University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2017 — Midterm — Thursday, March 16

- 3 Problems with point weightings shown. All 3 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.
- Final answers here.
- Additional workspace in "blue" book. Note where to find work in "blue" book if relevant.
- Sign Code of Academic Integrity statement at back of"blue" book.

Name: Answers

Mean: 74.5, Standard Deviation: 13.7

Common DTFT pairs:

Sequence	DTFT
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
1	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega+2\pi k)$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-\omega_0+2\pi k)$
$\alpha^n u[n], \lvert\alpha\rvert<1$	$\frac{1}{1-\alpha e^{-j\omega}}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega})=\begin{cases}1, & \lvert\omega\rvert\le\omega_c\\ 0, & \omega_c<\lvert\omega\rvert<\pi\end{cases}$

Common z-transform pairs:

Sequence	z-transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert<1$
$\delta[n-m]$	z^{-m}	All z except 0 (if m>0) or ∞ (if m<0)
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>\lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert<\lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert>\lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert<\lvert a\rvert$

Trigonometric Identity:

$$e^{jA} = cos(A) + jsin(A)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

DTFT Equations:

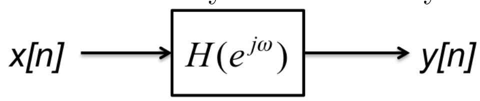
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

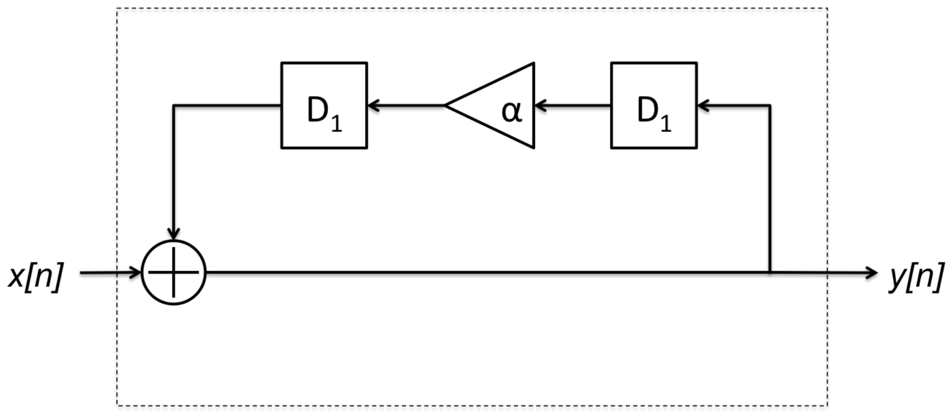
Upsampling/Downsampling:

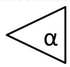
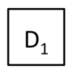
Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

1. (40 pts) Consider a causal LTI system defined by the following block diagram:





Above, the symbol  denotes multiplication by a scalar α: if the input is $w[n]$, the output is $\alpha w[n]$. The symbol  denotes a delay by one sample: if the input is $w[n]$, the output is $w[n-1]$.

(a) What is the impulse response $h[n]$ of the system? [10pts]

$$y[n] = x[n] + \alpha y[n-2]$$

The impulse response is the output when $x[n] = \delta[n]$, so the output $h[n] = 0$ for $n < 0$. For $n = 0$, $h[0] = \delta[0] = 1$. For $n > 0$, $x[n] = \delta[n] = 0$, so $h[n] = \alpha h[n-2]$. Putting this together, we can write:

$$h[n] = \begin{cases} \alpha^{n/2} & n \geq 0, \text{even} \\ 0 & \text{else} \end{cases} \tag{1}$$

(b) For what choices of α is the system stable? [10pts]
The system is stable if the impulse response is infinitely summable.

$$\begin{aligned} \sum_{0}^{\infty} |h[n]| &= \sum_{n \geq 0, \text{even}} |\alpha|^{n/2} \\ &= \sum_{k=0}^{\infty} |\alpha|^k = \begin{cases} \frac{1}{1-|\alpha|} & |\alpha| < 1 \\ \infty & \text{else} \end{cases} \end{aligned}$$

Therefore, the system is stable if $|\alpha| < 1$.

