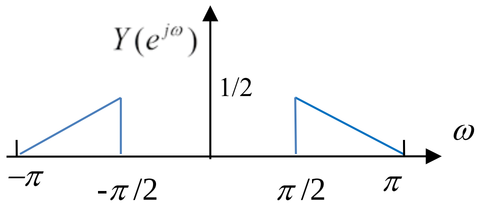
ESE 531 Solutions to Mid-term Exam October 15, 2015

Problem 1

(a)
$\{x[n]e^{j\pi n}\}$ has DTFT $X(e^{j(\omega+\pi)})$, shifted-by-π version. This is low-pass-filtered. The filter output DTFT is then shifted again by π and the result is the DTFT of y. This easily yields the following. The system acts as a high-pass filter.



(b) *One way* to do this is to express output in terms of input:

$$y[n] = b^n \cdot \left(\{h[n]\} * \{x[n]a^n\}\right) = b^n \sum_k h[n-k]x[k]a^k \qquad (1)$$

For $\boxed{b = a^{-1}}$

$$y[n] = \sum_k h[n-k]x[k]a^{-n}a^k = \sum_k h[n-k]a^{-(n-k)}x[k] \qquad (2)$$

$$\therefore\ y = \underbrace{\{h[n]a^{-n}\}}_{\text{Impulse Response of overall LTI system}} * \{x[n]\} \qquad (3)$$

<u>*Another way*</u> is to directly check linearity and time invariance.
Suppose input x produces output v from H and therefore system output $y = \{v[n]b^n\}$.

(i) Linearity:
Then sequence αx (where α is a scalar constant) clearly produces αy. If x_1 produces $y_1 = \{v_1[n]b^n\}$ and similarly x_2 produces y_2, then $x_1 + x_2$ at input will produce $y = \{(v_1[n] + v_2[n])b^n\} = y_1 + y_2$ *System is linear.*

(ii) Time Invariance:
Consider delayed version $x_{<m>}$ at input . Now $\{x_{<m>}[n]a^n\} = \{x_{<m>}[n]a^{n-m}\}a^m = \{x[n]a^n\}_{<m>} a^m$.
a^m is a constant. The output of H is $v_{<m>}a^m$ and system output is $\{v_{<m>}[n]b^n\}a^m$.
Since $\boxed{b^n = a^{-n}}$ we have $b^n a^m = a^{-(n-m)}$ and the system output is $\{v_{<m>}[n]a^{-(n-m)}\} = \{v[n]a^{-n}\}_{<m>}$
This the delayed version of $y = \{v[n]b^n\} = \{v[n]a^{-n}\}$. *System is time-invariant.*

Problem 2

(a)

$$y[n] = x[n] - 2\cos(\omega_0)x[n-1] + x[n-2], \quad \text{all } n$$

$$Y(z) = X(z) - 2\cos(\omega_0)z^{-1}X(z) + z^{-2}X(z) \;\Rightarrow\; \underline{\underline{H(z) = 1 - 2\cos(\omega_0)z^{-1} + z^{-2}}}$$

$$H(z) = \frac{z^2 - 2\cos(\omega_0)z + 1}{z^2} \;\Rightarrow\; \boxed{\text{2 poles at z = 0}}$$

For zeros:

$$z^2 - 2\cos(\omega_0)z + 1 = 0,$$

$$z = \left[2\cos(\omega_0) \pm \sqrt{4\cos^2(\omega_0) - 4}\right]/2 = \cos(\omega_0) \pm \sqrt{-\sin^2(\omega_0)} = \boxed{e^{\pm j\omega_0}} \text{ zeros}$$

(b)

Yes, there exists a causal inverse. The inverse system unit-sample response sequence h_{inv} is the inverse-Z transform of

$$\frac{1}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} = \frac{1}{\left(1 - z^{-1}\right)^2} \quad ROC: |z| > 1 \quad (\text{for } \omega_0 = 0)$$

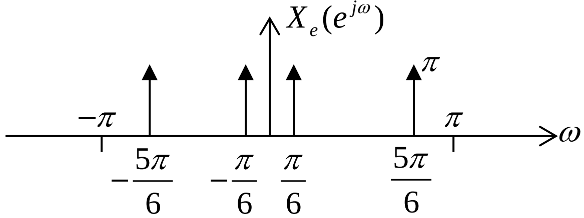
From given table, we find $h_{inv}[n] = (n+1)u[n+1] = \underline{\underline{(n+1)u[n]}}, \text{ all } n$

(c) No, a causal, stable inverse cannot exist. But can move the inverse system poles (which are on the unit circle) to slightly inside unit circle, say to 0.99, and this will yield (with ROC $|z| > 0.99$) an approximate inverse (as long as input signals have no dc component, which cannot be recovered).

Problem 3

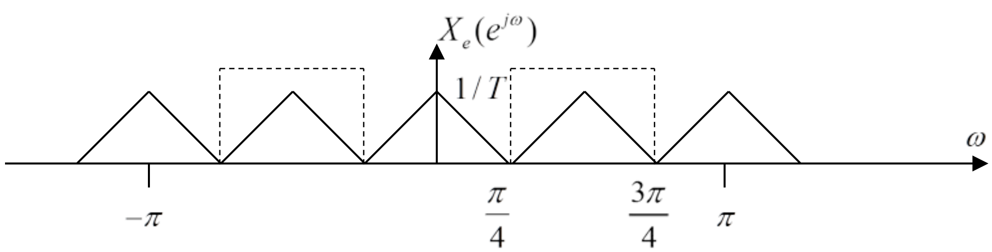
(a)

The freq. spectrum of $\{x[n]\} = \left\{\cos\left(\frac{\pi}{3}n\right)\right\}$ has 2 delta-functions of weight π at $\pm\pi/3$ between $-\pi$ and π. After rate expansion by L=2, this spectrum is compressed by a factor of 2, and has delta functions at locations $\pm\pi/6$ and $\pm 5\pi/6$.

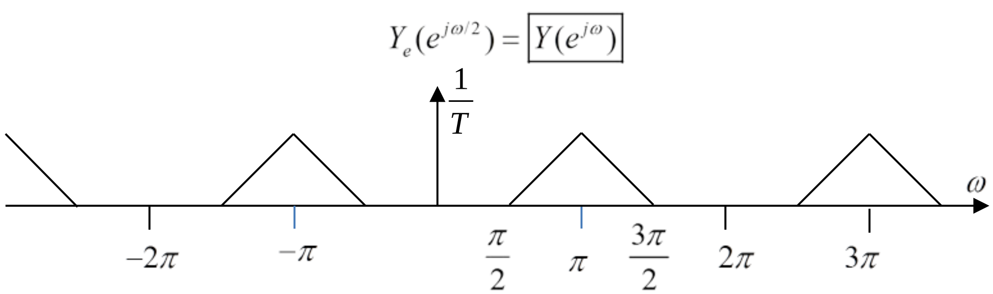


After the high-pass filter, only the upper frequency remains;
the output is $\underline{y_e[n] = \cos\left(\frac{5\pi}{6}n\right)}$ and therefore $\underline{y[n] = \cos\left(\frac{5\pi}{6}2n\right) = \cos\left(\frac{5\pi}{3}n\right)}$

(b)



Expand filtered spectrum by 2 ($Y_e(e^{j\omega/2})$ term): Its 2π shifted version gives the *same* result.



z^{-1} before compressor or downsampler is equivalent to phase $e^{-j\omega}$ in the filter. This gives an additional $\boxed{e^{-j\omega/2}}$ phase factor in the result for $Y(e^{j\omega})$.