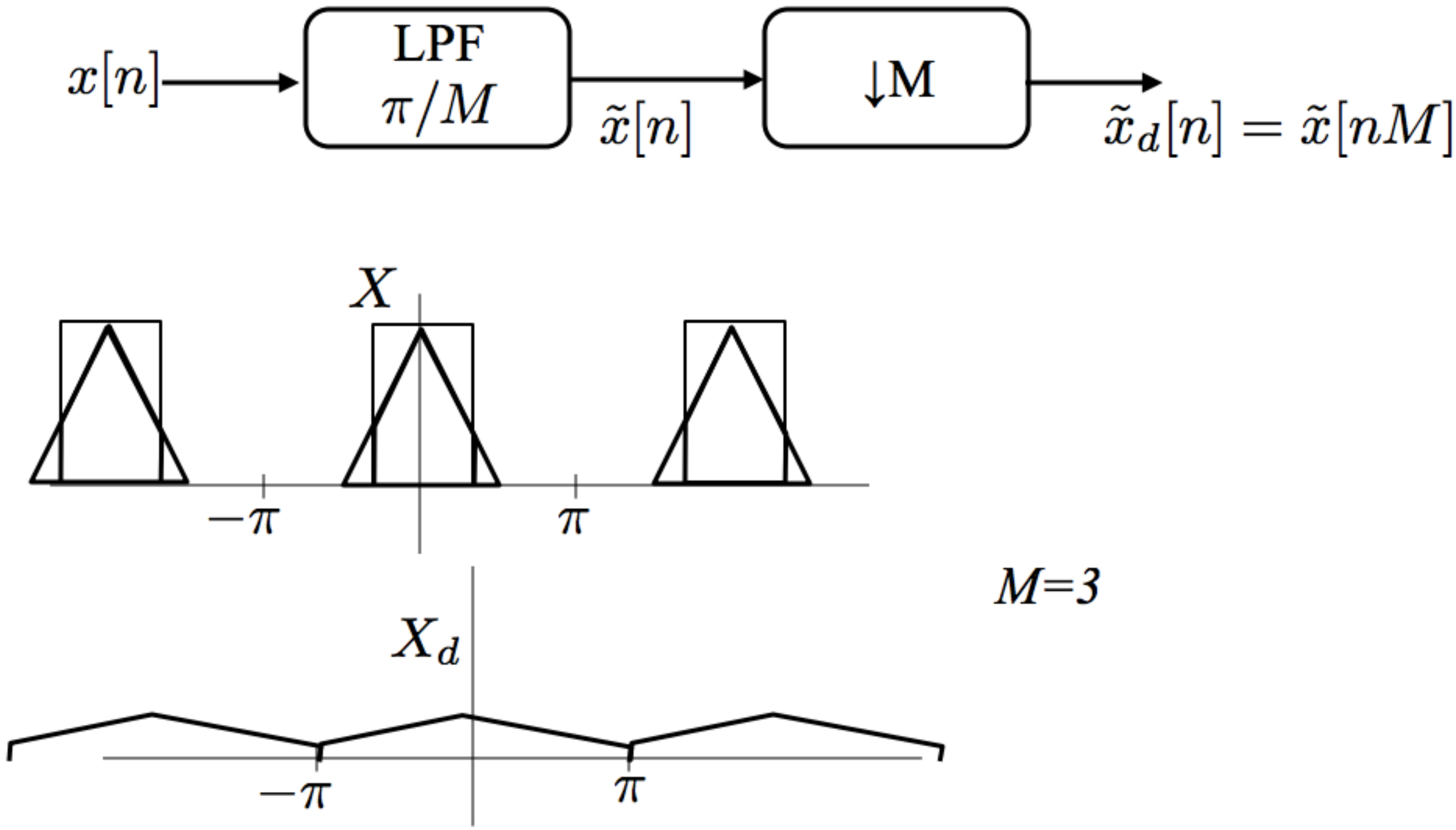
Example





M=3

Example



$x[n]$
LPF
π/M
$\tilde{x}[n]$
$\downarrow$M
$\tilde{x}_d[n] = \tilde{x}[nM]$
X
$-\pi$
π
M=3
X_d
$-\pi$
π

Upsampling

- Much like D/C converter
- Upsample by A LOT → almost continuous

- Intuition:
  - Recall our D/C model: $x[n] \rightarrow x_s(t) \rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

