ESE 531: Digital Signal Processing

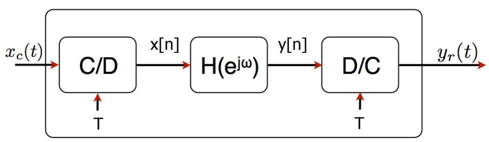
Lec 9: February 9th, 2017

CT Processing, Downsampling/Upsampling

Lecture Outline

- DT processing of CT signals
  - Impulse Invariance
- CT processing of DT signals (why??)
- Downsampling
- Upsampling

Discrete-Time Processing of Continuous Time

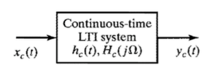


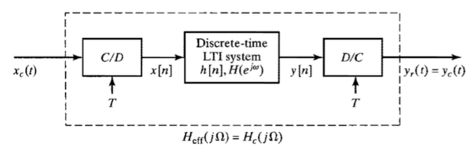
- If $x_c(t)$ is bandlimited by π/T
  - I.e $X_c(j\Omega) = 0$ for $|\Omega| > \pi/T$
- then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \pi/T \\ 0 & else \end{cases}$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time





Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$$

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$$\boxed{h[n] = Th_c(nT)}$$

Impulse Invariance

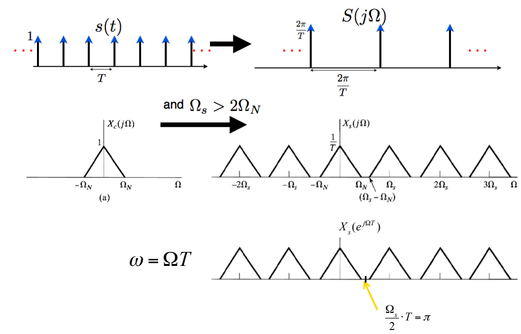
- Let,

$$h[n] = h_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

Frequency Domain Analysis



Impulse Invariance

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$$h[n] = h_c(nT)$$

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$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi$$

Impulse Invariance

- Let,

$$h[n] = T h_c(nT)$$

$$\boxed{H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T}$$

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$$H(e^{j\omega}) = \frac{T}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

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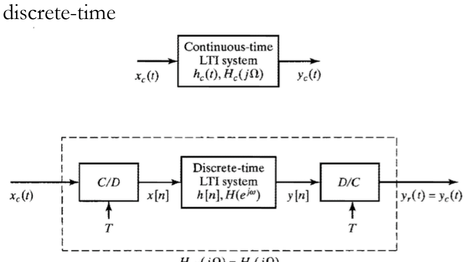
Impulse Invariance

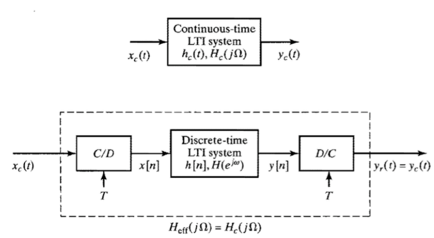
- Want to implement continuous-time system in discrete-time



$$\boxed{h[n] = T h_c(nT)}$$

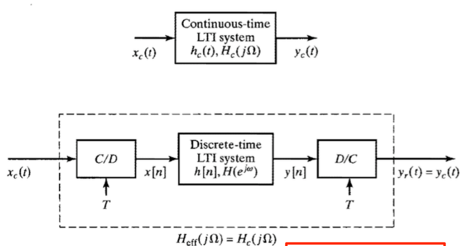
Example: DT Lowpass Filter

- We wish to implement a lowpass filter with cutoff frequency Ω_c on continuous time signal in discrete time with the following system



Impulse Invariance

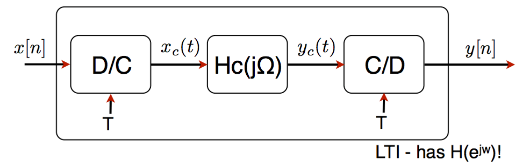
- Want to implement continuous-time system in discrete-time



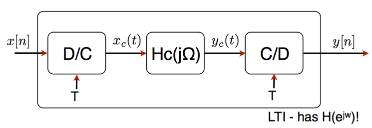
$$\boxed{h[n] = Th_c(nT)}$$

Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time

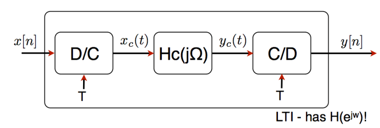


Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Continuous-Time Processing of Discrete-Time

