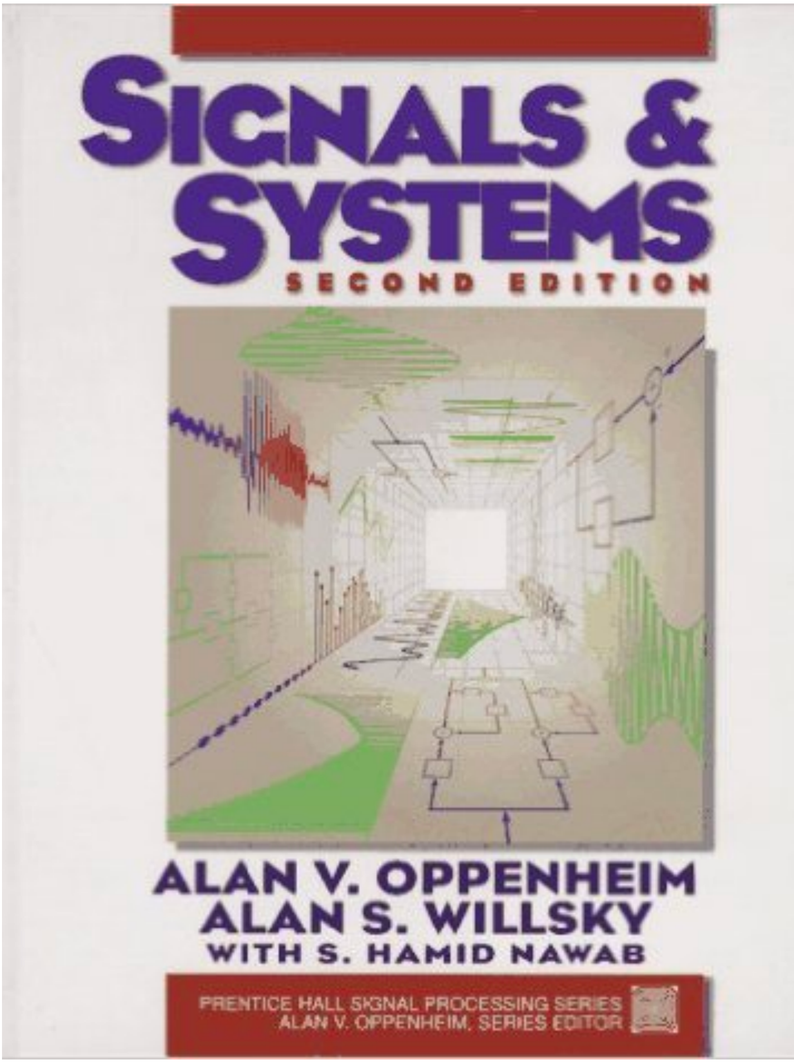
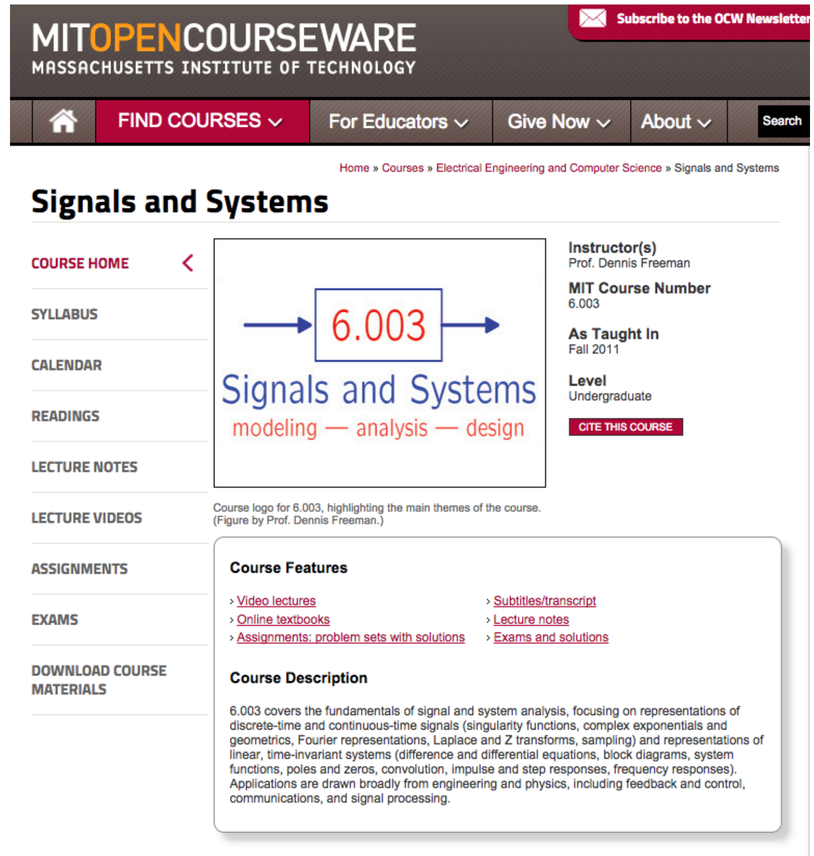
ESE 531: Digital Signal Processing

Lec 10: February 15th, 2018

Practical and Non-integer Sampling, Multi-rate Sampling

Signals and Systems Review



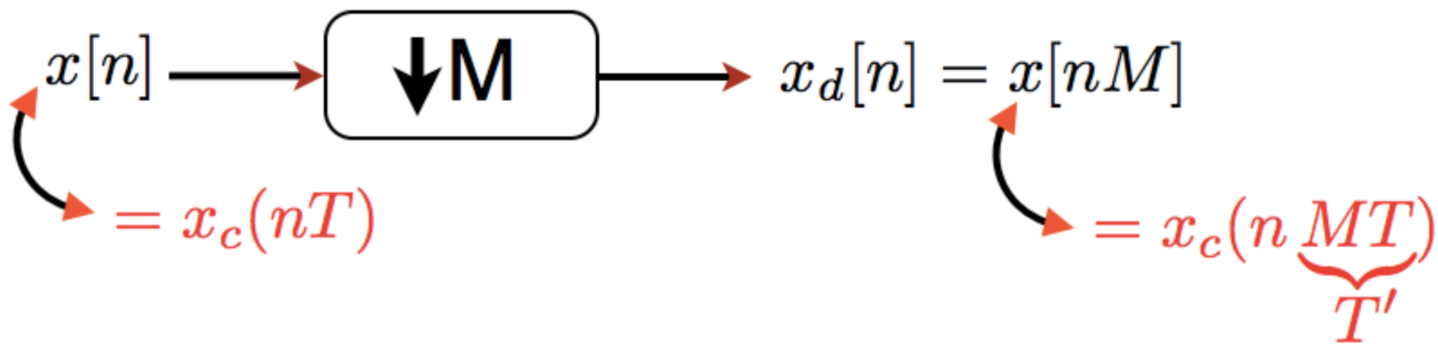


Lecture Outline

- Review: Downsampling/Upsampling
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks

Downsampling

- Definition: Reducing the sampling rate by an integer number



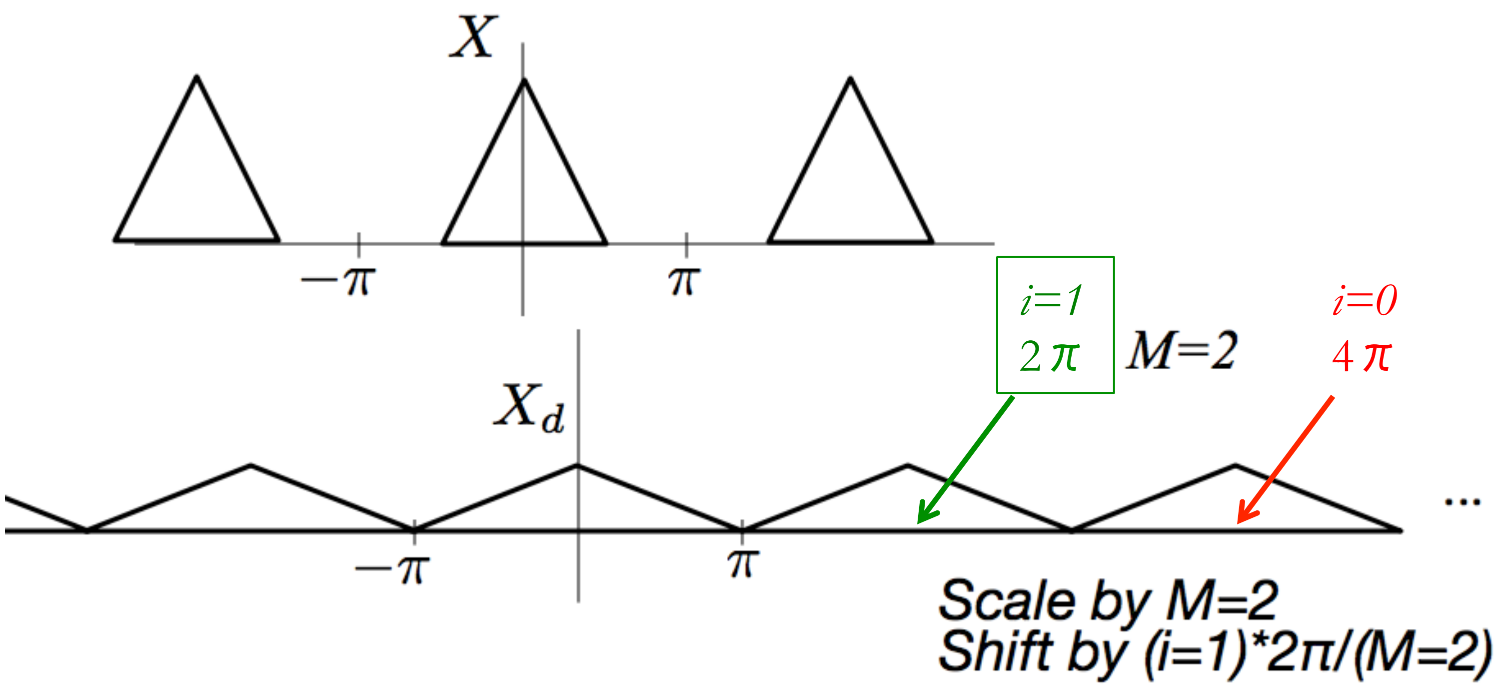
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M

replicate

Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



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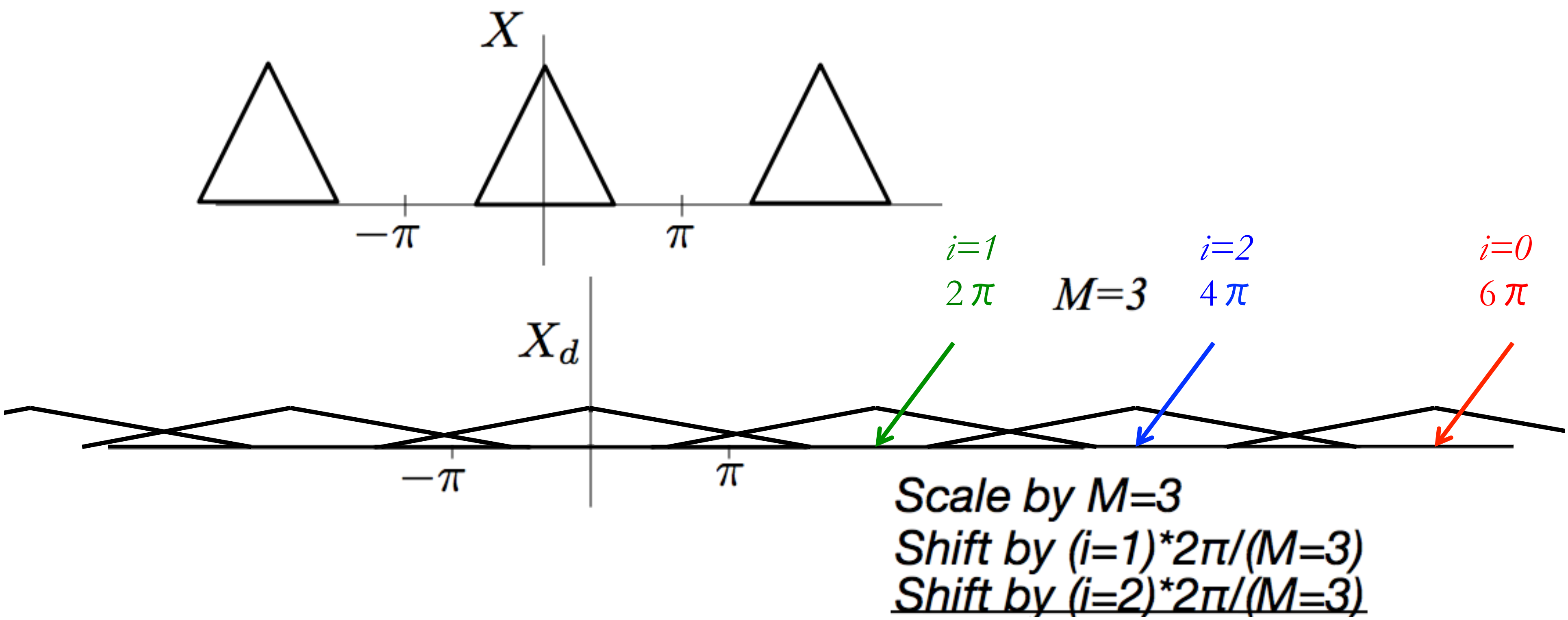
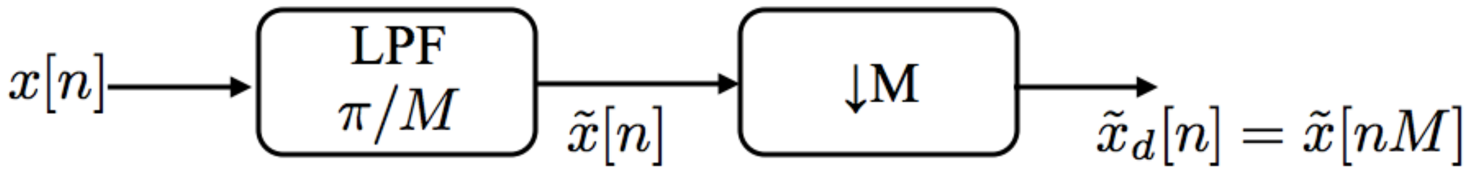
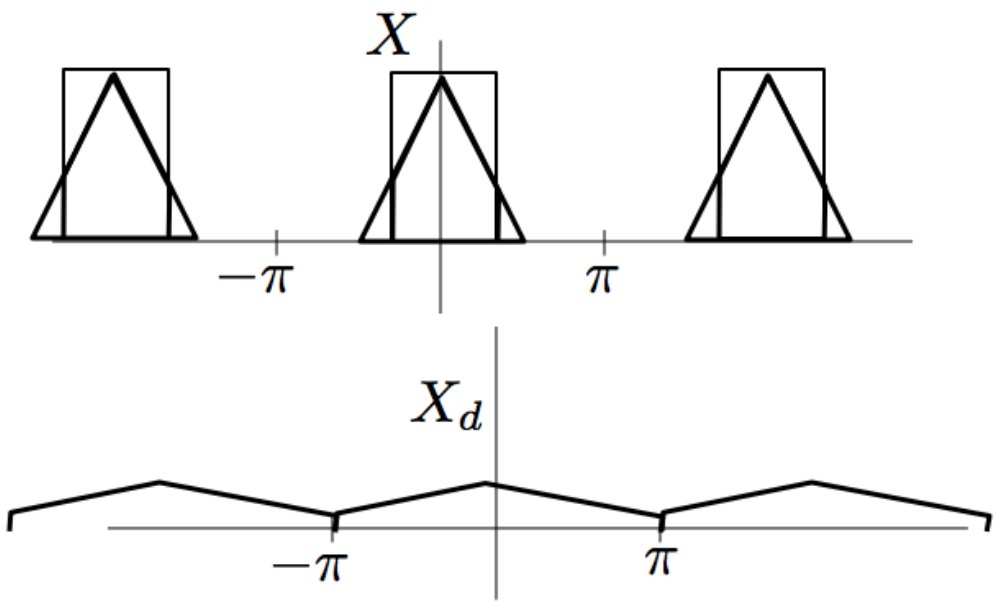


