University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2018 Final Monday, Apr 30

- 5 Problems with point weightings shown. All 5 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Sign Code of Academic Integrity statement at back of "blue" book.

Name:

Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Q5	
Total	

Mean: 69.3, Stdev: 15.6

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\|<1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\|<1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\|<1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(e^{j\omega})\|^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\|>r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\|>r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\|>0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

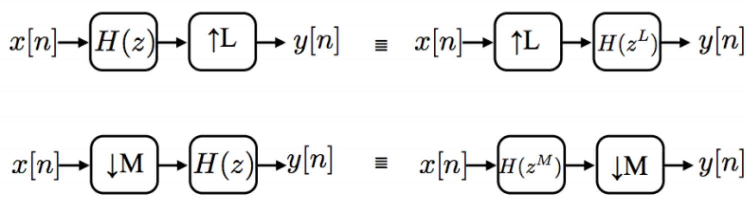
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

Interchange Identities:



DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum\limits_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum\limits_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

1. (40 points) Consider a single-pole, analog all-pass filter with transfer function (Laplace Transform of impulse response)

$$H_a(s) = \frac{s + p_a*}{s - p_a} \tag{1}$$

where the subscript a denotes analog. If one substitutes $s = j\Omega$, to get the frequency response (Fourier Transform of impulse response)

$$H_a(\Omega) = \frac{j\Omega + p_a*}{j\Omega - p_a} \tag{2}$$

it can be shown that the magnitude $|H_a(\Omega)| = 1$ for all Ω.

The transfer function for a digital filter is obtained via the bilinear transform

$$s = \frac{z-1}{z+1} \tag{3}$$

(a) Consider the case where the pole is located at $p_a = -0.2 + 0.4j$. Determine the values of the pole, p_d, and zero, z_d, of the digital filter obtained via the bilinear transform.

(b) Draw a pole-zero diagram for the resulting digital filter.

(c) Is the resulting digital filter stable? Explain your answer.

(d) Is the resulting digital filter an all-pass filter? Explain your answer.

(e) What digital frequency is the analog frequency $j\Omega = \sqrt{3}$ (in radians) mapped to?

(f) Determine and write the difference equation for the resulting digital filter.

a) $p_a = -0.2 + 0.4j$ and $z_a = -p_a^* = 0.2 + 0.4j$

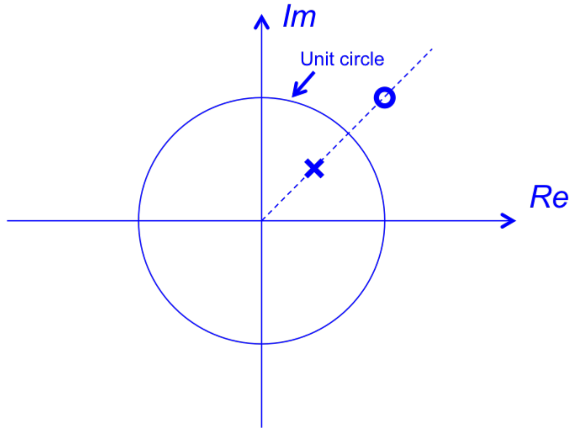
$$s = \frac{z-1}{z+1} \rightarrow z = \frac{1+s}{1-s}$$

Therefore:

$$z_d = \frac{1+z_a}{1-z_a} = \frac{1+(0.2+0.4j)}{1-(0.2+0.4j)} = \frac{1.2+0.4j}{0.8-0.4j} = \frac{3+j}{2-j} = 1+j$$

$$p_d = \frac{1+p_a}{1-p_a} = \frac{1+(-0.2+0.4j)}{1-(-0.2+0.4j)} = \frac{0.8+0.4j}{1.2-0.4j} = \frac{2+j}{3-j} = \frac{1}{2} + \frac{1}{2}j$$

b)



c) Yes, the filter is stable because all poles are inside the unit circle.

d) Yes, the filter is an all-pass filter. You can infer this from the fact that the bilinear transform maps the entire $j\Omega$ axis to one revolution around unit circle, therefore $|H(e^{j\omega})| = 1$ for all ω.

