**University of Pennsylvania**
**Department of Electrical and System Engineering**
**Digital Signal Processing**

ESE531, Spring 2018 | Final | Monday, Apr 30

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- 5 Problems with point weightings shown. All 5 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Sign Code of Academic Integrity statement at back of "blue" book.

**Name:**

Grade:

| | |
|---|---|
| Q1 | |
| Q2 | |
| Q3 | |
| Q4 | |
| Q5 | |
| Total | |

**TABLE 2.3** FOURIER TRANSFORM PAIRS

| Sequence | Fourier Transform |
|---|---|
| 1. $\delta[n]$ | $1$ |
| 2. $\delta[n-n_0]$ | $e^{-j\omega n_0}$ |
| 3. $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$ |
| 4. $a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| 5. $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$ |
| 6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| 7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$ | $\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$ |
| 8. $\frac{\sin\omega_c n}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$ |
| 9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$ |
| 10. $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$ |
| 11. $\cos(\omega_0 n+\phi)$ | $\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$ |

**TABLE 2.2** FOURIER TRANSFORM THEOREMS

| Sequence | Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $y[n]$ | $Y(e^{j\omega})$ |
| 1. $ax[n]+by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 2. $x[n-n_d] \quad (n_d$ an integer$)$ | $e^{-j\omega n_d} X(e^{j\omega})$ |
| 3. $e^{j\omega_0 n} x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. $x[-n]$ | $X(e^{-j\omega})$<br>$X^*(e^{j\omega})$ if $x[n]$ real. |
| 5. $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 6. $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 7. $x[n]y[n]$ | $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| Parseval's theorem: | |
| 8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$ | |
| 9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$ | |

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $r^n\cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $r^n\sin(\omega_0 n)u[n]$ | $\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

**TABLE 3.2** SOME $z$-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n]+bx_2[n]$ | $aX_1(z)+bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 2 | 3.4.2 | $x[n-n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\lvert z_0\rvert R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\frac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z)+X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z)-X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |

## Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

## Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

## DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

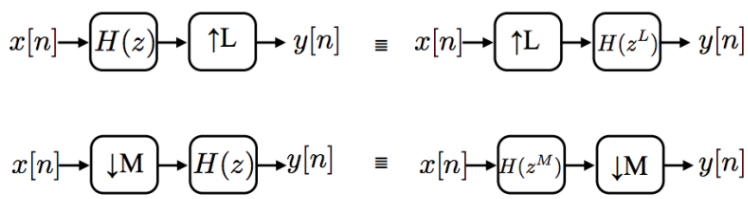
**Z-Transform Equations:**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

**Upsampling/Downsampling:**

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$
Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

**Interchange Identities:**




**DFT Equations:**

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

1. (40 points) Consider a single-pole, analog all-pass filter with transfer function (Laplace Transform of impulse response)

$$H_a(s) = \frac{s + p_a*}{s - p_a} \tag{1}$$

where the subscript a denotes analog. If one substitutes $s = j\Omega$, to get the frequency response (Fourier Transform of impulse response)

$$H_a(\Omega) = \frac{j\Omega + p_a*}{j\Omega - p_a} \tag{2}$$

it can be shown that the magnitude $|H_a(\Omega)| = 1$ for all $\Omega$.

The transfer function for a digital filter is obtained via the bilinear transform

$$s = \frac{z-1}{z+1} \tag{3}$$

(a) Consider the case where the pole is located at $p_a = -0.2 + 0.4j$. Determine the values of the pole, $p_d$, and zero, $z_d$, of the digital filter obtained via the bilinear transform.

(b) Draw a pole-zero diagram for the resulting digital filter.

(c) Is the resulting digital filter stable? Explain your answer.

(d) Is the resulting digital filter an all-pass filter? Explain your answer.

(e) What digital frequency is the analog frequency $j\Omega = \sqrt{3}$ (in radians) mapped to?

(f) Determine and write the difference equation for the resulting digital filter.