Example





M=3

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

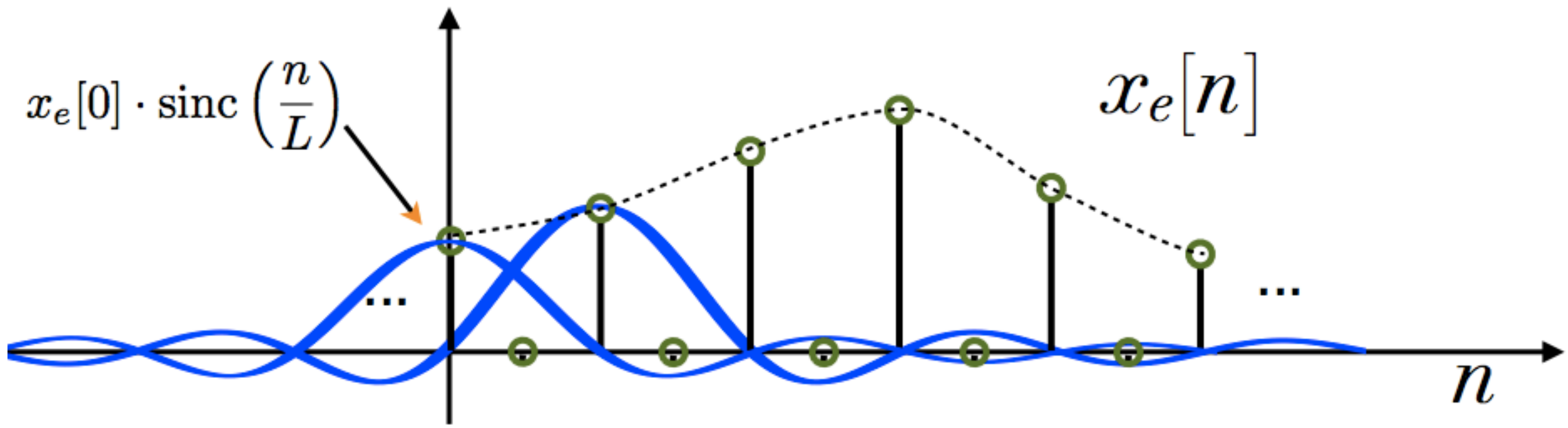
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

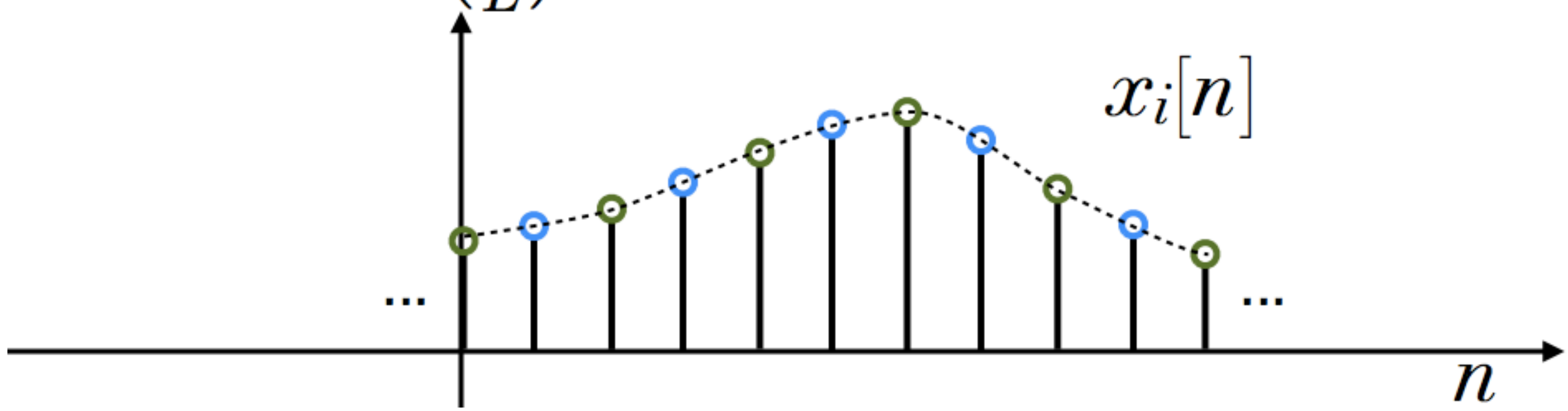
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

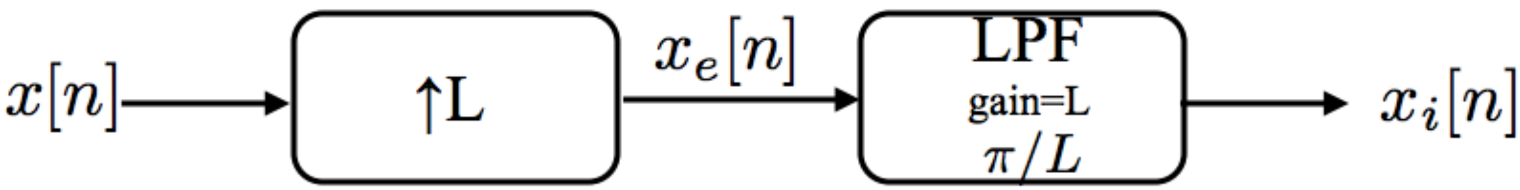


$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$

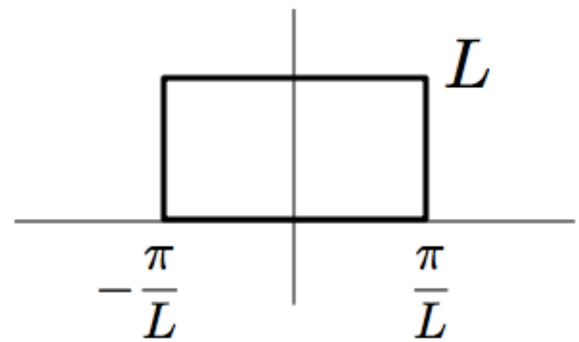


Frequency Domain Interpretation

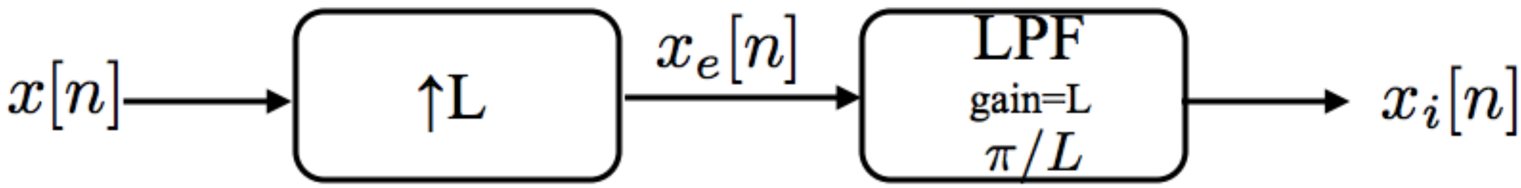
$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$



$\mathrm{sinc}(n/L)$ DTFT ⇒



Frequency Domain Interpretation

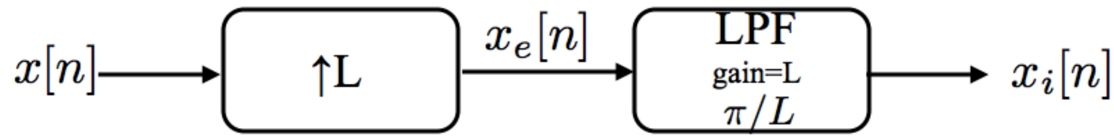


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

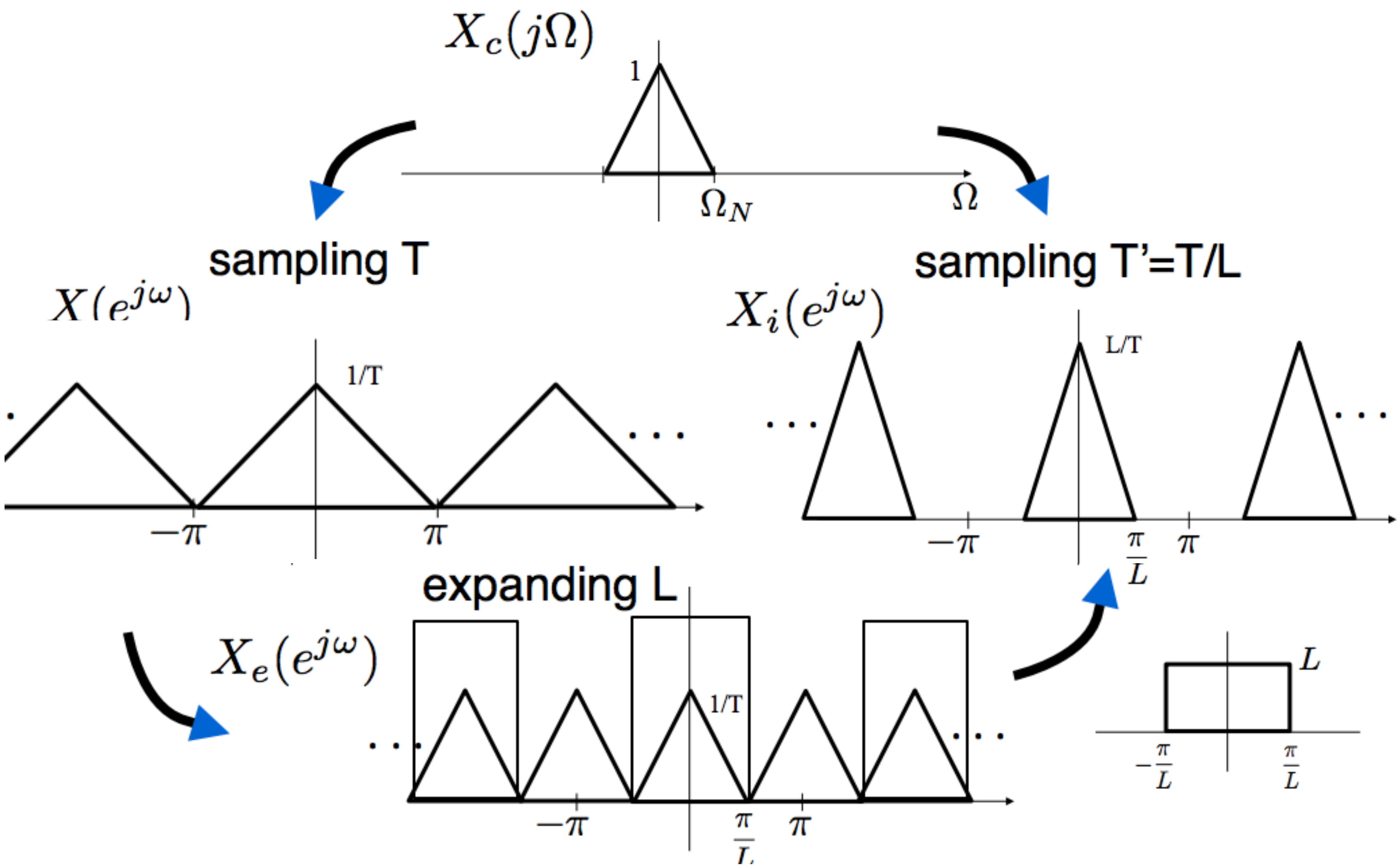
$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = \boxed{X(e^{j\omega L})}$$

Compress DTFT by a factor of L!