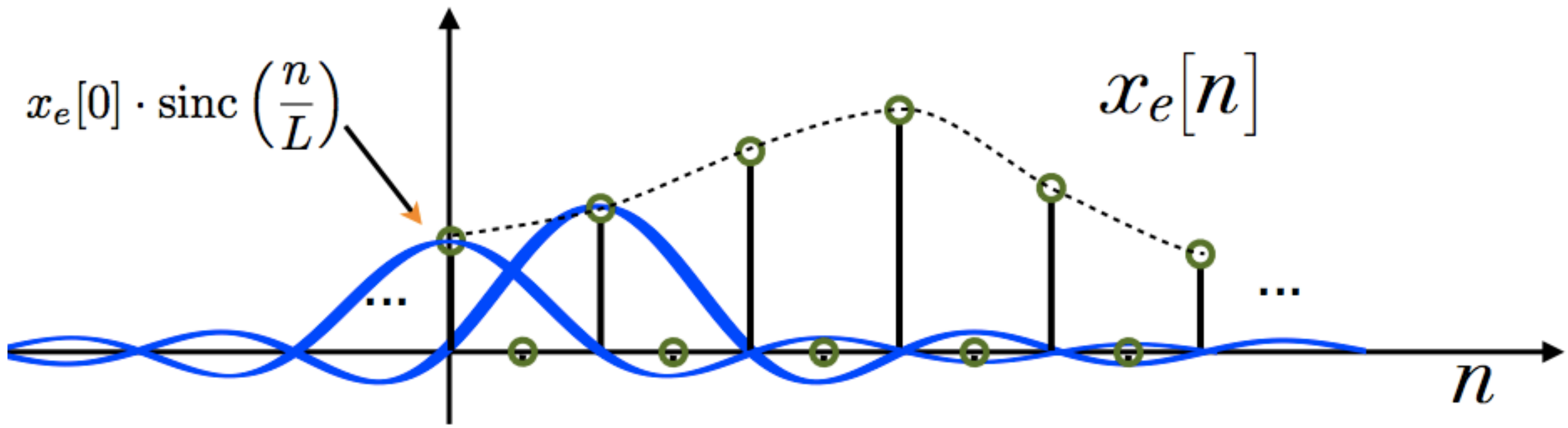
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

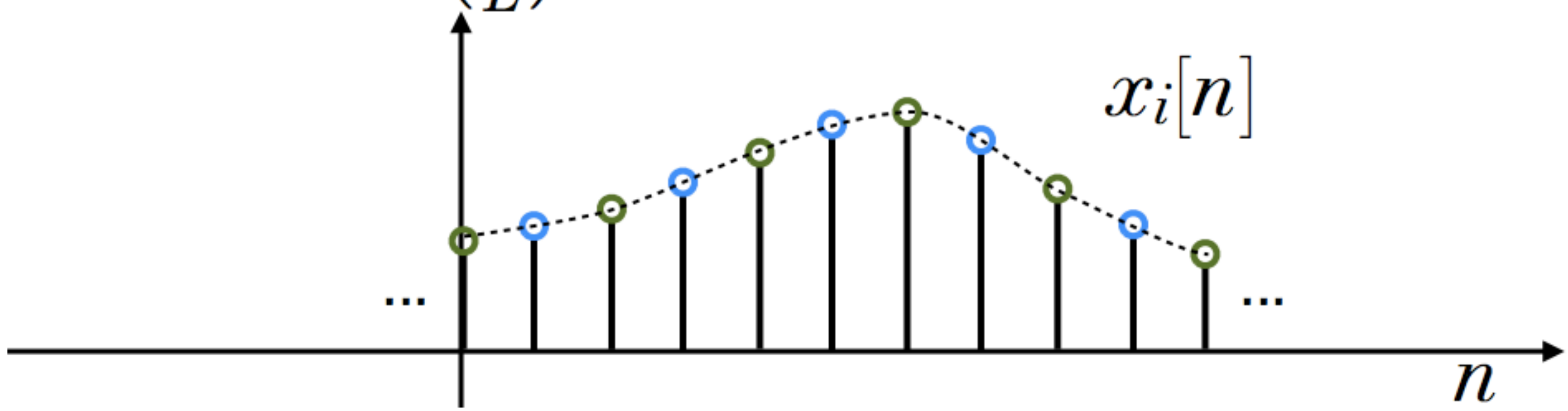
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$



Upsampling

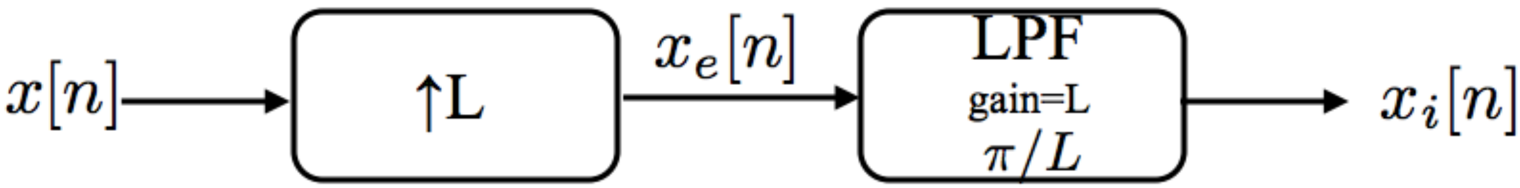
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