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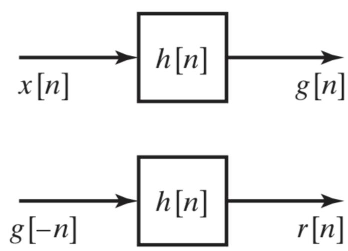
2. (20 points) In many filtering problems, we would prefer that the phase characteristics be zero or linear. For causal filters, it is impossible to have zero phase. However, for many filtering applications, it is not necessary that the impulse response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

   One technique commonly used in discrete-time filtering when the data to be filtered are of finite duration and are stored, for example, in computer memory is to process the data forward and then backward through the same filter.

   Let $h[n]$ be the impulse response of a causal filter with an arbitrary phase characteristic. Assume that $h[n]$ is **real**, and denote its Fourier transform by $H(e^{j\omega})$. Let $x[n]$ be the data that we want to filter.

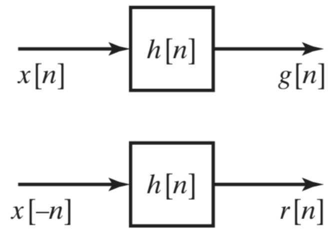
   (a) Method A: The filtering operation is performed as shown below:




$$s[n] = r[-n]$$

   Determine the overall impulse response $h_1[n]$ that relates $x[n]$ to $s[n]$ expressed in terms of $h[n]$, and $H_1(e^{j\omega})$ expressed in terms of $H(e^{j\omega})$.

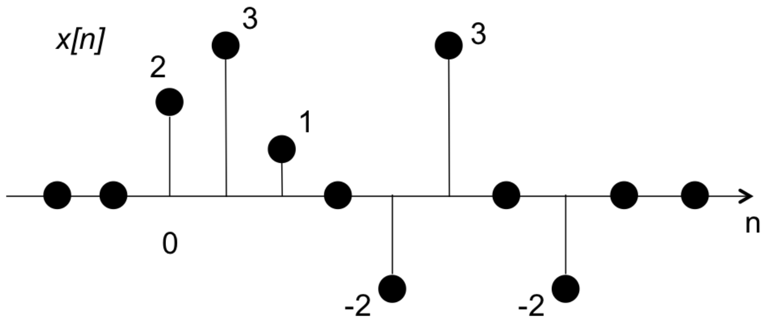
(b) Method B: As depicted below process $x[n]$ through the filter $h[n]$ to get $g[n]$. Also, process $x[n]$ backward through $h[n]$ to get $r[n]$. The output $y[n]$ is then taken as the sum of $g[n]$ and $r[-n]$. This composite set of operations can be represented by a filter with input $x[n]$, output $y[n]$, and impulse response $h_2[n]$.




$$y[n] = g[n] + r[-n]$$

Determine the overall impulse response $h_2[n]$ that relates $x[n]$ to $y[n]$ expressed in terms of $h[n]$, and $H_2(e^{j\omega})$ expressed in terms of $H(e^{j\omega})$.

3. (20 points) Each part of this problem may be solved independently. All parts use the signal $x[n]$ shown below.

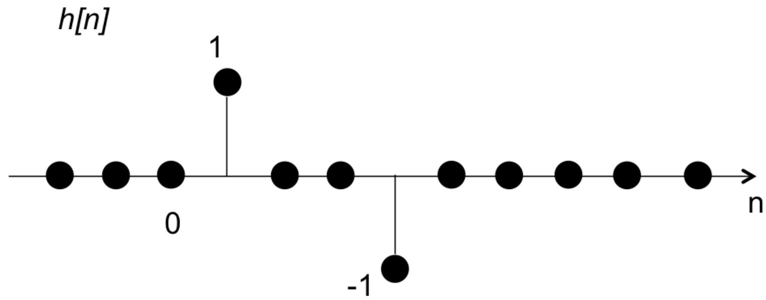



(a) Let $X(e^{j\omega})$ be the DTFT of $x[n]$. Define $R[k]$ such that

$$R[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{4}},\ 0 \leq k \leq 3 \tag{4}$$

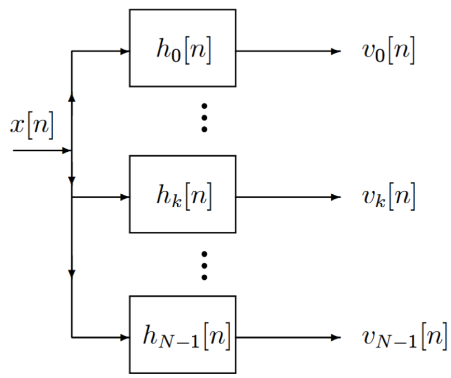
Sketch the signal $r[n]$, the four-point inverse DFT of $R[k]$.