Example



$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

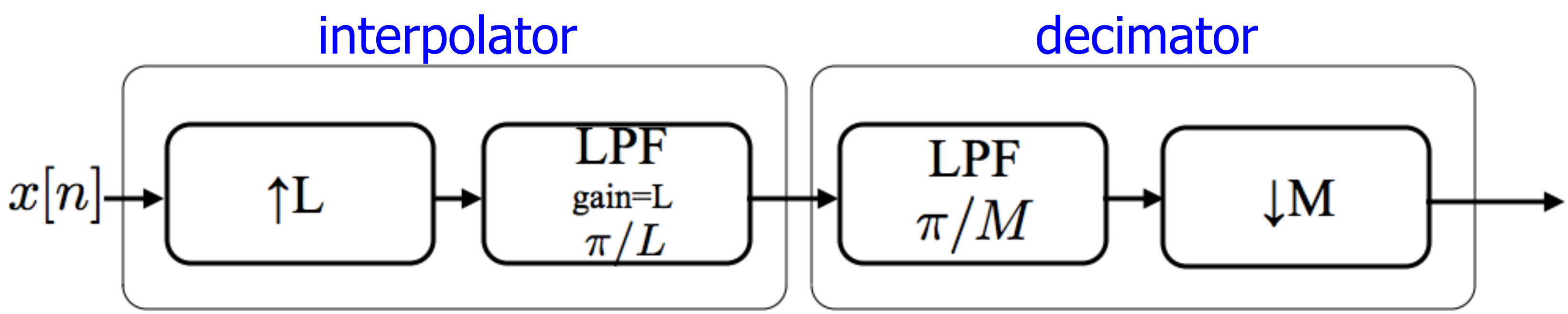
Non-integer Resampling

Non-integer Resampling

- T'=TM/L

Non-integer Resampling

- T'=TM/L
  - Upsample by L, then downsample by M



Non-integer Resampling

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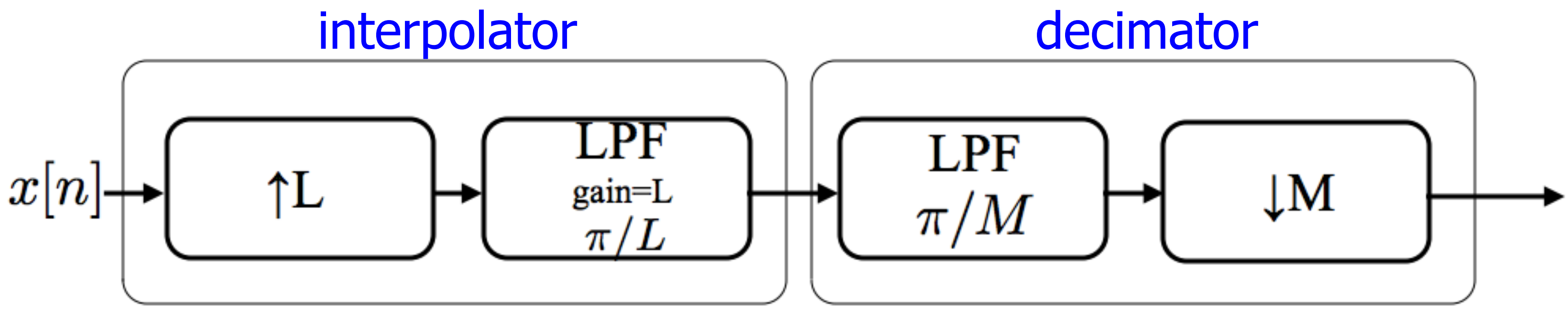
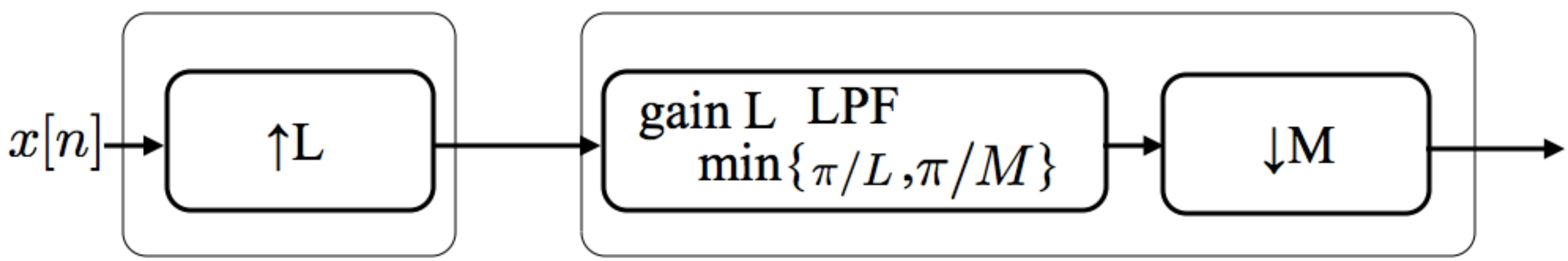
  - Upsample by L, then downsample by M

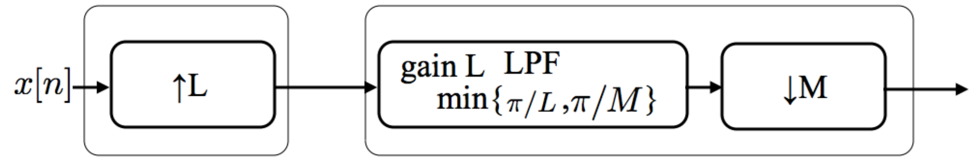


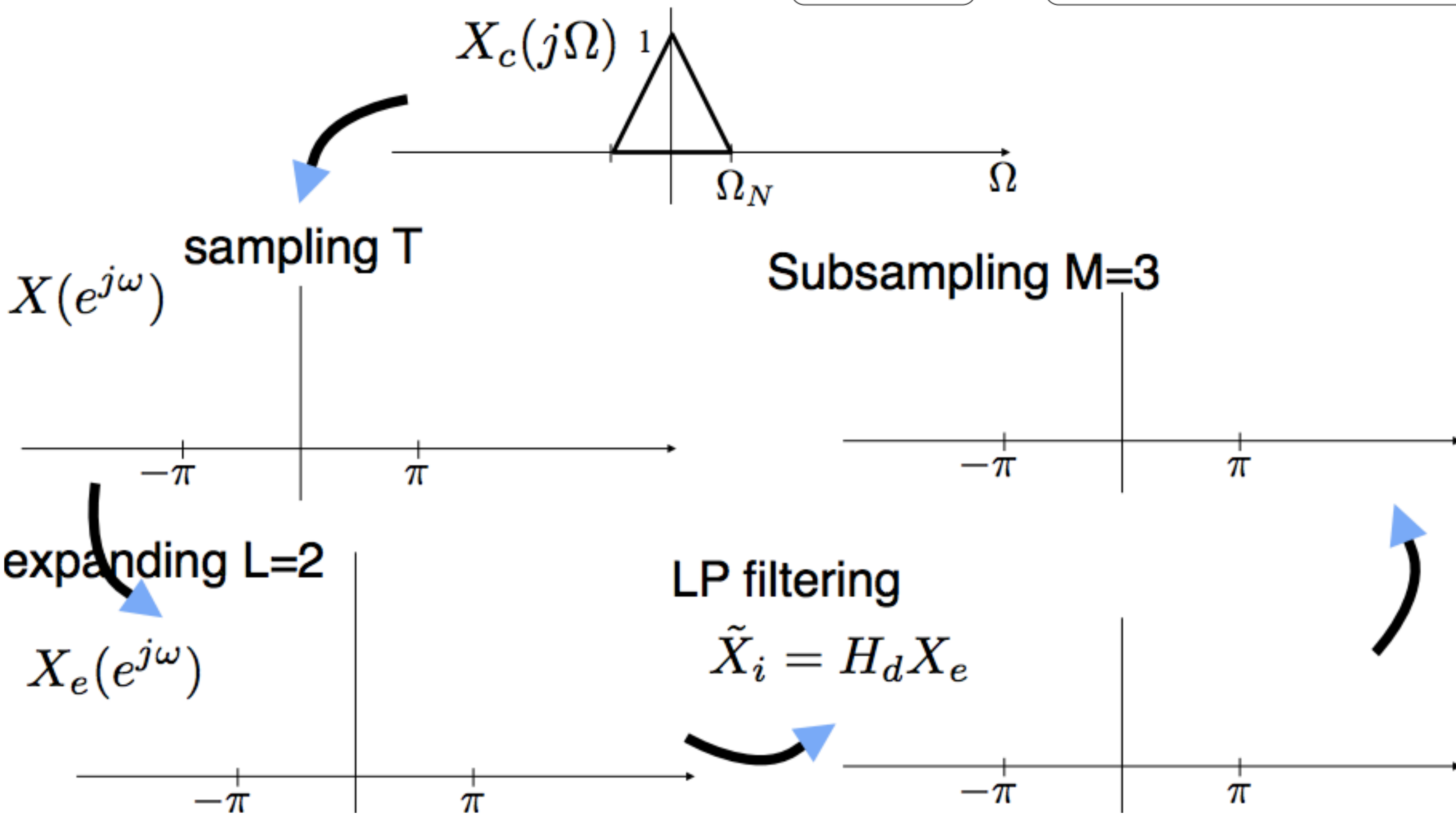
Or,



Example

- T'=3/2T → L=2, M=3





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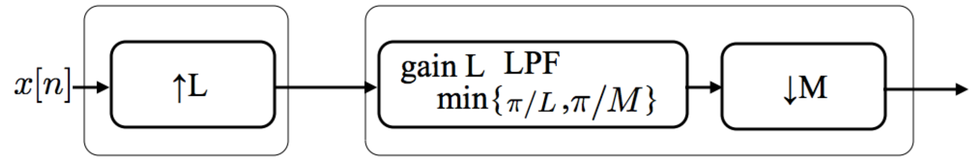
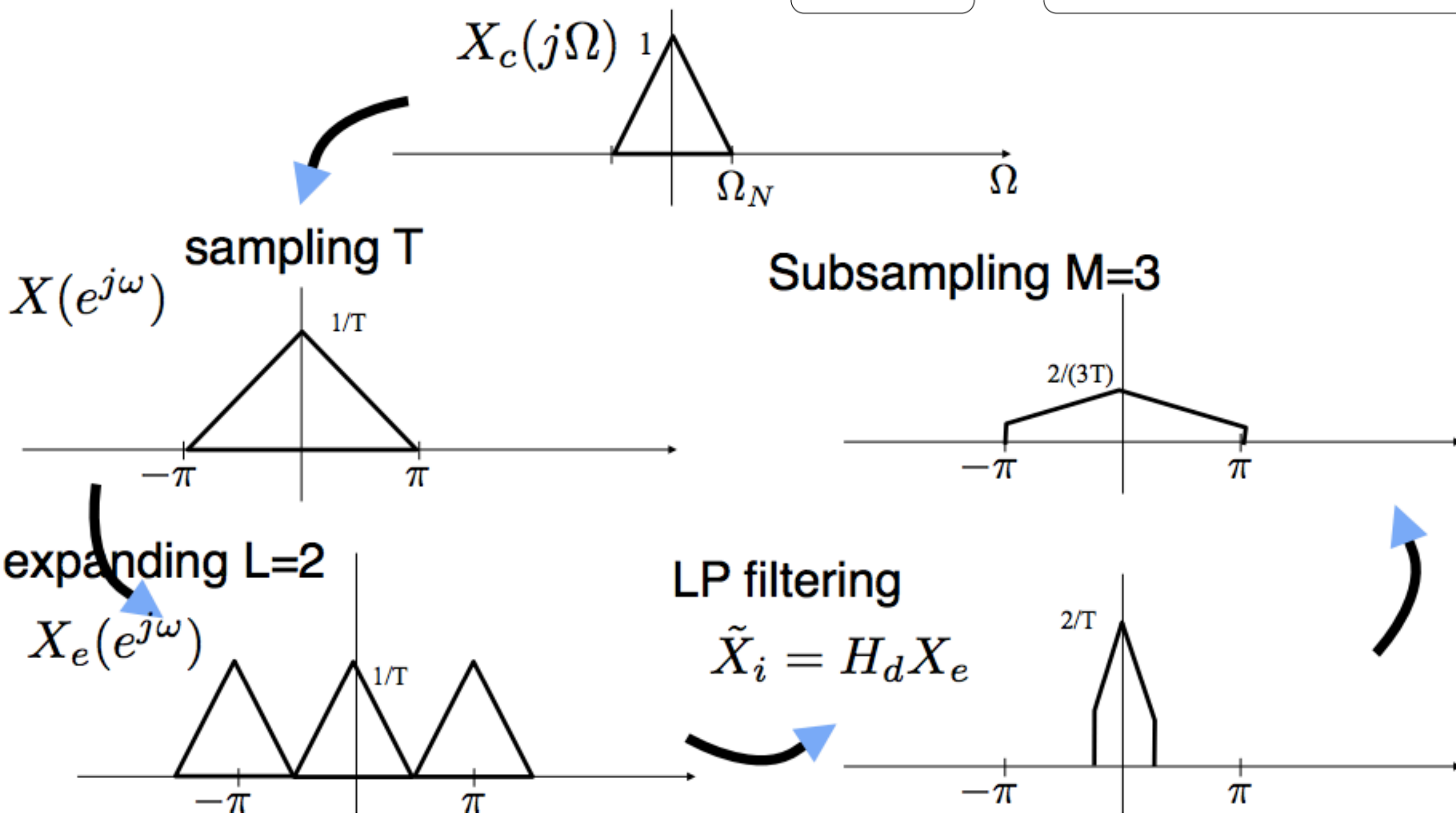
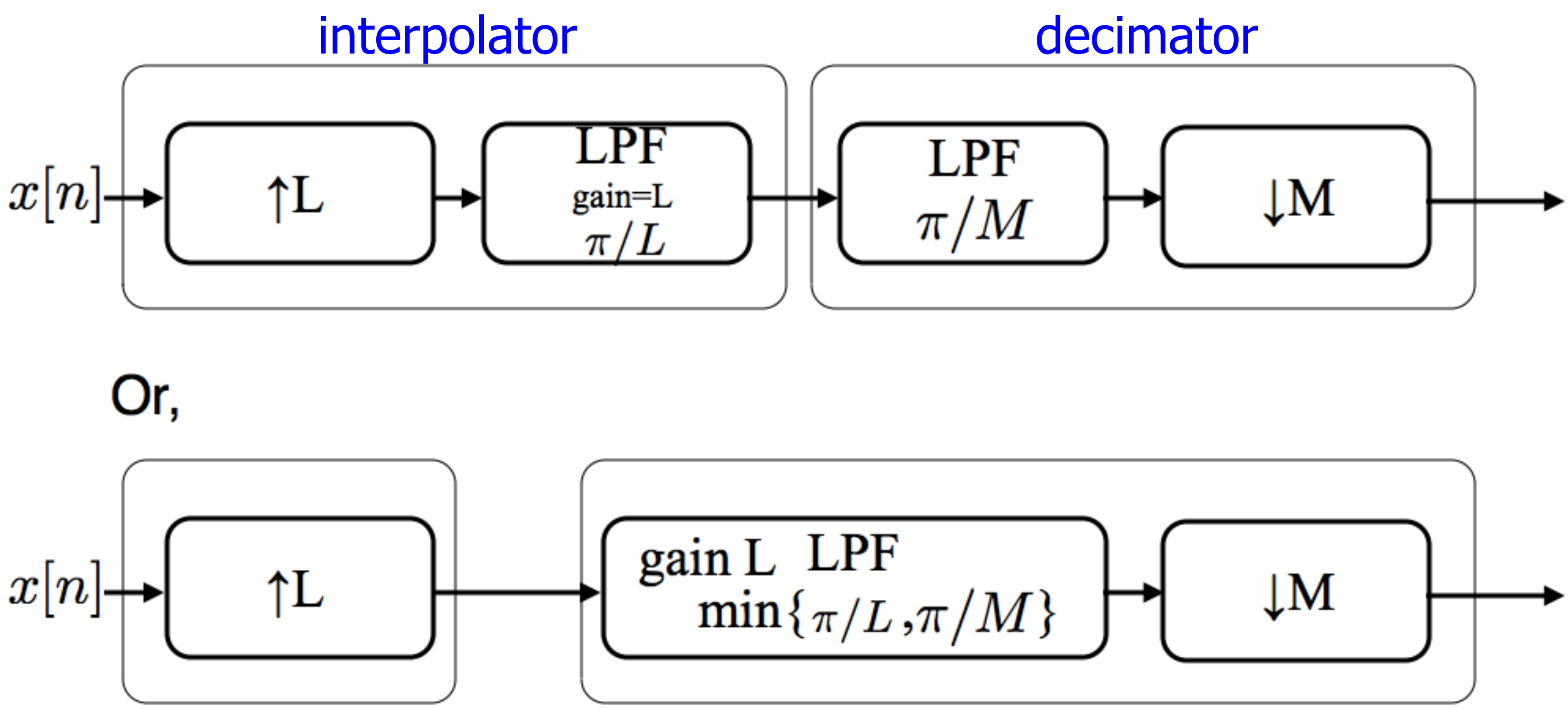




Non-integer Sampling

- T’=TM/L
  - Downsample by M, then upsample by L?