(c) For any α such that the system is stable, what is the frequency response $H(e^{j\omega})$ of the system? [10pts]

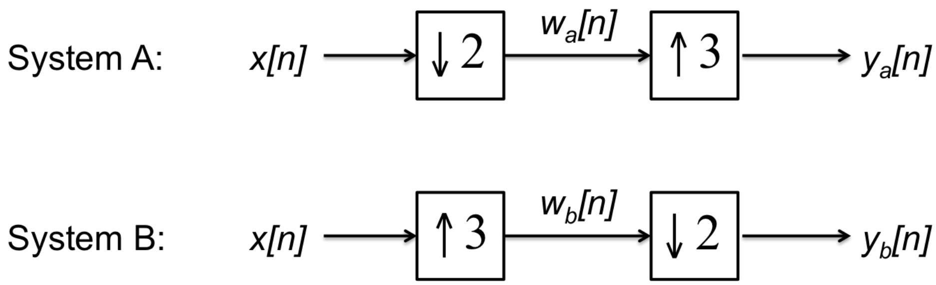
$$\begin{aligned} H(e^{j\omega}) &= \sum_{n\geq 0,\text{even}} \alpha^{n/2} e^{-j\omega n} \\ &= \sum_{k=0}^{\infty} \alpha^k e^{-j\omega(2k)} \\ &= \sum_{k=0}^{\infty} (\alpha e^{-j2\omega})^k \\ &= \frac{1}{1-\alpha e^{-j2\omega}} \end{aligned}$$

(d) What is the output $y[n]$ when the input is the oscillatory signal $x[n] = (-1)^n$? [10pts]

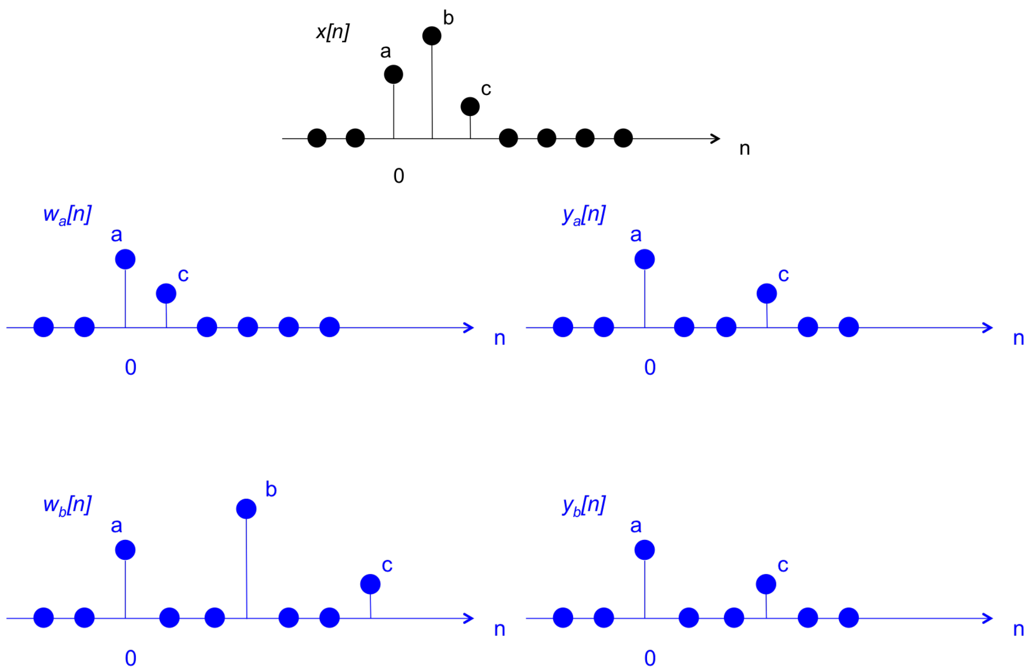
$$x[n] = (-1)^n = e^{j\pi n}$$

$$\begin{aligned} y[n] &= H(e^{j\pi})e^{j\pi n} \\ &= \frac{1}{1-\alpha e^{-j2\pi}} \cdot e^{j\pi n} \\ &= \frac{1}{1-\alpha} \cdot (-1)^n \end{aligned}$$

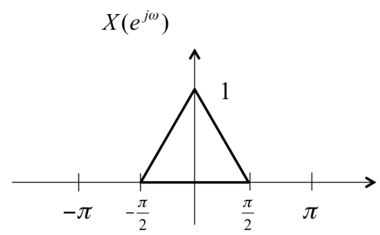
2. (30 pts)Below are two systems consisting of a compressor and expander.