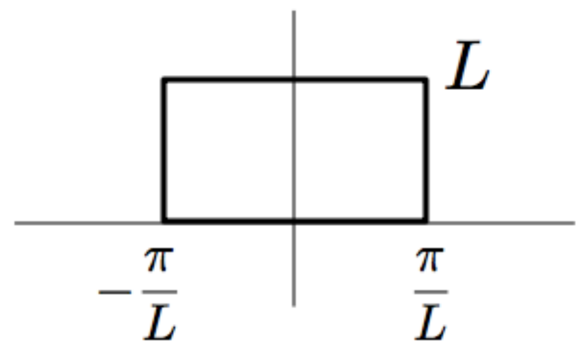
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}(\frac{n - kL}{L})$$

Frequency Domain Interpretation

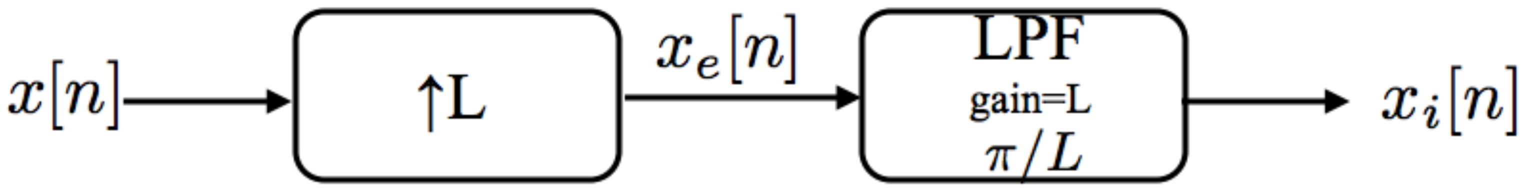
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ **DTFT ⇒**

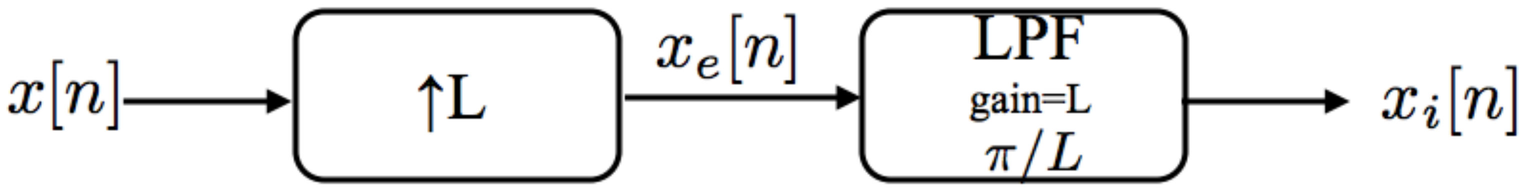


Frequency Domain Interpretation



$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

Frequency Domain Interpretation

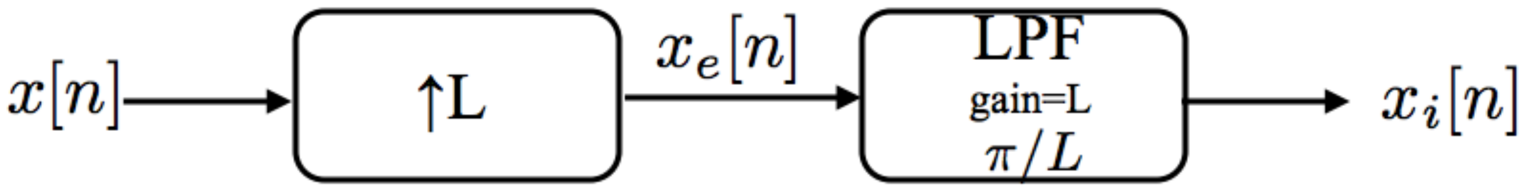


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL}$$

Compress DTFT by a factor of L!