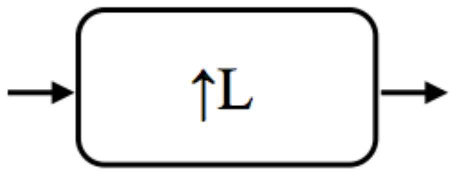


Multi-Rate Signal Processing

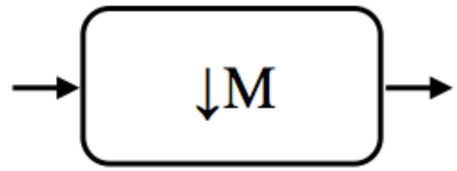
- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter $\pi/101$ ($$$$$)
  - Downsample by M=101

- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations



"expander"



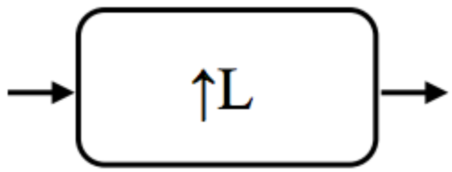
"compressor"

not LTI!

Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Expander



"expander"

Upsampling
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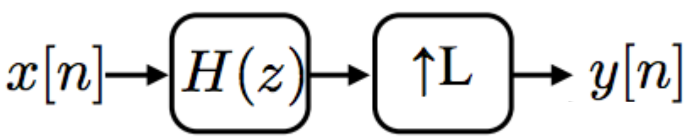
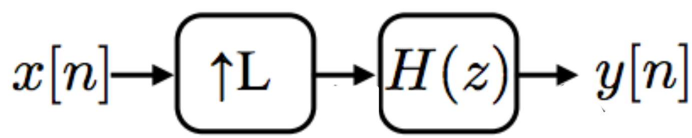
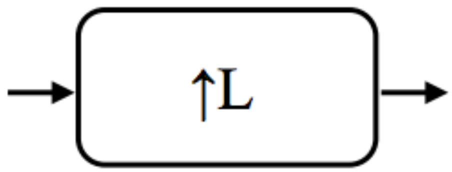


?



Interchanging Operations - Expander



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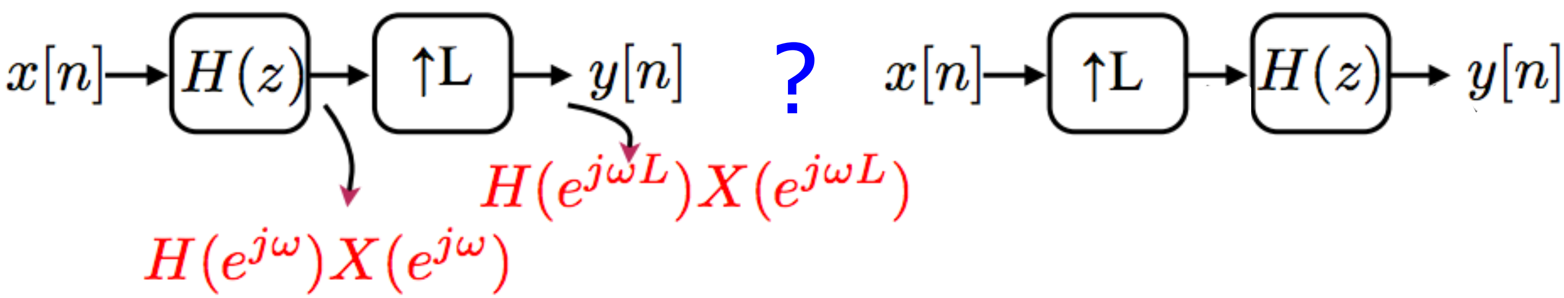
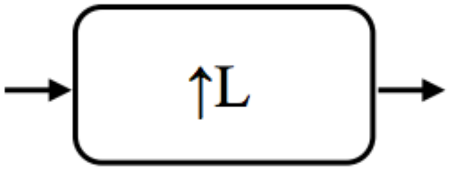


Interchanging Operations - Expander



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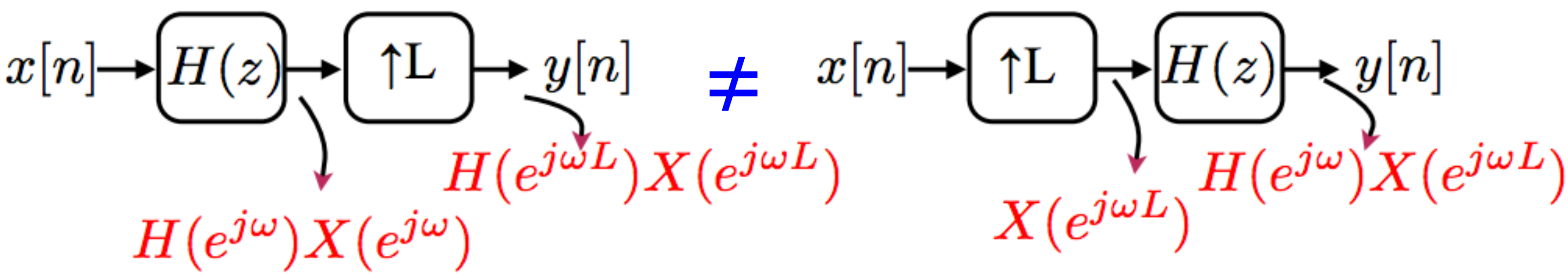

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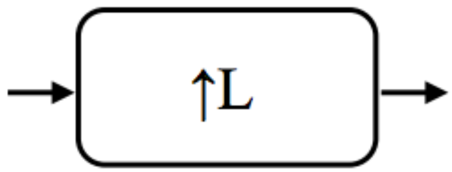

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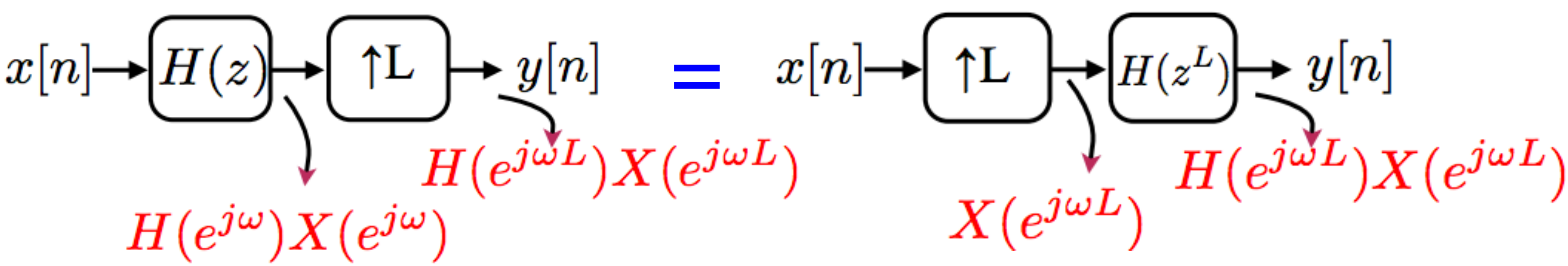
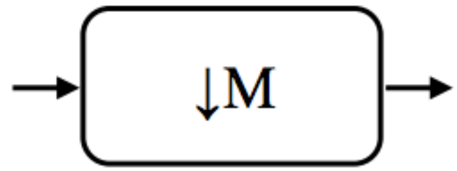


Interchanging Operations - Compressor



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"compressor"

Downsampling
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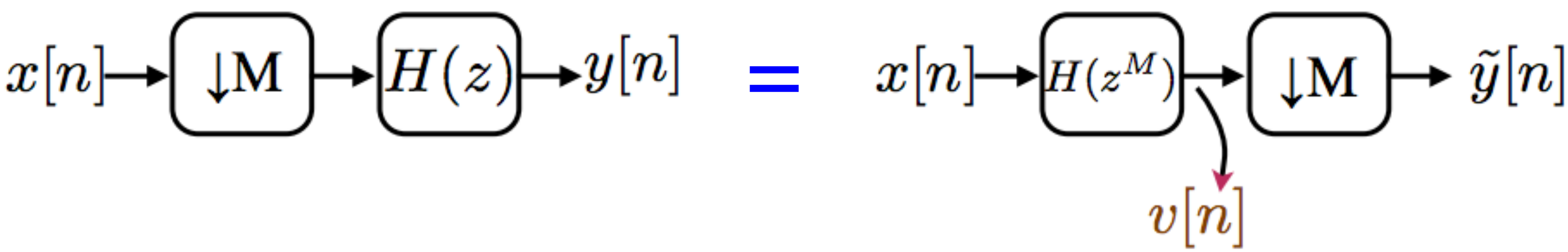
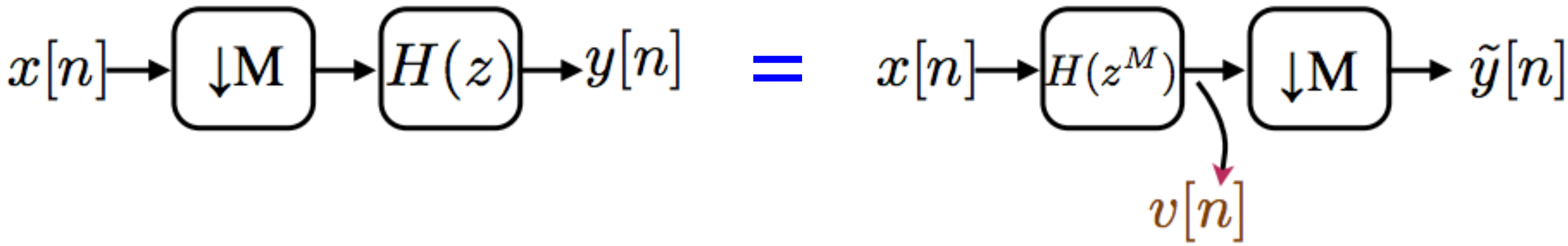
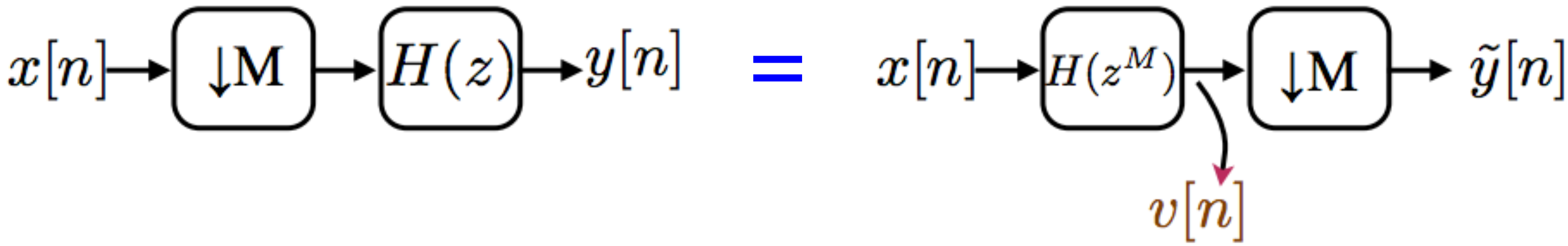


Interchanging Operations - Compressor



$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)\right)$$

Interchanging Operations - Compressor



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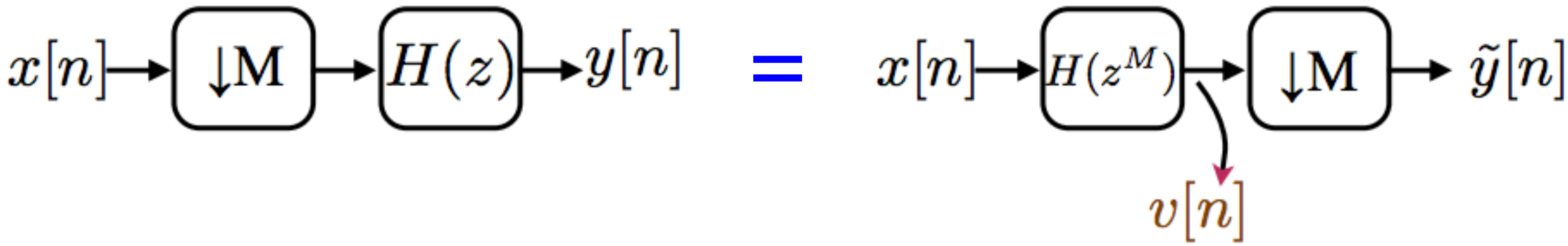
&= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H\left(e^{j(\omega-2\pi i)}\right)}_{H(e^{j\omega})} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)
\end{aligned}
$$

Interchanging Operations - Compressor



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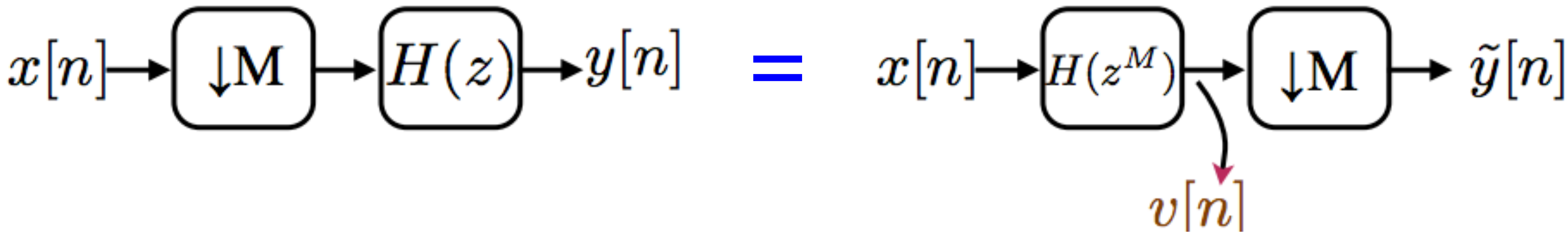
&= \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right) X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)
\end{aligned}
$$

Interchanging Operations - Compressor



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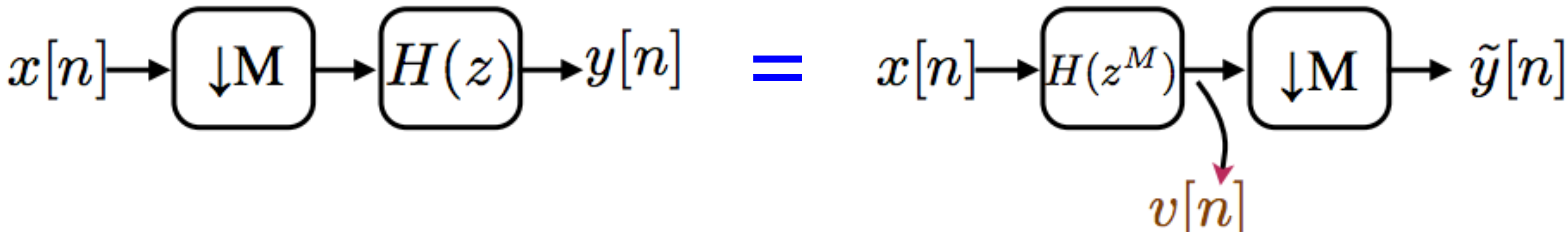