e) For the bilinear transformation $\Omega = tan\left(\frac{\omega}{2}\right)$.

$$\sqrt{3} = tan\left(\frac{\omega}{2}\right) \rightarrow \omega = 2tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$$

f)

$$\frac{Y(z)}{X(z)} = H(z) = \frac{z - z_d}{z - z_a} = \frac{z - (1 + j)}{z - (\frac{1}{2} + \frac{1}{2}j)} = \frac{1 - (1 + j)z^{-1}}{1 - (\frac{1}{2} + \frac{1}{2}j)z^{-1}}$$

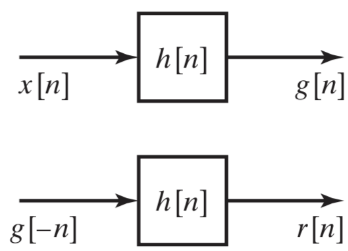
$$y[n] - (\frac{1}{2} + \frac{1}{2}j)y[n-1] = x[n] - (1 + j)x[n-1]$$

2. (20 points) In many filtering problems, we would prefer that the phase characteristics be zero or linear. For causal filters, it is impossible to have zero phase. However, for many filtering applications, it is not necessary that the impulse response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

   One technique commonly used in discrete-time filtering when the data to be filtered are of finite duration and are stored, for example, in computer memory is to process the data forward and then backward through the same filter.

   Let $h[n]$ be the impulse response of a causal filter with an arbitrary phase characteristic. Assume that $h[n]$ is **real**, and denote its Fourier transform by $H(e^{j\omega})$. Let $x[n]$ be the data that we want to filter.

   (a) Method A: The filtering operation is performed as shown below:



$$s[n] = r[-n]$$

Determine the overall impulse response $h_1[n]$ that relates $x[n]$ to $s[n]$ expressed in terms of $h[n]$, and $H_1(e^{j\omega})$ expressed in terms of $H(e^{j\omega})$.

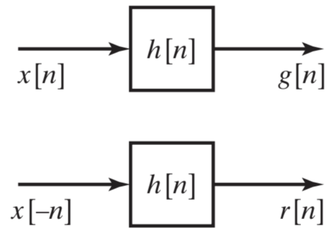
$$g[n] = x[n] * h[n]$$
$$r[n] = g[-n] * h[n]$$
$$s[n] = r[-n] = g[n] * h[-n] = x[n] * (h[n] * h[-n])$$

Therefore, $h_1[n] = h[n] * h[-n]$.

The frequency response, $H_1(e^{j\omega})$, is the product of the individual frequency responses. Since $h[n]$ is real, we can use the identity from Table 2.2 line 4 in the data sheets:

$$H_1(e^{j\omega}) = H(e^{j\omega}) \cdot H^*(e^{j\omega})$$
$$H_1(e^{j\omega}) = |H(e^{j\omega})|^2$$

(b) Method B: As depicted below process $x[n]$ through the filter $h[n]$ to get $g[n]$. Also, process $x[n]$ backward through $h[n]$ to get $r[n]$. The output $y[n]$ is then taken as the sum of $g[n]$ and $r[-n]$. This composite set of operations can be represented by a filter with input $x[n]$, output $y[n]$, and impulse response $h_2[n]$.



$$y[n] = g[n] + r[-n]$$

Determine the overall impulse response $h_2[n]$ that relates $x[n]$ to $y[n]$ expressed in terms of $h[n]$, and $H_2(e^{j\omega})$ expressed in terms of $H(e^{j\omega})$.