Frequency Domain Interpretation



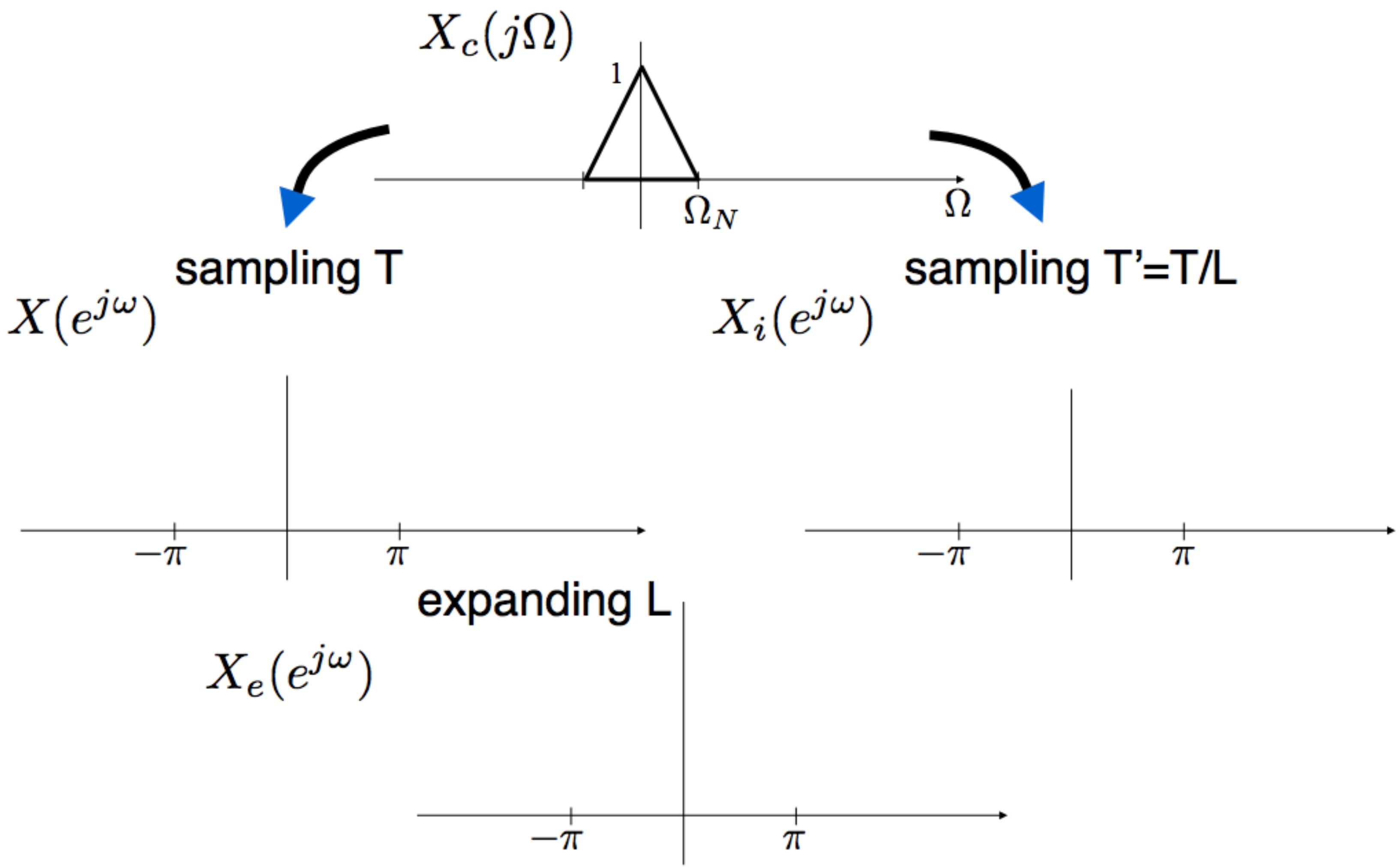
$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{} e^{-j\omega n}$$

≠ 0 only for n=mL (integer m)

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \qquad = X(e^{j\omega L})$$

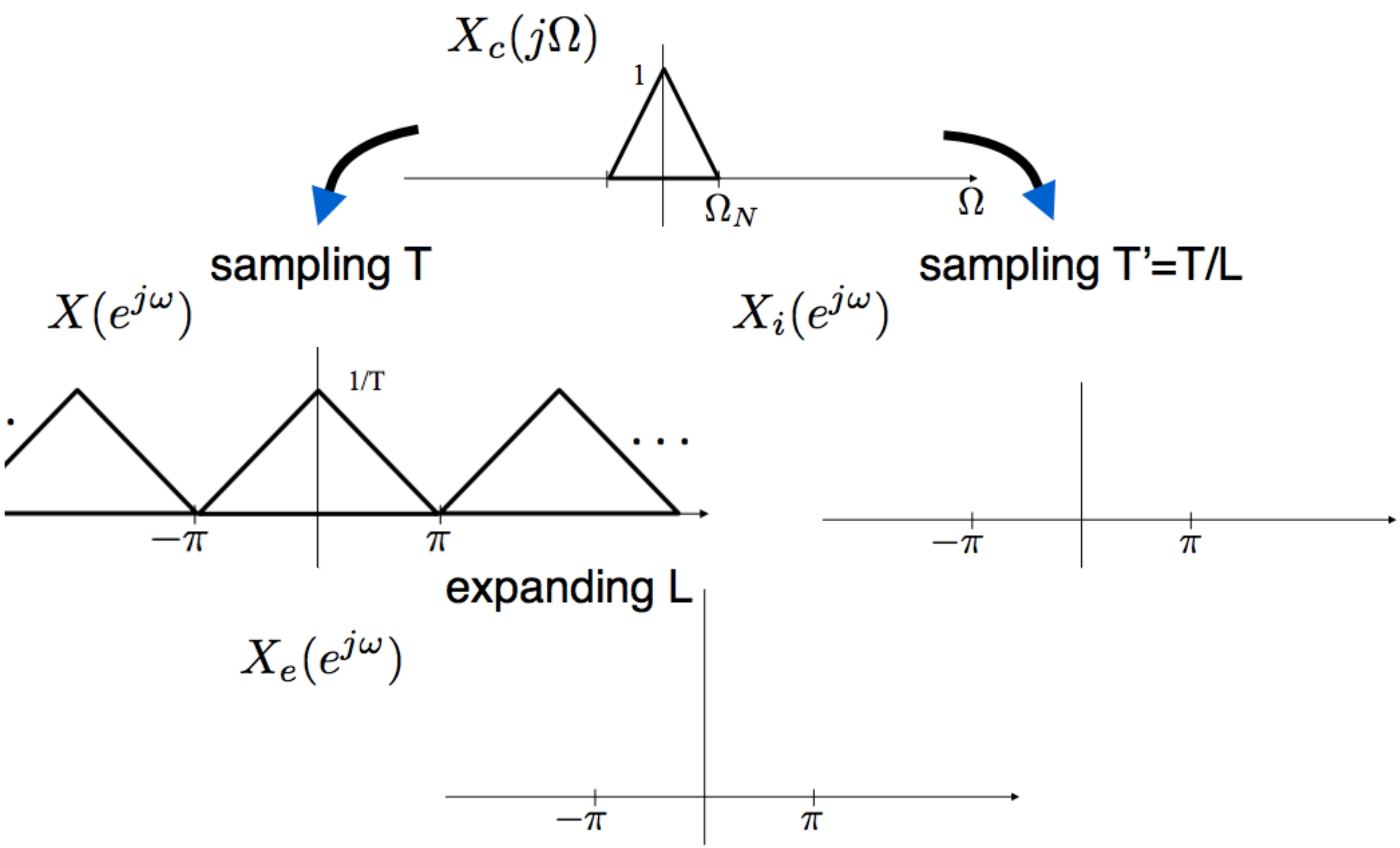
Compress DTFT by a factor of L!