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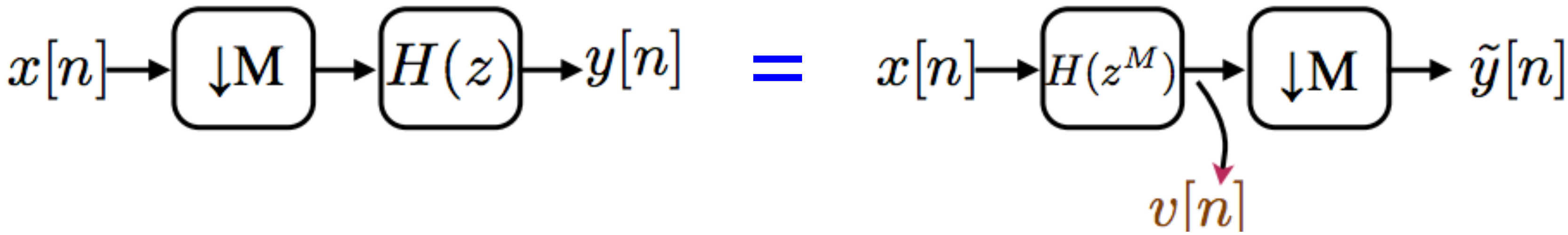
$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)$$

Interchanging Operations - Compressor



$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right) X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)$$

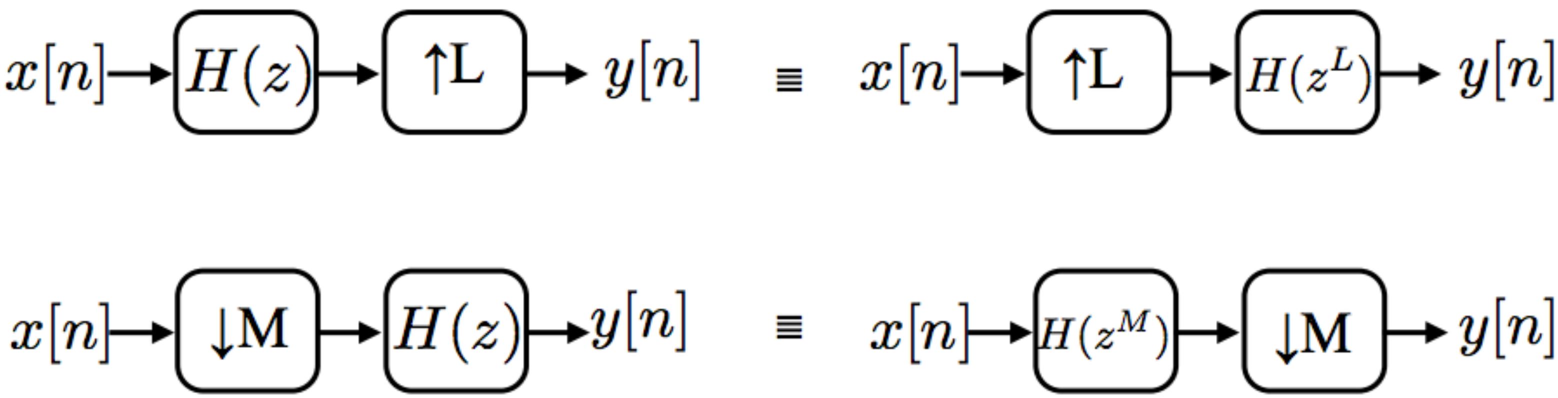
$$V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)$$

=

Interchanging Operations - Summary

Filter and expander | Expander and expanded filter*



Compressor and filter | Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter $\pi/101$ ($$$$$)
  - Downsample by M=101

- Fortunately there are ways around it!
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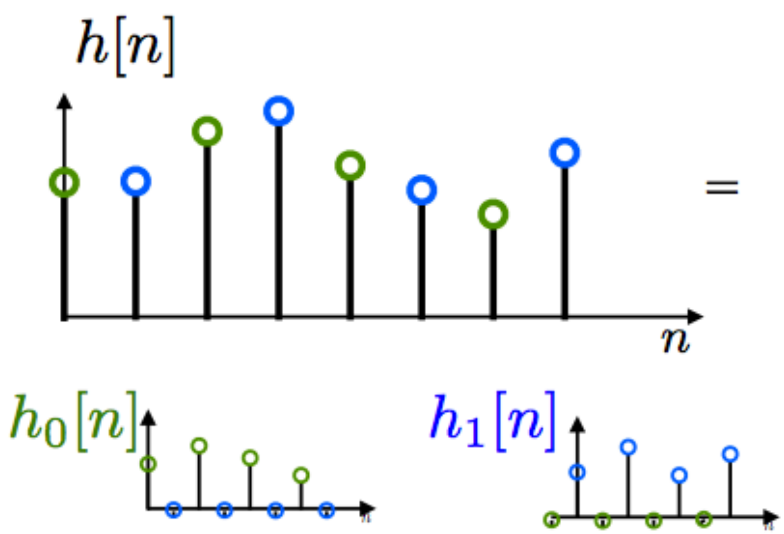
Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$



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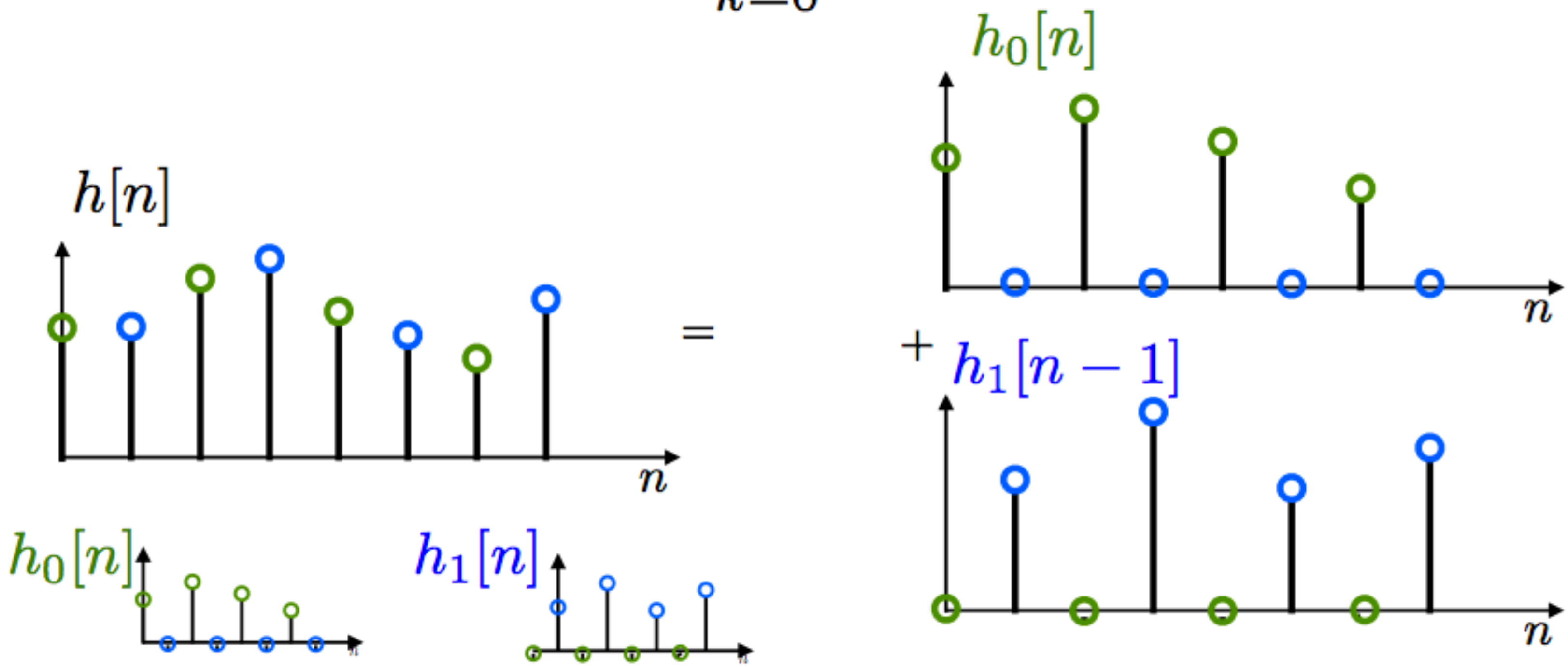
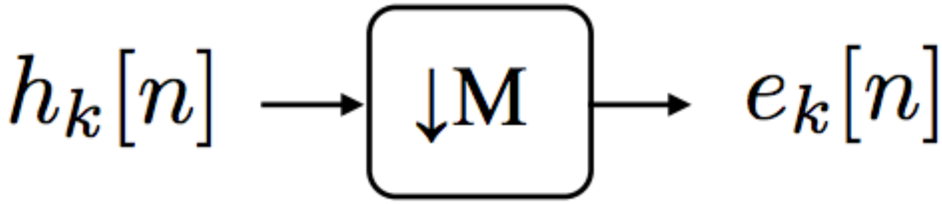
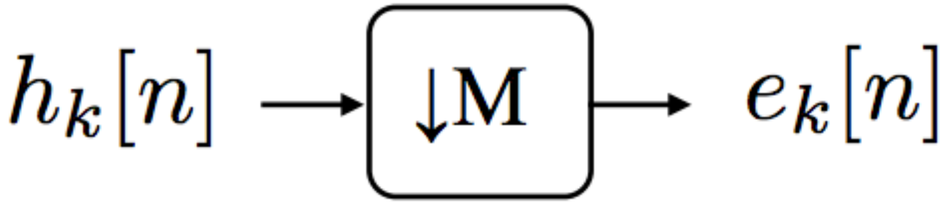


Polyphase Decomposition



$$e_k[n] = h_k[nM]$$

Polyphase Decomposition



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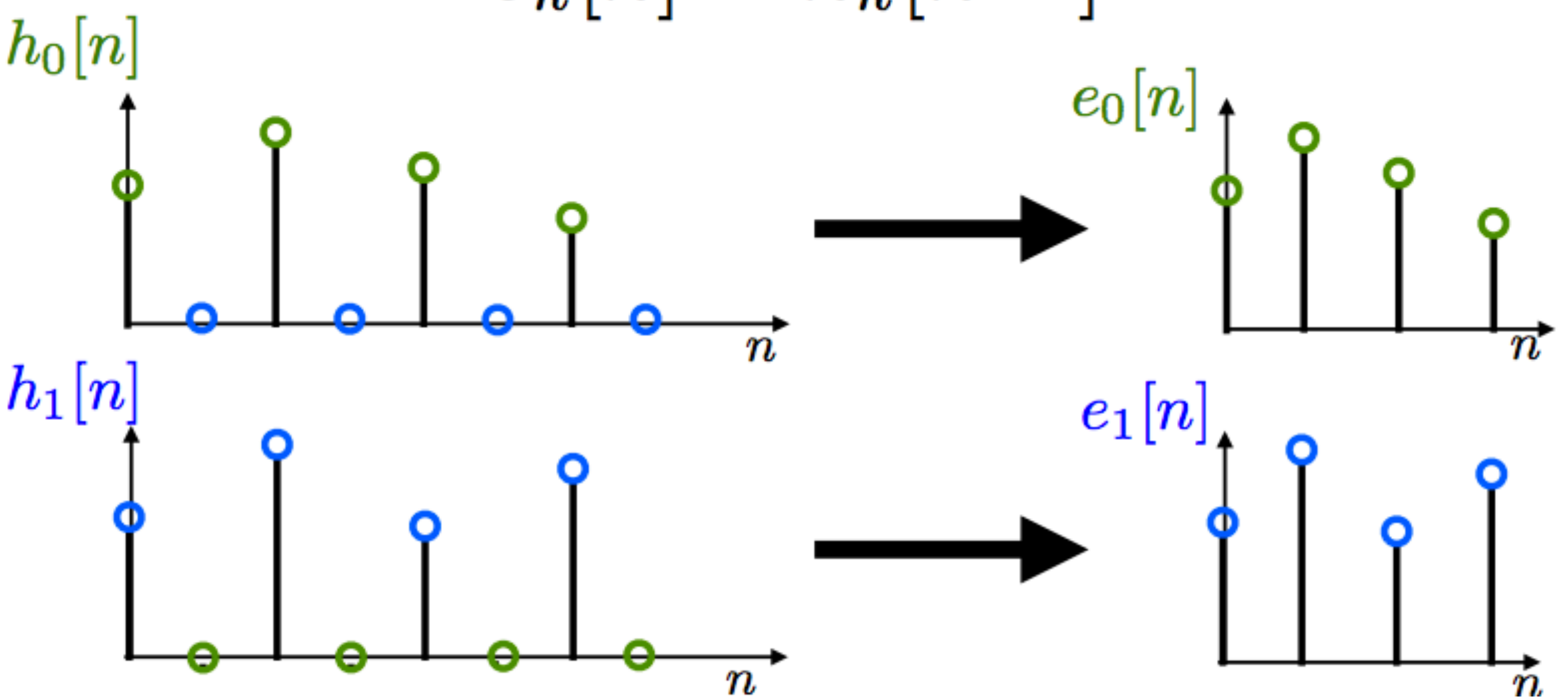
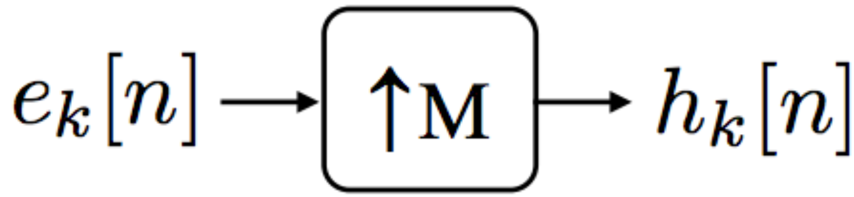


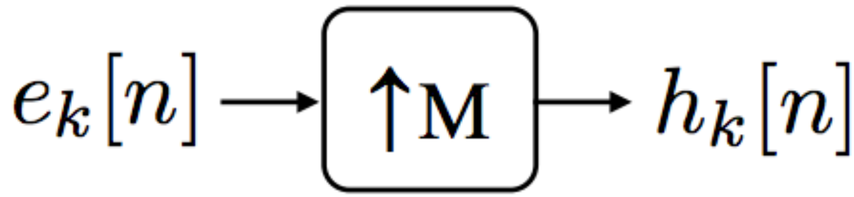
Polyphase Decomposition



recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Polyphase Decomposition



recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

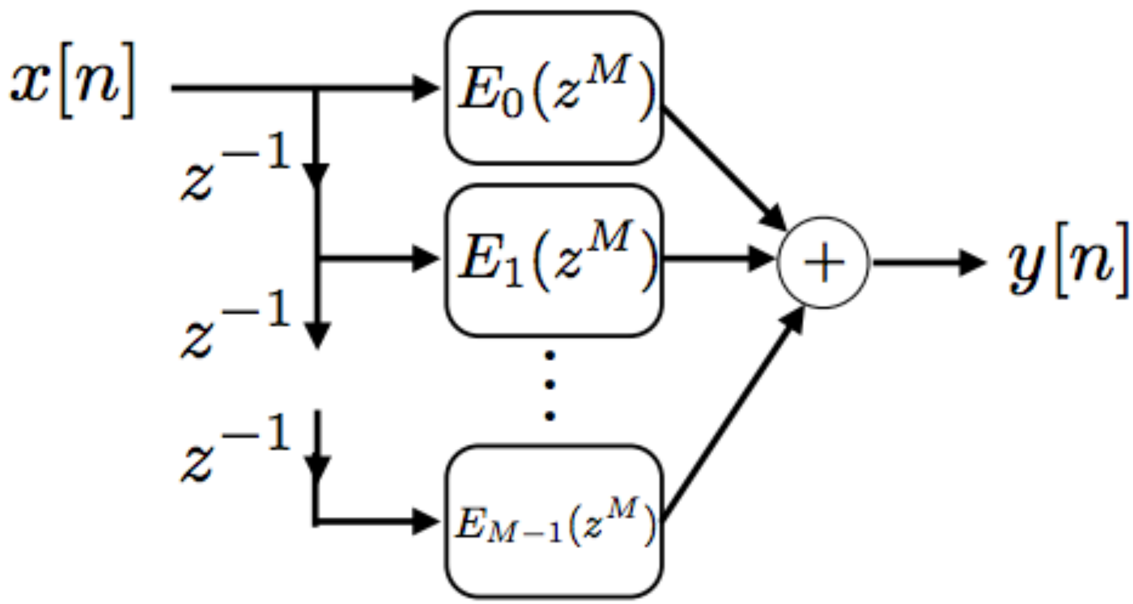
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