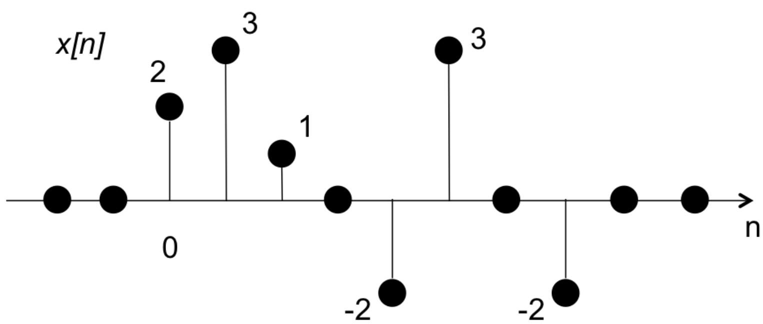
$$g[n] = x[n] * h[n]$$
$$r[n] = x[-n] * h[n]$$
$$y[n] = g[n] + r[-n] = x[n] * h[n] + x[n] * h[-n]$$
$$y[n] = x[n] * (h[n] + h[-n])$$

Therefore, $h_2[n] = h[n] + h[-n]$.

The frequency response, $H_2(e^{j\omega})$, is the sum of the individual frequency responses. Since $h[n]$ is real, we can again use the identity from Table 2.2 line 4 in the data sheets:

$$H_2(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$$
$$H_2(e^{j\omega}) = 2\,\mathrm{Re}\{H(e^{j\omega})\}$$

3. (20 points) Each part of this problem may be solved independently. All parts use the signal $x[n]$ shown below.

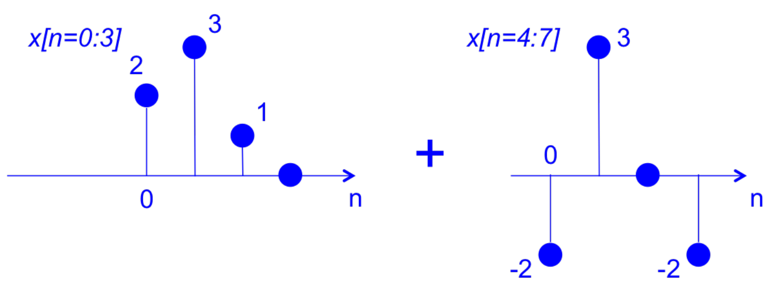


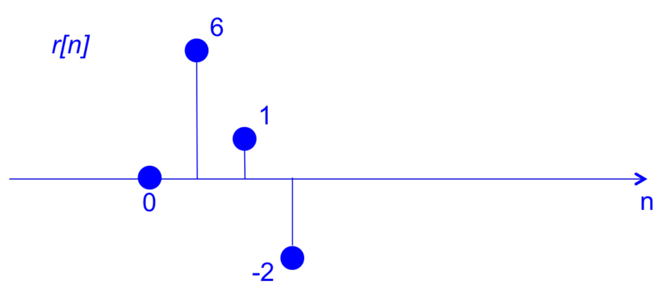
(a) Let $X(e^{j\omega})$ be the DTFT of $x[n]$. Define $R[k]$ such that

$$R[k] = X(e^{j\omega})|_{\omega=\frac{2\pi k}{4}},\ 0 \leq k \leq 3 \tag{4}$$

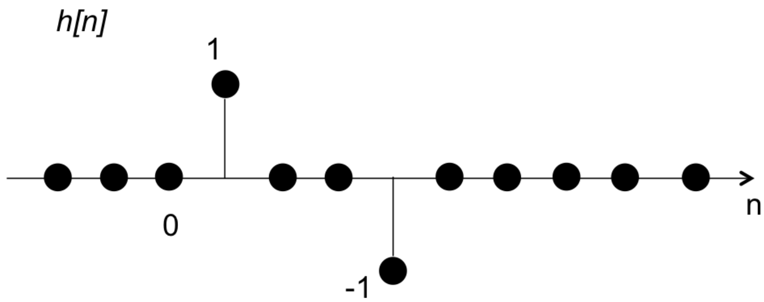
Sketch the signal $r[n]$, the four-point inverse DFT of $R[k]$.

R[k] is the 4pt DFT of x[n]. Inverting R[k] to get r[n] is x[n] with time-aliasing.