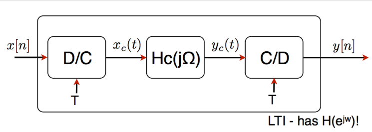


$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Also bandlimited

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

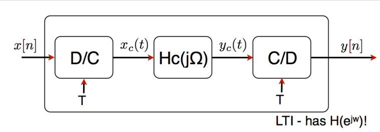
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

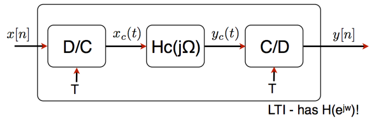
$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left[j(\Omega - k\Omega_s)\right]\Bigg|_{\Omega=\omega/T} = \frac{1}{T}Y_c(j\Omega)\Bigg|_{\Omega=\omega/T}$$

Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T}Y_c(j\Omega)\Bigg|_{\Omega=\omega/T}$$

Continuous-Time Processing of Discrete-Time

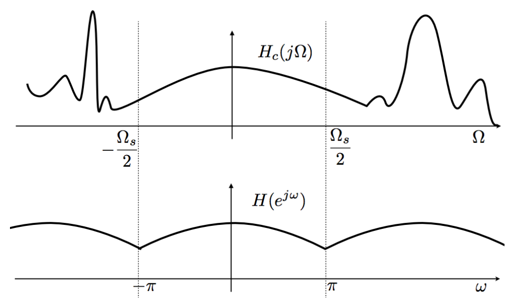


$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T}Y_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$Y(e^{j\omega}) = \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T} X_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\underbrace{H_c(j\Omega)\Big|_{\Omega=\omega/T}}_{\boxed{H(e^{j\omega})}}(TX(e^{j\omega})) \qquad |\omega| < \pi$$

Example



Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

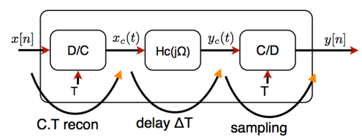
$$\boxed{\begin{array}{c}\delta[n] \leftrightarrow 1 \\ \delta[n-n_d] \leftrightarrow e^{-j\omega n_d}\end{array}}$$

Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



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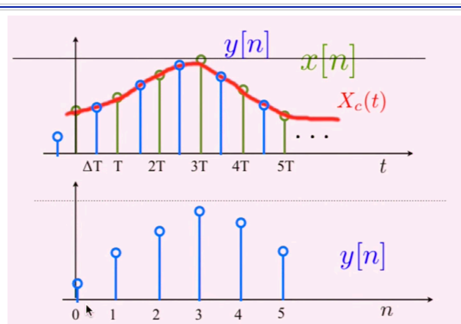
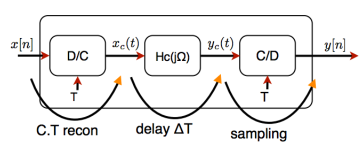


Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$y_c(t) = x_c(t - T\Delta)$$

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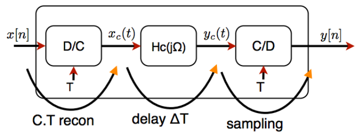

$$y_c(t) = x_c(t - T\Delta)$$

$$y[n] = y_c(nT) = x_c(nt - T\Delta)$$

$$= \sum_k x[k]\,\mathrm{sinc}\left(\frac{t - kT - T\Delta}{T}\right)\bigg|_{t=nT}$$

$$= \sum_k x[k]\,\mathrm{sinc}\left(n - k - \Delta\right)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$h[n] = \mathrm{sinc}(n - \Delta)$$

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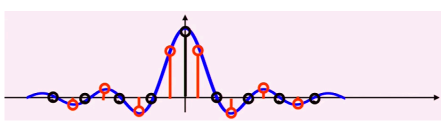

Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects
- If your discrete time signal is finely sample almost like a CT signal
  - Downsampling is just like sampling (C/D conversion)

Downsampling

- Definition: Reducing the sampling rate by an integer number

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT) \qquad x_d[n] = x_c(n\underbrace{MT}_{T'})$$

Downsampling

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The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Downsampling

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- Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$
- Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)

Downsampling

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- k=rM+i
  - i = 0, 1, …, M-1
  - r = -∞, …, ∞