Example



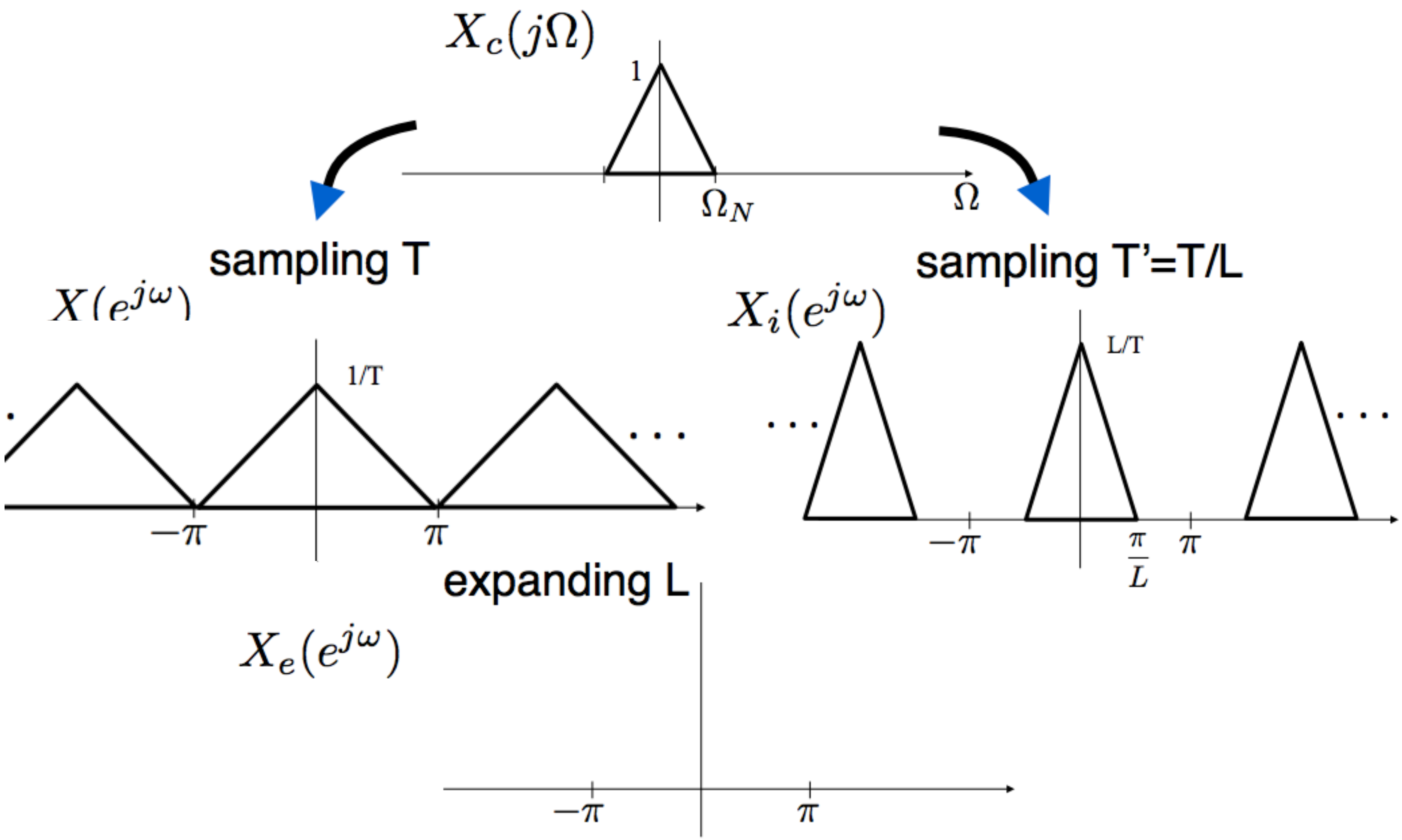
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