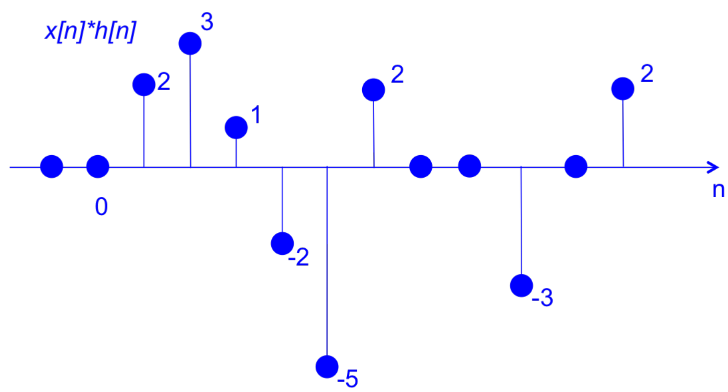


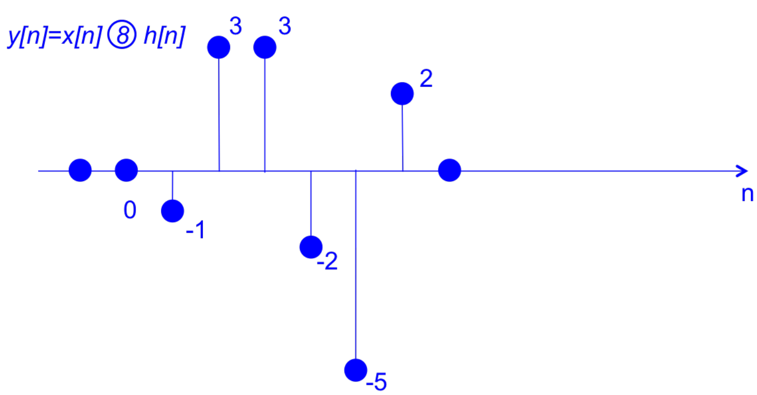
(b) Let $X[k]$ be the eight-point DFT of $x[n]$, and let $H[k]$ be the eight-point DFT of the impulse response h[n] shown below. Define $Y[k] = X[k]H[k]$ for $0 \leq k \leq 7$.



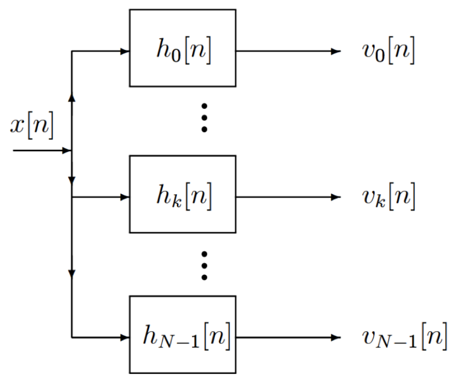
Sketch $y[n]$, the eight-point inverse DFT of $Y[k]$.

$y[n]$ is the 8-pt circular convolution of $x[n]$ and $h[n]$, which can be seen as the linear convolution of $x[n]$ and $h[n]$ with time-aliasing.



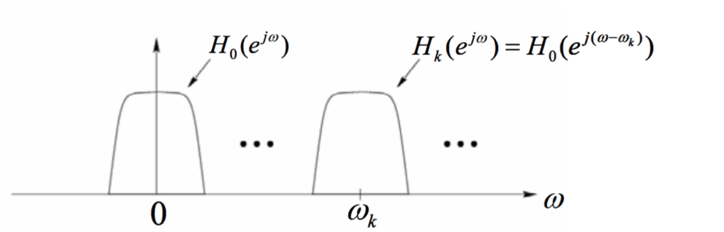


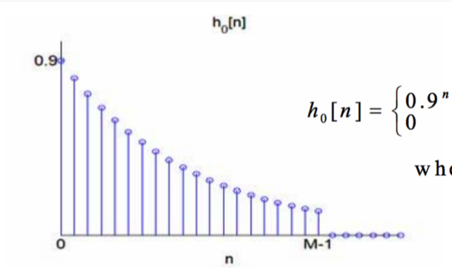
4. (10 points) The system shown below, uses a modulated filter bank for spectral analysis. The system is comprised of N systems with the frequency and impulse responses indicated.