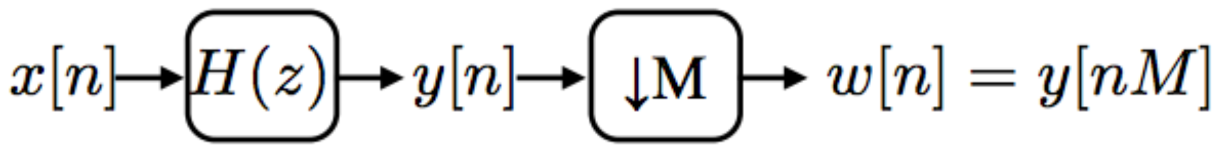
$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$

Polyphase Decomposition

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$

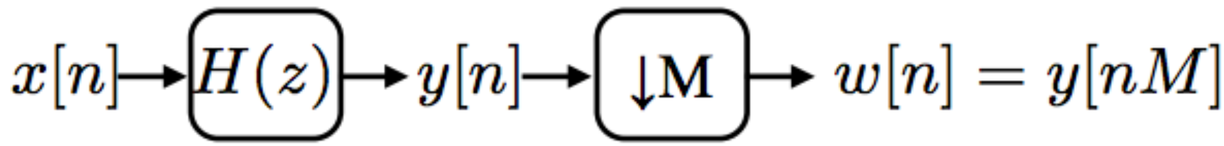


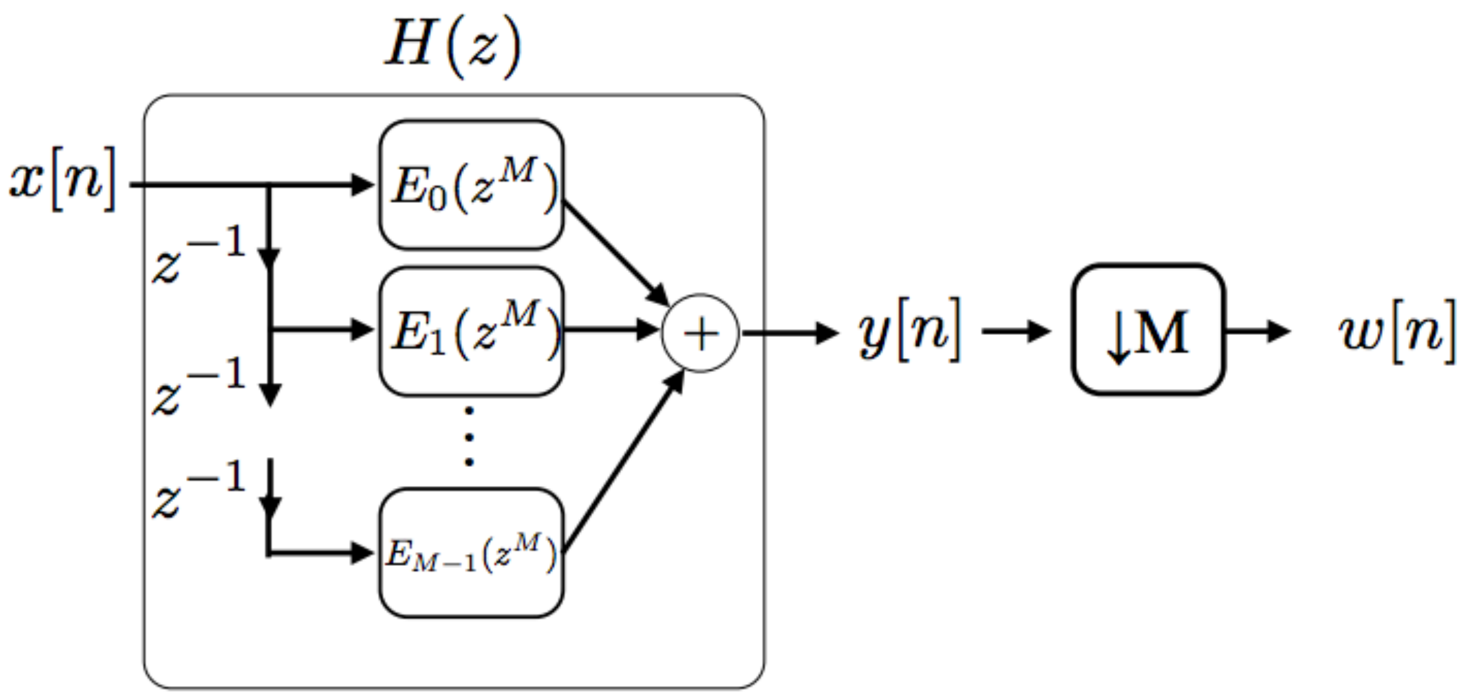
Polyphase Implementation of Decimation



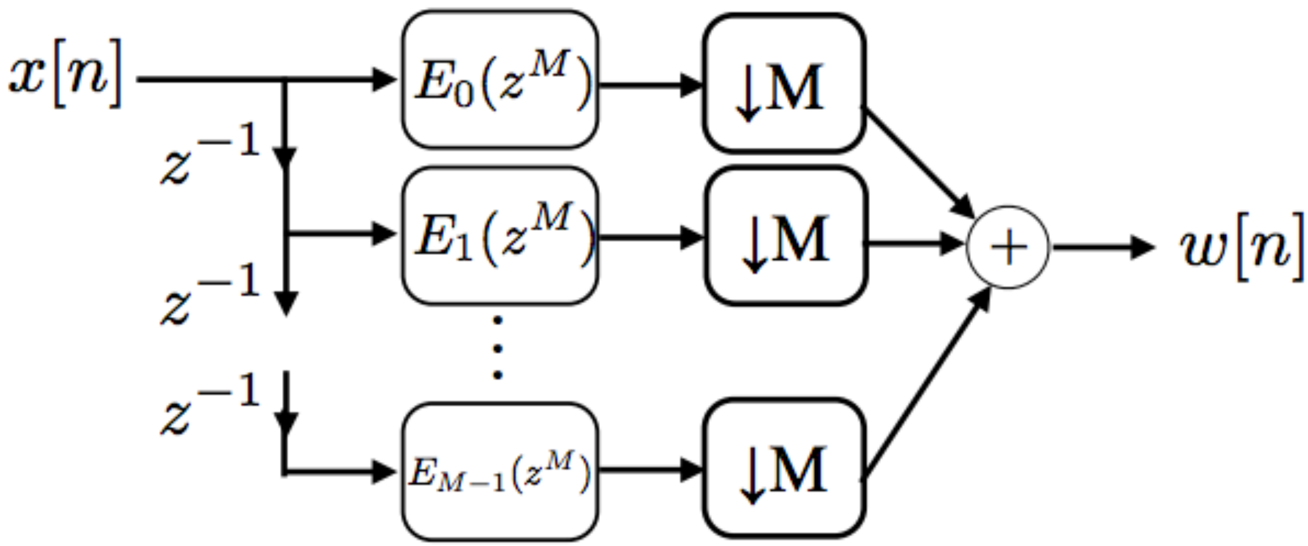
- Problem:
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N mults/unit time

Polyphase Implementation of Decimation



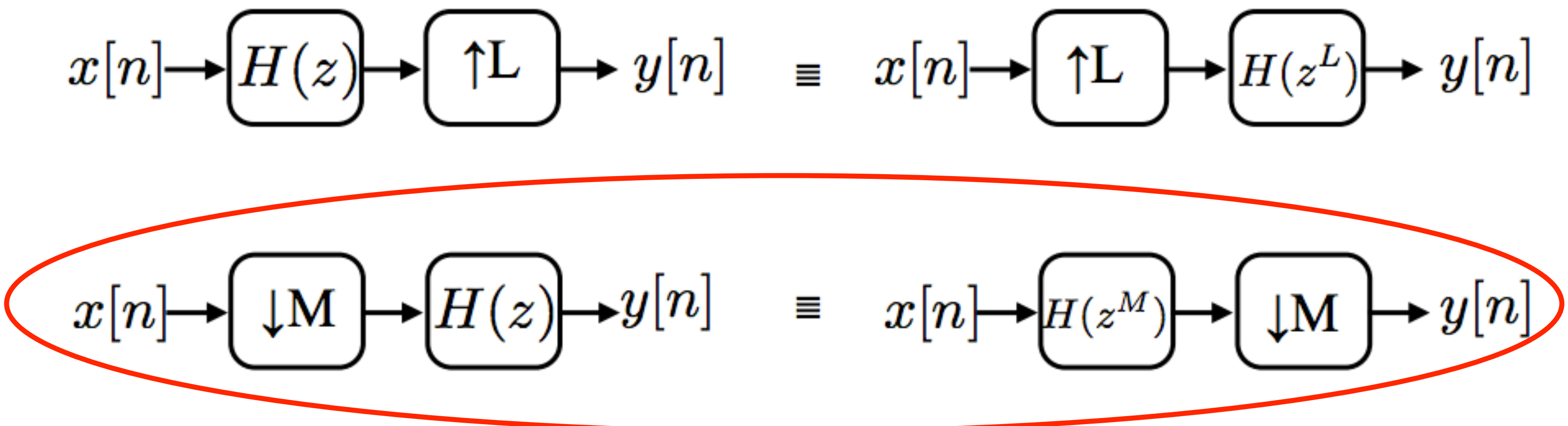


Polyphase Implementation of Decimation



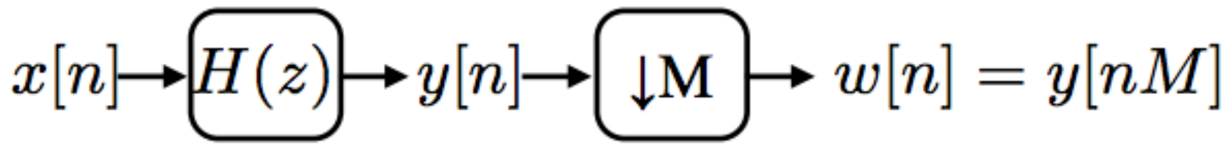
Interchanging Operations - Summary

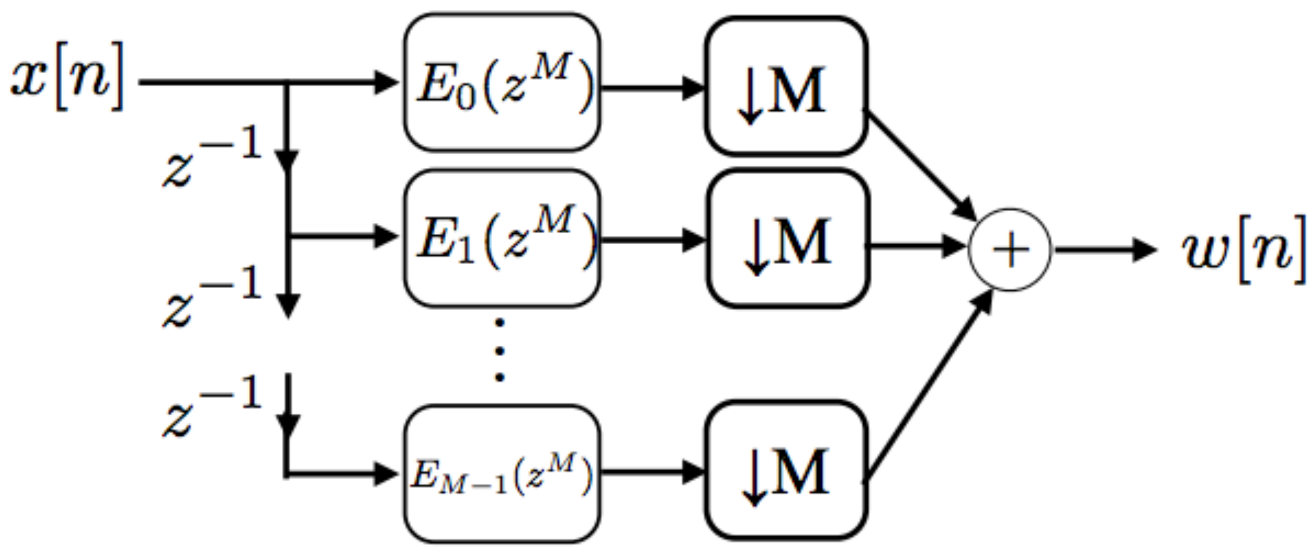
Filter and expander | Expander and expanded filter



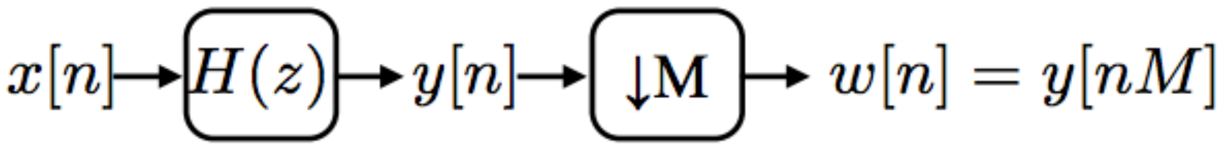
Compressor and filter | Expanded filter and compressor