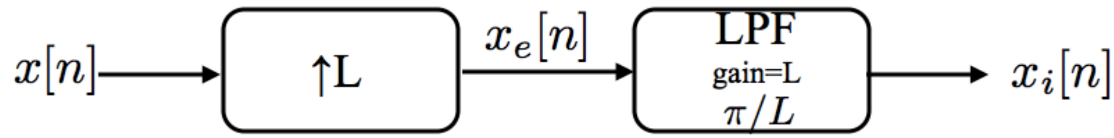


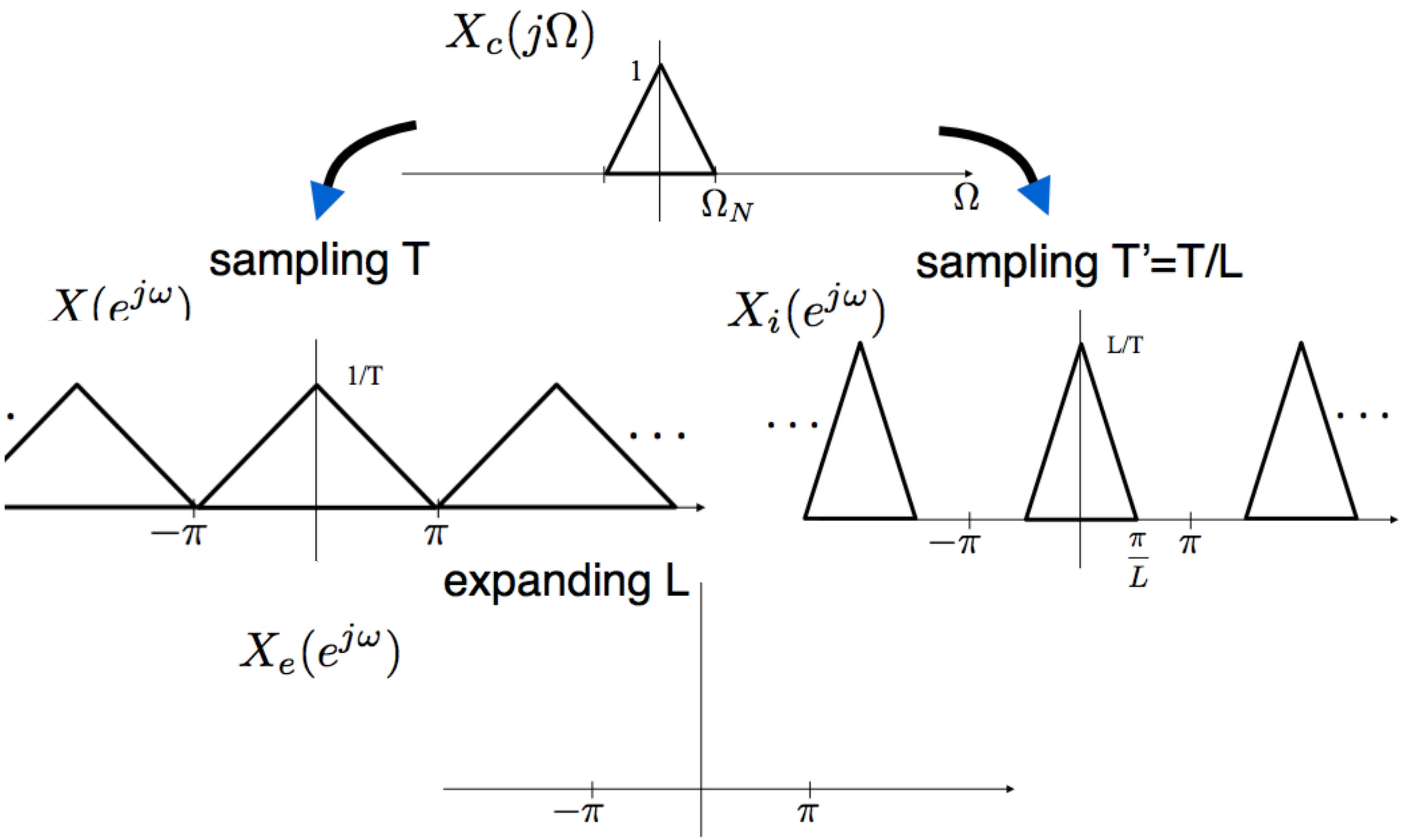
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
. . .
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