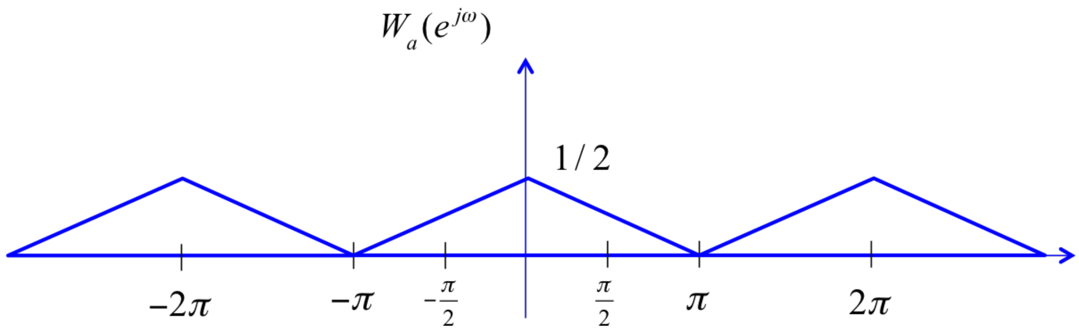


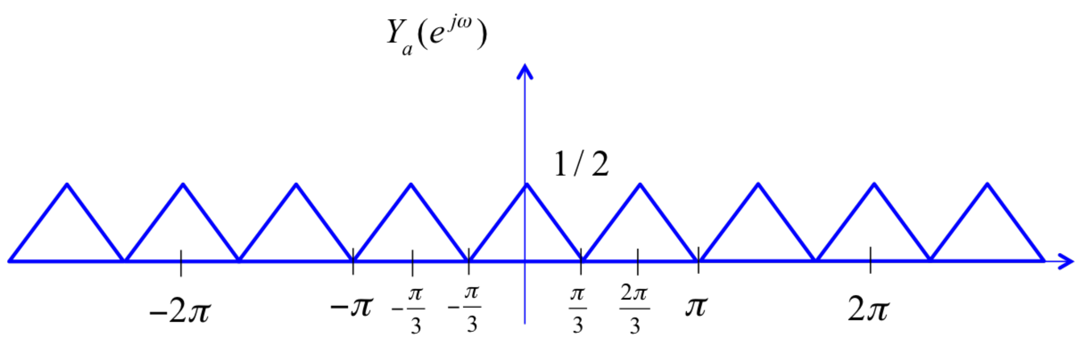
(a) For $x[n]$ shown below, sketch $y_a[n]$ and $y_b[n]$ (assume $x[n] = 0$ outside the range shown). [10pts]



(b) For $X(e^{j\omega})$ shown below, sketch $W_a(e^{j\omega})$ and $Y_a(e^{j\omega})$. [10pts]







(c) $X(e^{j\omega})$ is now the Fourier transform of an arbitrary signal $x[n]$. Express $Y_b(e^{j\omega})$ in terms of $X(e^{j\omega})$. Your answer should be in the form of an equation, not a sketch. [5pts]

$$W_b(e^{j\omega}) = X(e^{j3\omega})$$

$$\begin{aligned}
Y_b(e^{j\omega}) &= \frac{1}{2}\left[W_b(e^{j\frac{\omega}{2}}) + W_b(e^{j\left(\frac{\omega}{2}-\pi\right)})\right] \\
Y_b(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j3\left(\frac{\omega}{2}-\pi\right)})\right] \\
Y_b(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j\left(\frac{3\omega}{2}-3\pi\right)})\right] \\
Y_b(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j\left(\frac{3\omega}{2}-\pi\right)})\right]
\end{aligned}$$

(d) For any arbitrary $x[n]$, will $y_a[n] = y_b[n]$? If your answer is yes, algebraically justify your answer. If your answer is no, clearly explain or show a counterexample. [5pts]

Yes.