Downsampling

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right) \end{aligned}$$

Downsampling

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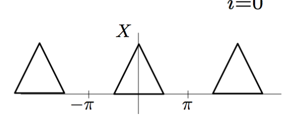
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)$$

stretch by M

replicate

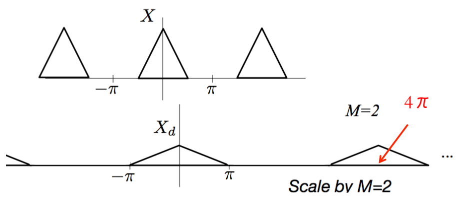
Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)$$



Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M}-\frac{2\pi}{M}i\right)}\right)$$



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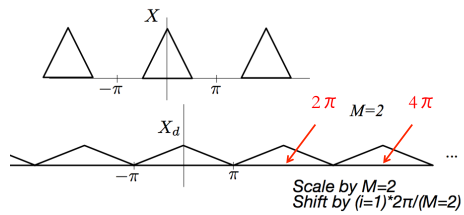

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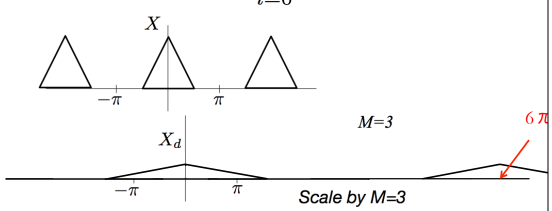

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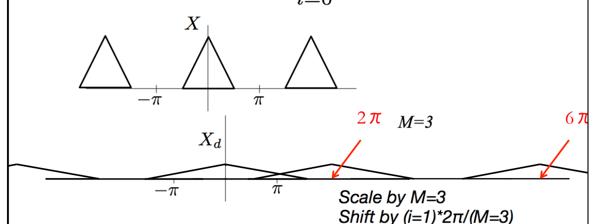

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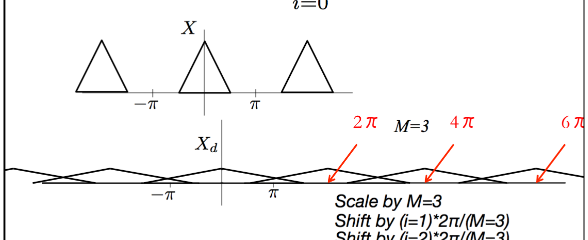
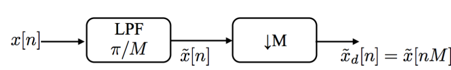
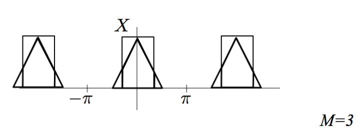
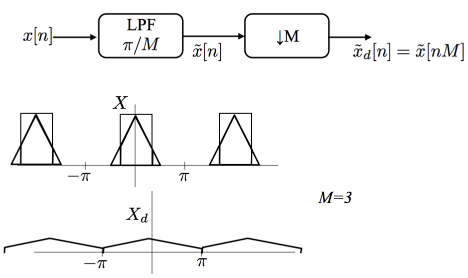


Example





Example



Upsampling

- Much like D/C converter
- Upsample by A LOT → almost continuous

- Intuition:
  - Recall our D/C model: x[n] → $x_s(t)$→$x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

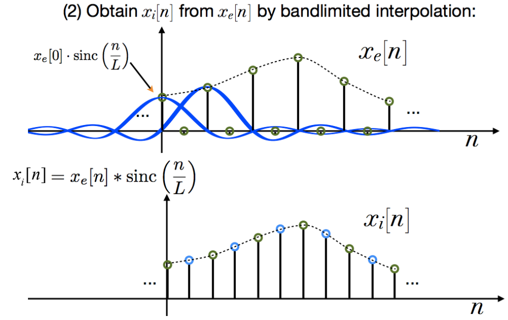
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



Upsampling

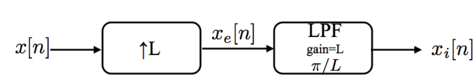
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

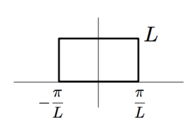
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}\left(\frac{n-kL}{L}\right)$$

Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ DTFT ⇒



Frequency Domain Interpretation

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{x_e[n]} \boxed{\begin{matrix}\text{LPF}\\ \text{gain=L}\\ \pi/L\end{matrix}} \longrightarrow x_i[n]$$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

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$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL}$$

Compress DTFT by a factor of L!