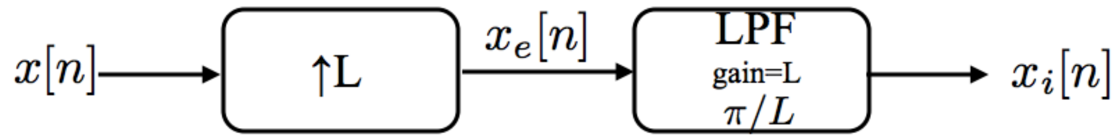

$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

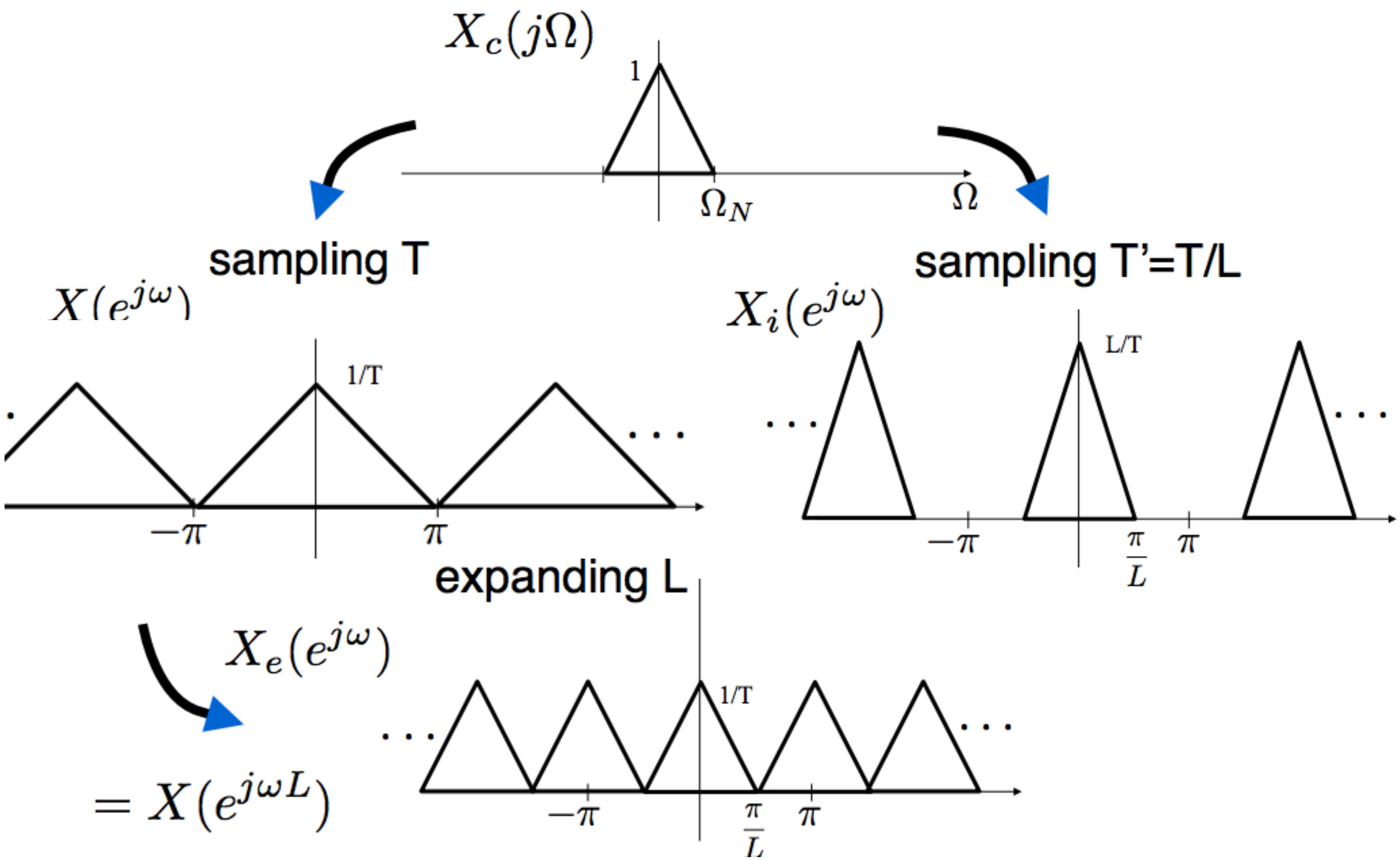

$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

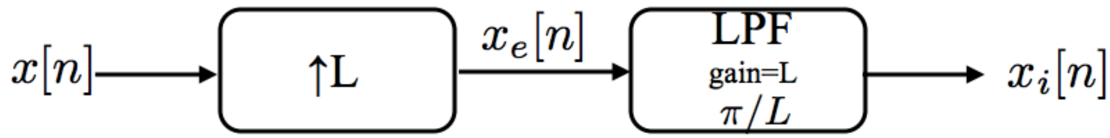


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$

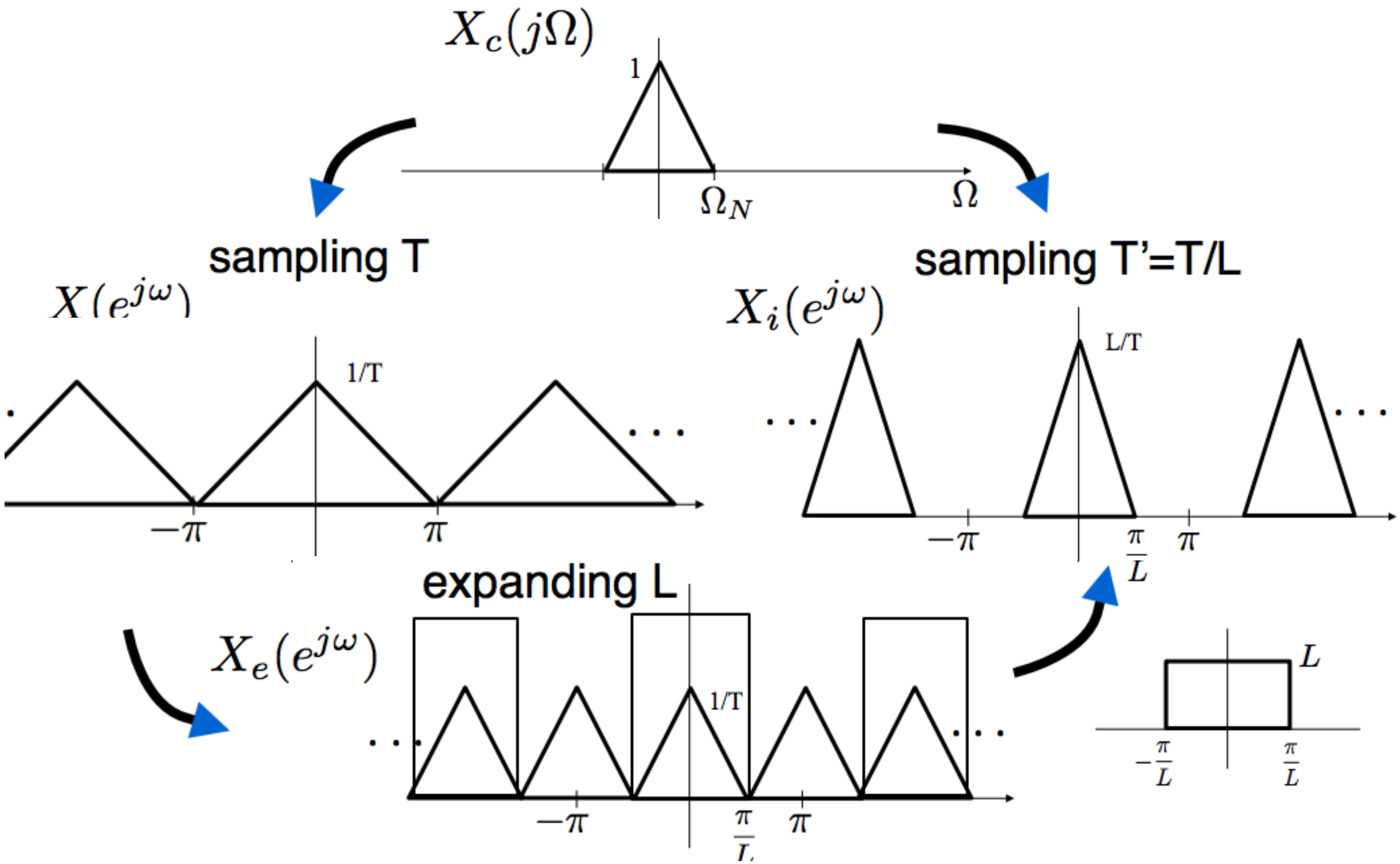


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$= X(e^{j\omega L})$
1/T
$-\pi$
$\frac{\pi}{L}$
π

Example



x[n]
↑L
x_e[n]
LPF
gain=L
π/L
x_i[n]



X_c(jΩ)
1
Ω_N
Ω
sampling T
X(e^{jω})
1/T
−π
π
sampling T'=T/L
X_i(e^{jω})
L/T
−π
π/L
π
expanding L
X_e(e^{jω})
1/T
−π
π/L
π
L
−π/L
π/L

Big Ideas

- DT processing of CT signals
  - Impulse Invariance to design DT systems for CT signals
- CT processing of DT signals
  - Allows for interpretation of DT systems
- Downsampling
  - Like a C/D converter
- Upsampling
  - Like a D/C converter

Admin

- HW 3 due Friday
- HW 4 posted after class