(b) Let $X[k]$ be the eight-point DFT of $x[n]$, and let $H[k]$ be the eight-point DFT of the impulse response h[n] shown below. Define $Y[k] = X[k]H[k]$ for $0 \leq k \leq 7$.



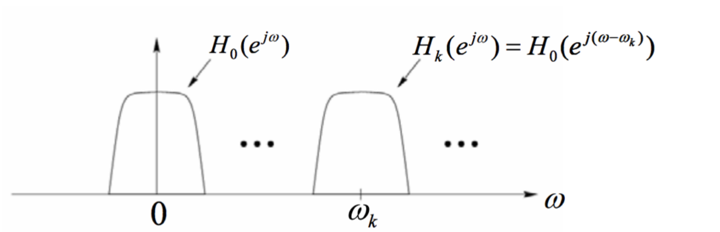

Sketch $y[n]$, the eight-point inverse DFT of $Y[k]$.

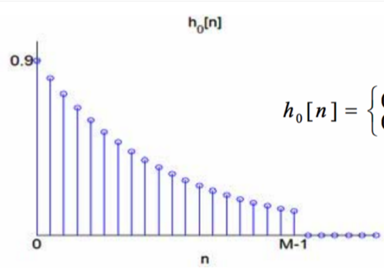
4. (10 points) The system shown below, uses a modulated filter bank for spectral analysis. The system is comprised of N systems with the frequency and impulse responses indicated.




$$h_k[n] = e^{j\omega_k n} h_0[n], \qquad \omega_k = \frac{2\pi k}{N}, \qquad \text{where } k = 0,\ 1,\ \ldots,\ N-1$$

$$h_0[n] = \text{lowpass prototype filter}, \qquad H_k(z) = H_0(e^{-j\frac{2\pi k}{N}} z)$$

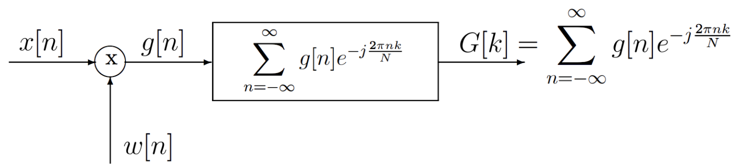



$$h_0[n] = \begin{cases} 0.9^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

where, $M < N$

An alternative system for spectral analysis is shown below.

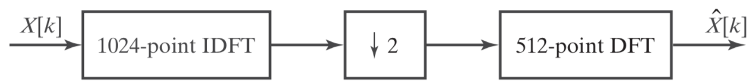



Determine $w[n]$ so that $G[k] = v_k[0]$, for $k = 0, 1, ..., N-1$.

5. (10 points) Consider a 1024-point sequence $x[n]$ constructed by interleaving two 512-point sequences $x_e[n]$ and $x_o[n]$. Specifically,

$$x[n] = \begin{cases} x_e[n/2] & \text{if } n = 0, 2, 4, ..., 1022 \\ x_o[(n-1)/2] & \text{if } n = 1, 3, 5, ..., 1023 \\ 0 & \text{else} \end{cases} \tag{5}$$

Let $X[k]$ denote the 1024-point DFT of $x[n]$ and $X_e[k]$ and $X_o[k]$ denote the 512-point DFTs of $x_e[n]$ and $x_o[n]$, respectively. Given $X[k]$ we would like to obtain $X_e[k]$ from $X[k]$ in a computationally efficient way where computational efficiency is measured in terms of the total number of complex multiplies and adds required. One not-very-efficient approach is as shown below




Specify the most efficient algorithm that you can to obtain $X_e[k]$ from $X[k]$.