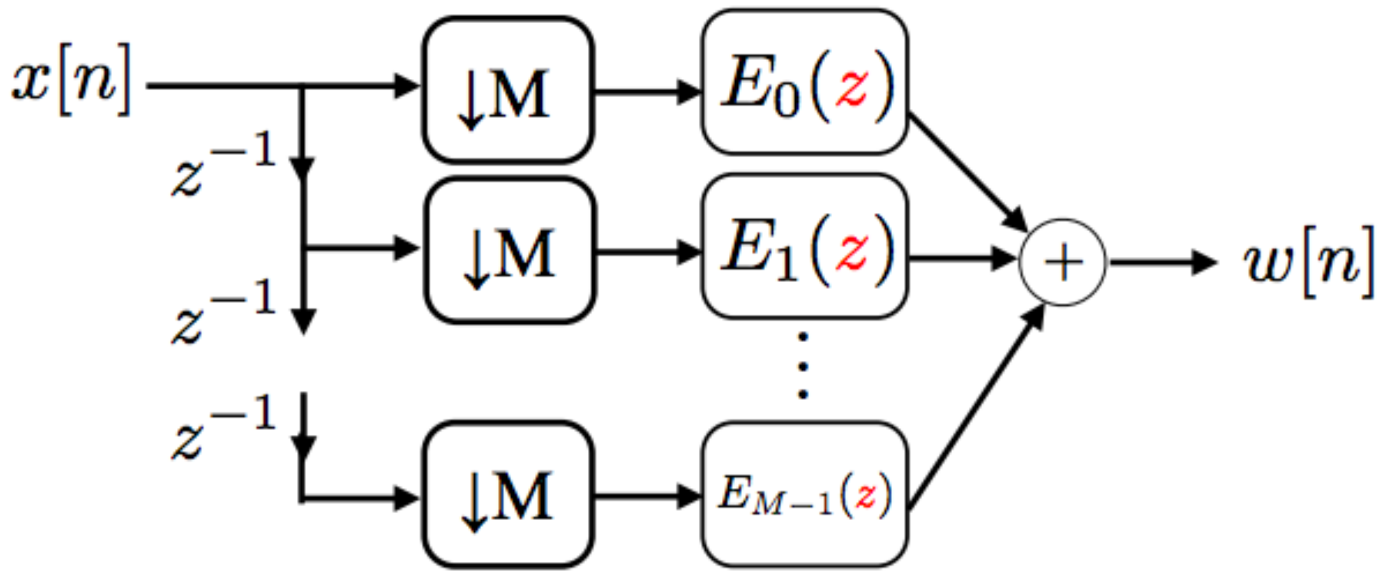
Polyphase Implementation of Decimation



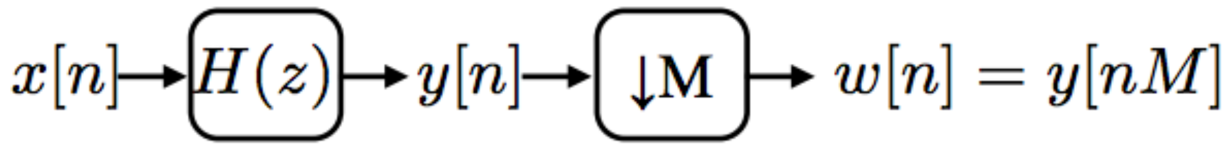


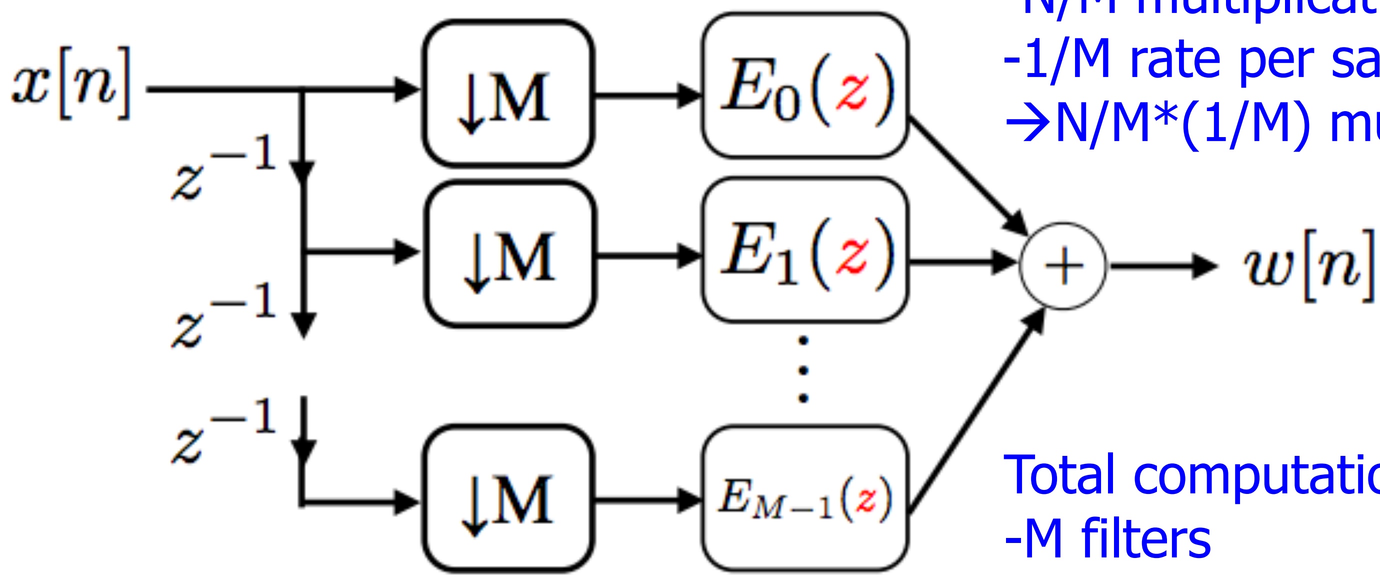
Polyphase Implementation of Decimation





Polyphase Implementation of Decimation





Each filter computation:
- N/M multiplications
- 1/M rate per sample

→N/M*(1/M) mults/unit time

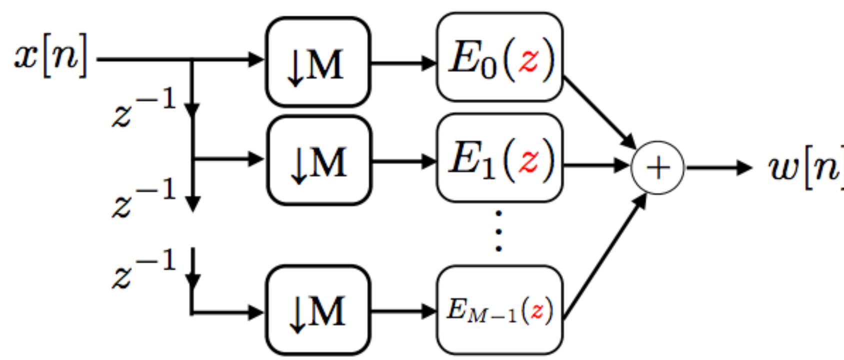
Total computation:
- M filters

→N/M mults/unit time

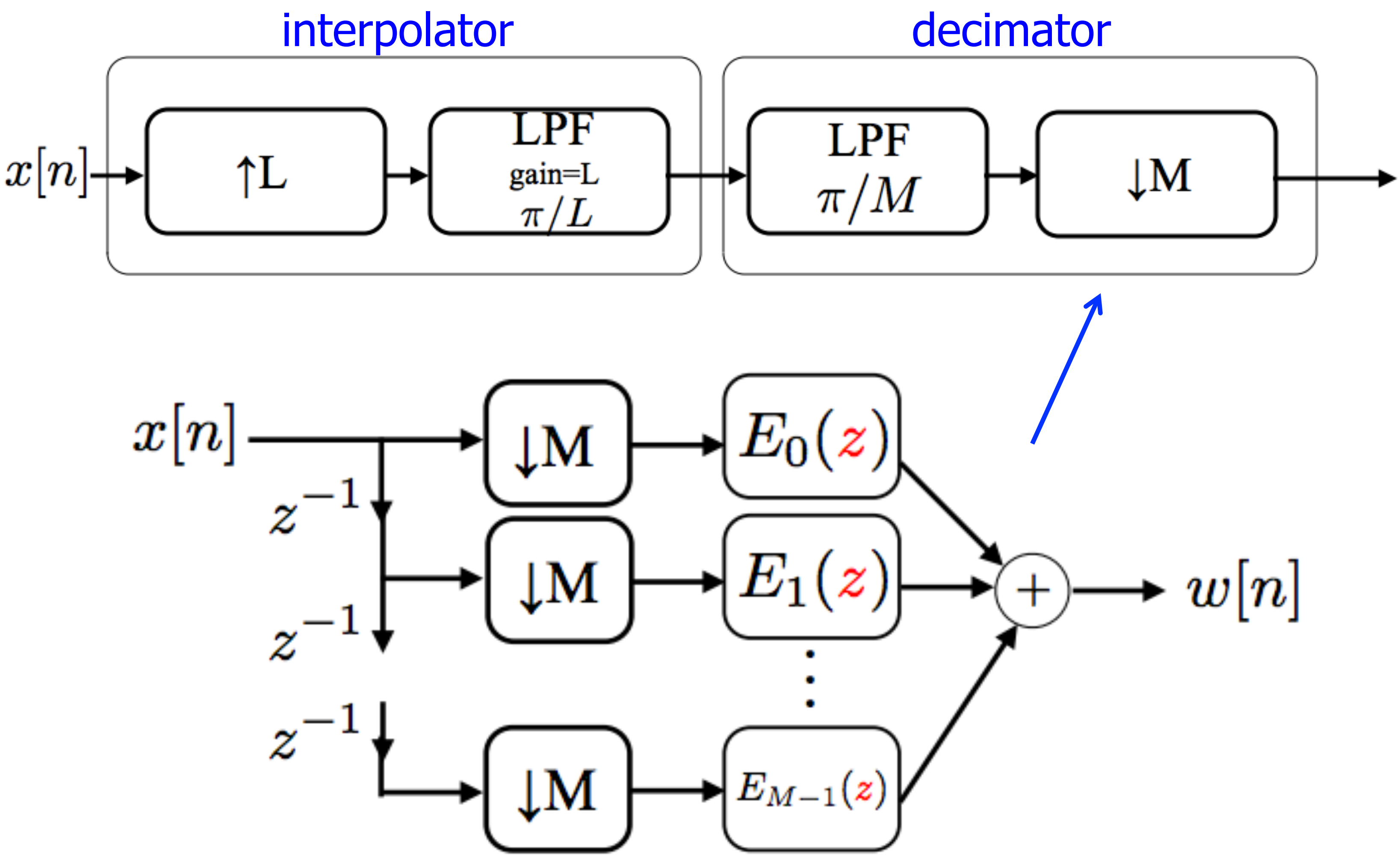
Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter $\pi/101$ ($$$$$)
  - Downsample by M=101

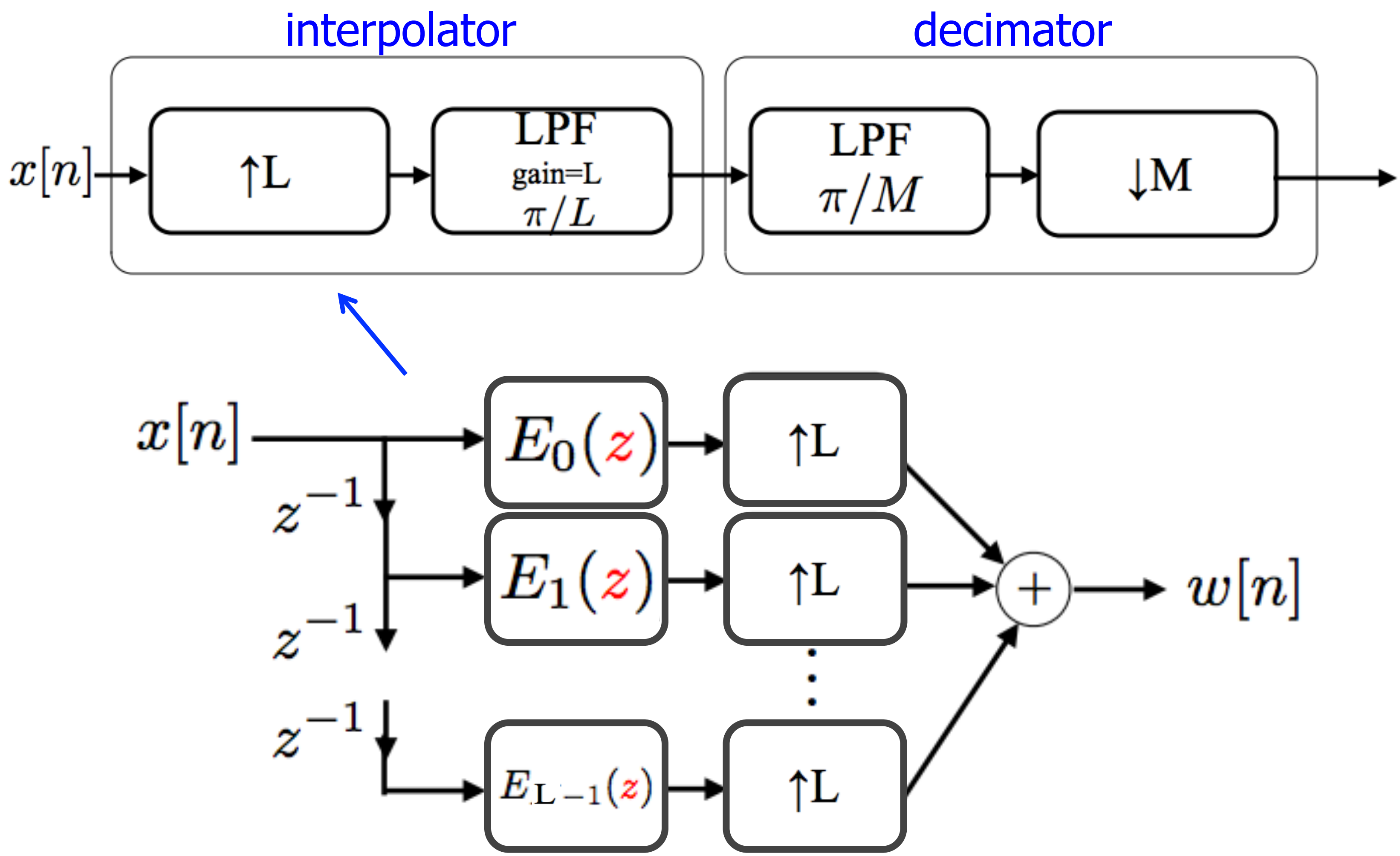
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering



Polyphase Implementation of Decimator



Polyphase Implementation of Interpolation



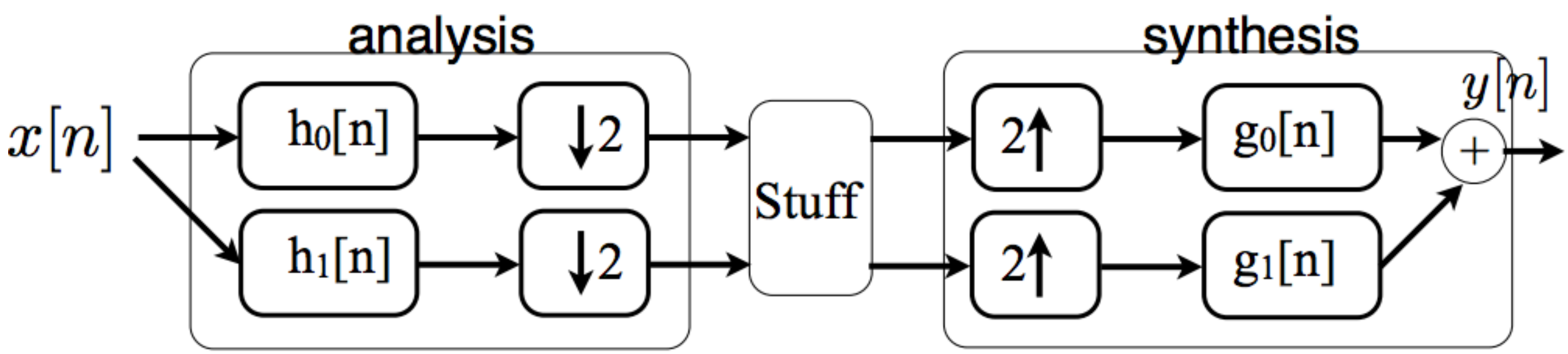
Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