$$h_k[n] = e^{j\omega_k n} h_0[n], \qquad \omega_k = \frac{2\pi k}{N}, \qquad \text{where } k = 0,\ 1,\ \ldots,\ N-1$$

$$h_0[n] = \text{lowpass prototype filter}, \qquad H_k(z) = H_0(e^{-j\frac{2\pi k}{N}} z)$$

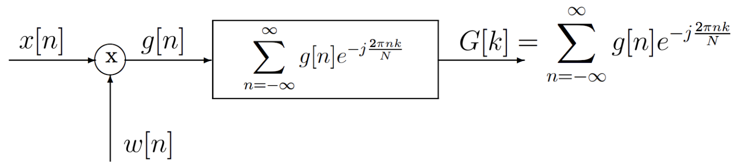




$$h_0[n] = \begin{cases} 0.9^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

where, $M < N$

An alternative system for spectral analysis is shown below.



Determine $w[n]$ so that $G[k] = v_k[0]$, for $k = 0, 1, ..., N-1$.

$$G[k] = \sum_{n=-\infty}^{\infty} g[n]e^{-j\frac{2\pi nk}{N}} = \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-j\frac{2\pi nk}{N}} \quad 1$$
$$H_k[z] = H_0(e^{-j\frac{2\pi k}{N}}z) \rightarrow h_k[n] = h_0[n]e^{j\frac{2\pi nk}{N}}$$
$$v_k[m] = \sum_{n=-\infty}^{\infty} x[n]h_k[m-n] = \sum_{n=-\infty}^{\infty} x[n]h_0[m-n]e^{j\frac{2\pi(m-n)k}{N}}$$
$$v_k[0] = \sum_{n=-\infty}^{\infty} x[n]h_0[-n]e^{-j\frac{2\pi nk}{N}} = G[k] \quad 2$$

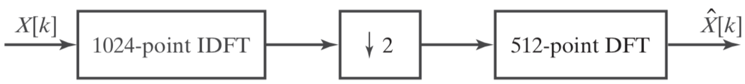
Combining equations 1 and 2:

$$w[n] = h_0[-n] = \begin{cases} 0.9^{-n} & \text{if } -M+1 \leq n \leq 0 \\ 0 & \text{else} \end{cases} \tag{5}$$

5. (10 points) Consider a 1024-point sequence $x[n]$ constructed by interleaving two 512-point sequences $x_e[n]$ and $x_o[n]$. Specifically,

$$x[n] = \begin{cases} x_e[n/2] & \text{if } n = 0, 2, 4, ..., 1022 \\ x_o[(n-1)/2] & \text{if } n = 1, 3, 5, ..., 1023 \\ 0 & \text{else} \end{cases} \tag{6}$$

Let $X[k]$ denote the 1024-point DFT of $x[n]$ and $X_e[k]$ and $X_o[k]$ denote the 512-point DFTs of $x_e[n]$ and $x_o[n]$, respectively. Given $X[k]$ we would like to obtain $X_e[k]$ from $X[k]$ in a computationally efficient way where computational efficiency is measured in terms of the total number of complex multiplies and adds required. One not-very-efficient approach is as shown below



Specify the most efficient algorithm that you can to obtain $X_e[k]$ from $X[k]$.

The idea is that since we want the DFT of the even samples and the decimation-in-time algorithm computes the DFT of the even and odd samples as an intermediate step, we just want to undo this last step.

Let N=1024. Using the IDFT equation for $x \leftrightarrow X$ and the fact that $x_e[n]$ is a down-sampled version of $x[n]$,

$$\begin{aligned} x_e[n] &= x[2n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-k2n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_{N/2}^{-kn} \\ &= \frac{1}{N}\sum_{k=0}^{N/2-1} (X[k]+X[k+N/2])W_{N/2}^{-kn} \text{ (since } W_{N/2}^{-kn} \text{ is a periodic function with period } N/2.\text{)} \end{aligned}$$

Comparing to the IDFT of the equation for $x_e \leftrightarrow X_e$:

$$x_e[n] = \frac{1}{N/2}\sum_{k=0}^{N/2-1} X_e[k]W_{N/2}^{-kn}$$

Therefore,

$$X_e[k] = \frac{1}{2}(X[k]+X[k+N/2])$$