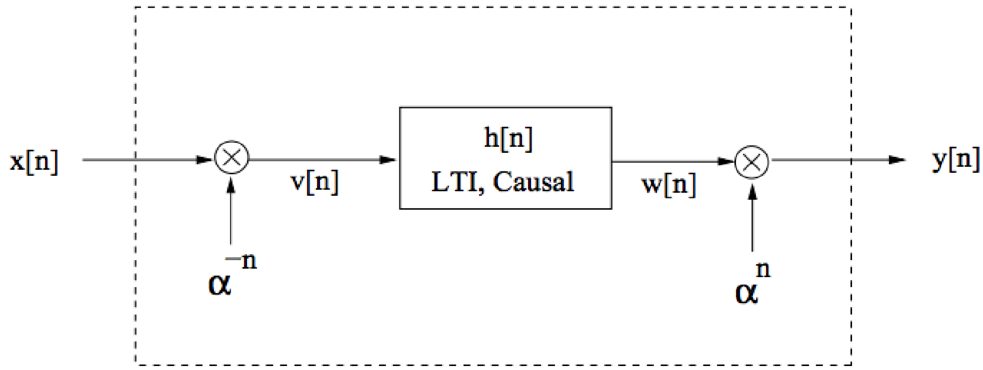
$$\begin{aligned}
W_a(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X(e^{j\left(\frac{\omega}{2}-\pi\right)})\right] \\
Y_a(e^{j\omega}) &= W_a(e^{j3\omega}) = \frac{1}{2}\left[X(e^{j\frac{3\omega}{2}}) + X(e^{j\left(\frac{3\omega}{2}-\pi\right)})\right]
\end{aligned}$$

Therefore, $Y_a(e^{j\omega}) = Y_b(e^{j\omega})$ and $y_a[n] = y_b[n]$.

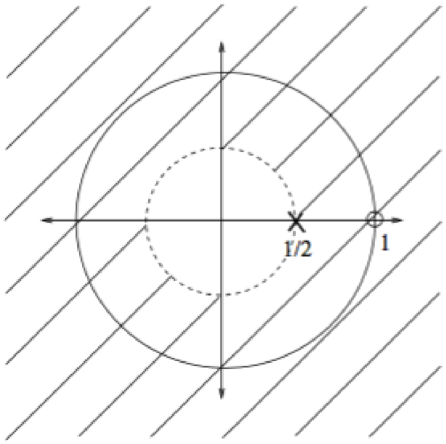
3. (30 pts)Consider a causal linear time-invariant system with impulse response $h[n]$. The z-transform of $h[n]$ is

$$H(z) = \frac{1 - z^{-1}}{1 - 0.5z^{-1}} \tag{2}$$

Consider the cascade configuration given in the figure below. Here we assume that α is a real number.



(a) Draw the pole-zero plot of $H(z)$. Also, sketch the region of convergence. Is this system stable? [10pts]



ROC: $|z| > \frac{1}{2}$. The ROC includes the unit circle, so the system is stable.

(b) Assume that $\alpha = \frac{1}{3}$ and $x[n] = \left(\frac{1}{4}\right)^n u[n]$. Compute the output $y[n]$ for the given input $x[n]$. [10pts]

From the system diagram:

$$v[n] = \alpha^{-n} x[n]$$
$$w[n] = v[n] * h[n]$$
$$y[n] = w[n]\alpha^n$$

Therefore,

$$\begin{aligned} y[n] &= (v[n] * h[n])\alpha^n \\ &= x[n] * (\alpha^n h[n]) \end{aligned}$$

Taking the z-transform:

$$\begin{aligned} Y(z) &= X(z)H(\alpha^{-1}z) \\ &= \frac{1}{1-\frac{1}{4}z^{-1}} \cdot \frac{1-\alpha z^{-1}}{1-0.5\alpha z^{-1}} \\ &= \frac{1}{1-\frac{1}{4}z^{-1}} \cdot \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{6}z^{-1}} \\ &= \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{6}z^{-1}} \end{aligned}$$

The ROC is intersection of ROC of X(z) and H(z), so ROC: $|z| > \frac{1}{4}$. We can take the inverse z-transform and get:

$$y[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{6}\right)^n u[n]$$

(c) Find the response $g[n]$ of the overall system within the enclosed box to the input $x[n] = \delta[n]$. [5pts]
From part (b), we know

$$y[n] = x[n] * (\alpha^n h[n])$$

Therefore,

$$g[n] = \delta[n] * (\alpha^n h[n])$$
$$g[n] = \alpha^n h[n]$$
$$g[n] = \alpha^n \left[\left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{2}\right)^{n-1} u[n-1]\right]$$

(d) Plot the pole-zero plot of $G(z)$, the z-transform of $g[n]$. Also, sketch the region of convergence. [5pts]

We can use the transform property $z_0 x[n] \longleftrightarrow X\left(\frac{z}{z_0}\right)$ to take the transform of $g[n]$:

$$\begin{aligned} g[n] &= \alpha^n h[n] \\ G(z) &= H\left(\frac{z}{\alpha}\right) \\ &= \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{6}z^{-1}}, ROC: |z| > \frac{1}{6} \end{aligned}$$

The pole zero plot of $G(z)$ is