Frequency Domain Interpretation

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{x_e[n]} \boxed{\begin{matrix}\text{LPF}\\ \text{gain=L}\\ \pi/L\end{matrix}} \longrightarrow x_i[n]$$

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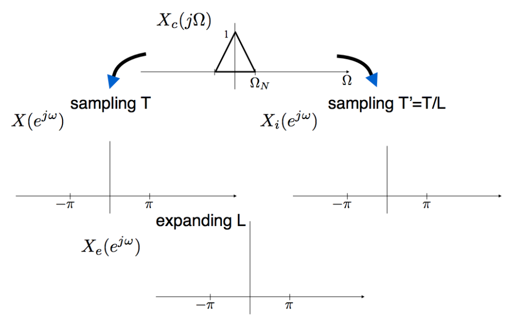
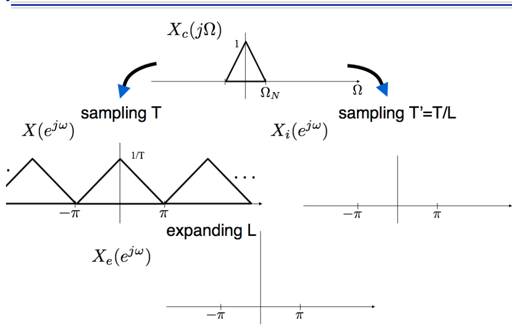
$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = X(e^{j\omega L})$$

Compress DTFT by a factor of L!

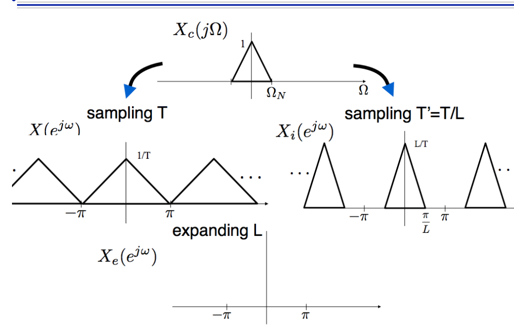
Example



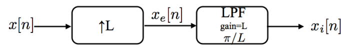
Example

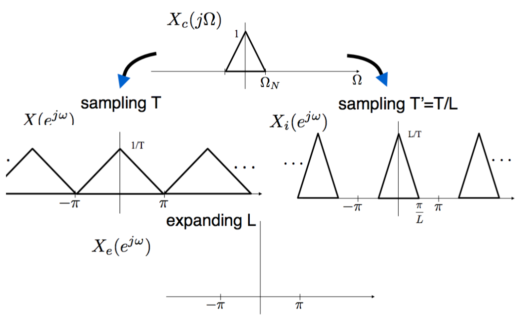


Example

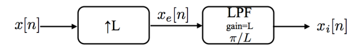


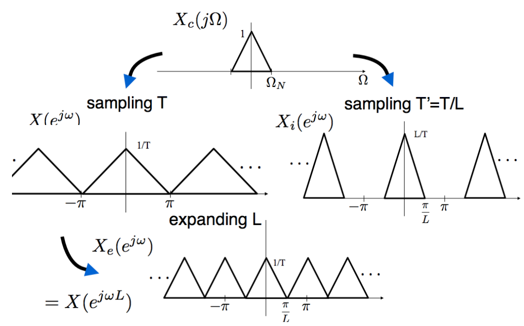
Example



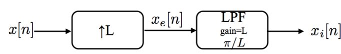


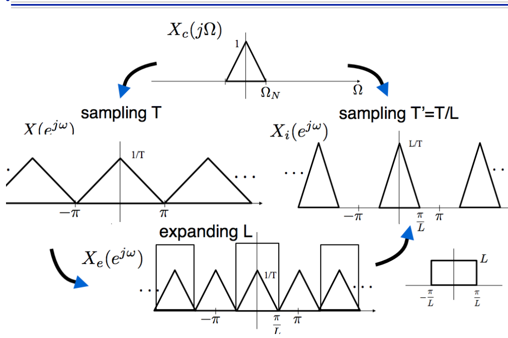
Example





Example





Big Ideas

- DT processing of CT signals
  - Impulse Invariance to design DT systems for CT signals
- CT processing of DT signals
  - Allows for interpretation of DT systems
- Downsampling
  - Like a C/D converter
- Upsampling
  - Like a D/C converter

Admin

- HW 3 due Friday
- HW 4 posted after class