Multi-Rate Filter Banks

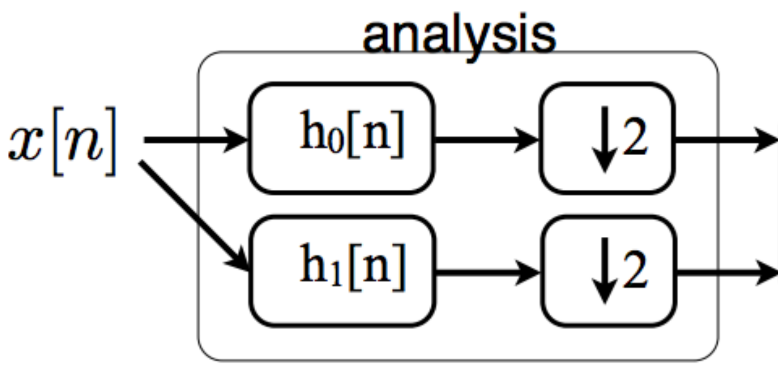
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

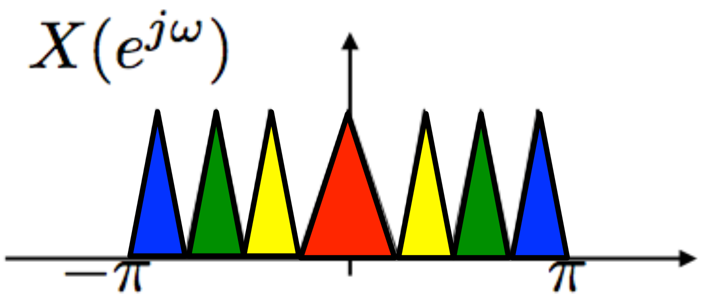
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$





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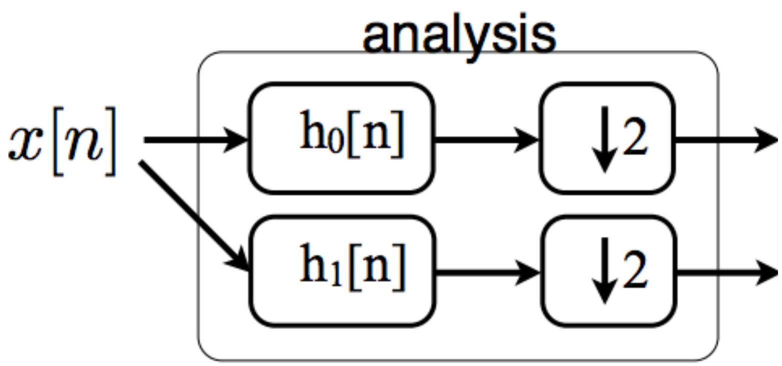

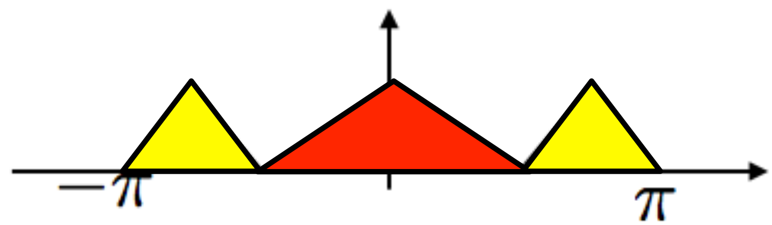

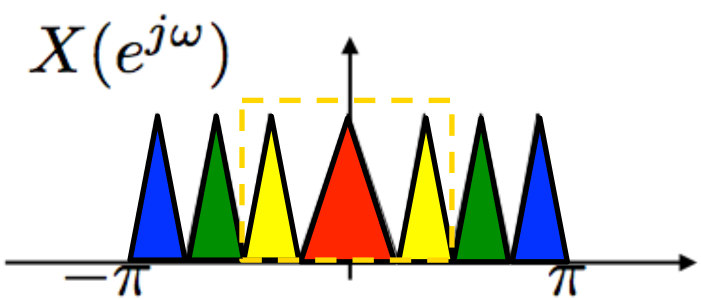

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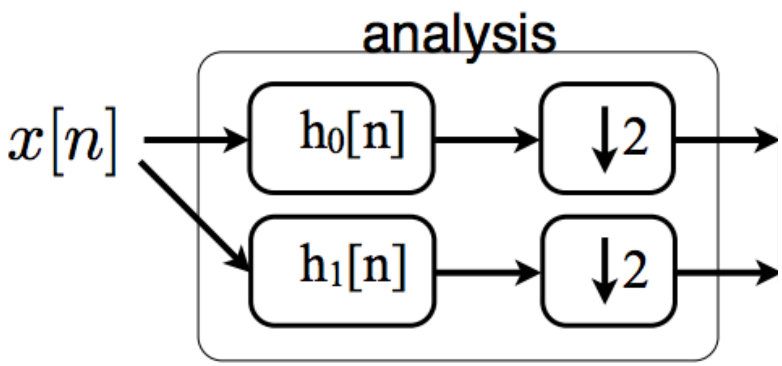

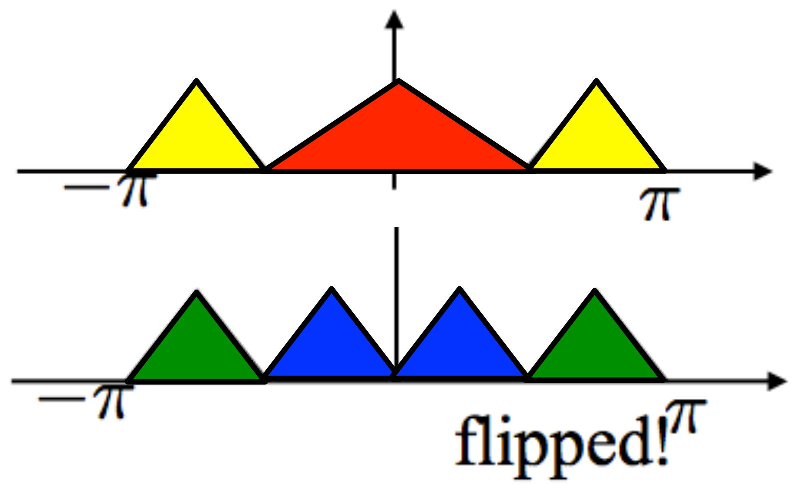

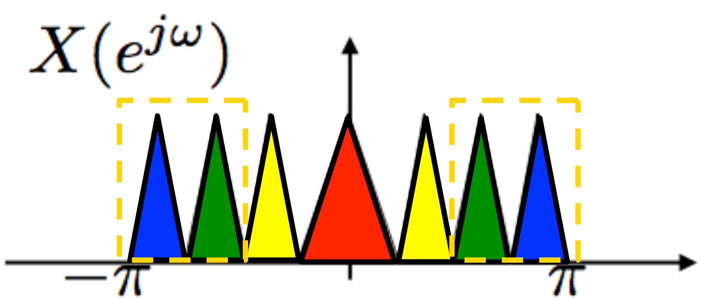

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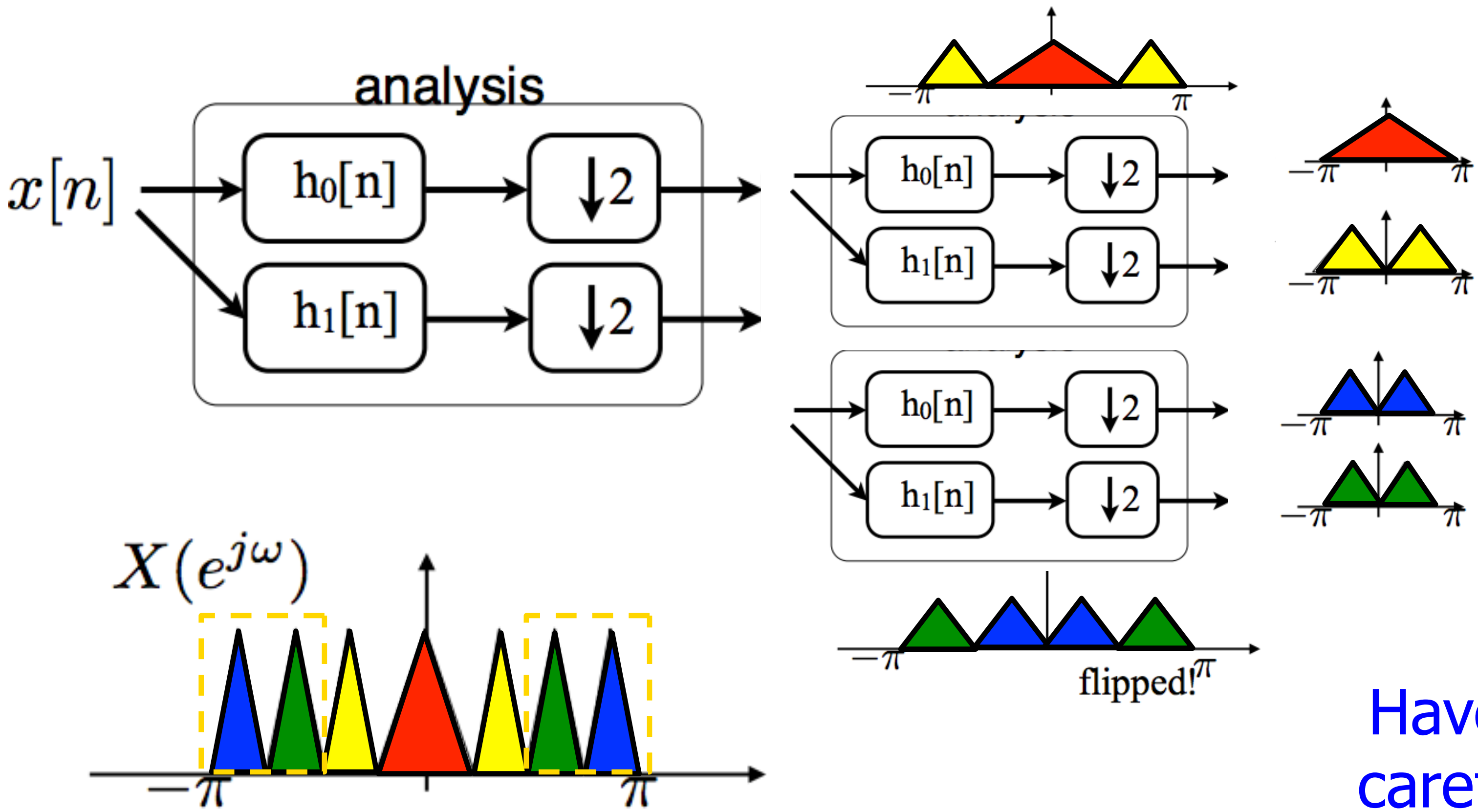
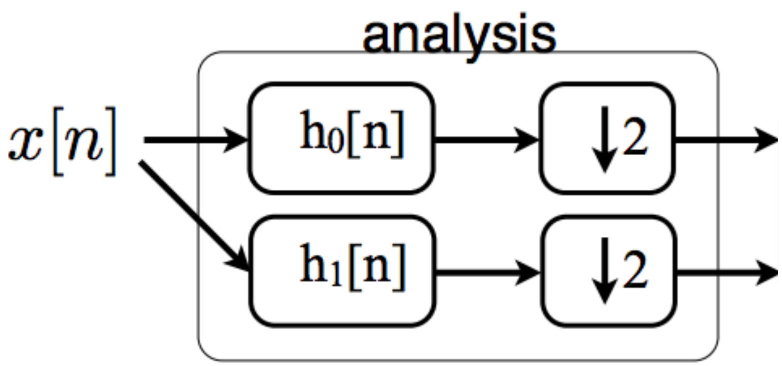
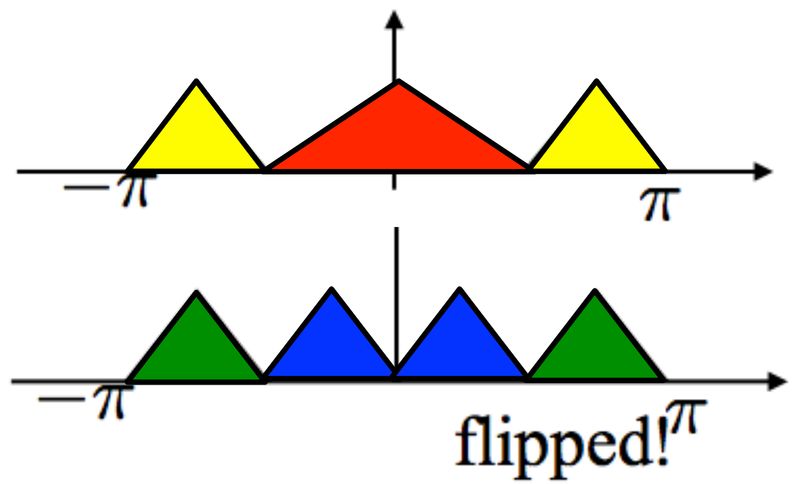


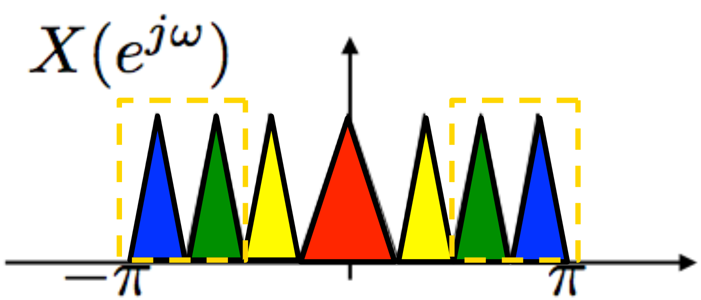
Have to be careful with order!

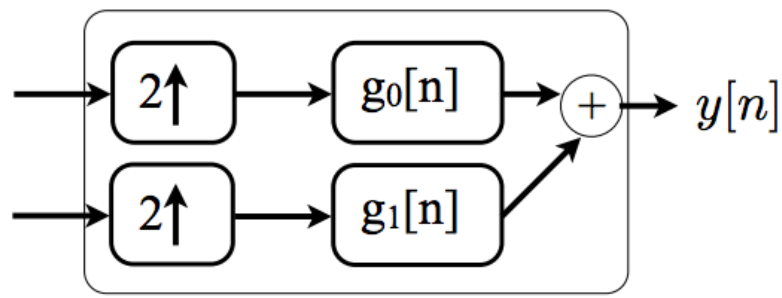
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$









Multi-Rate Filter Banks

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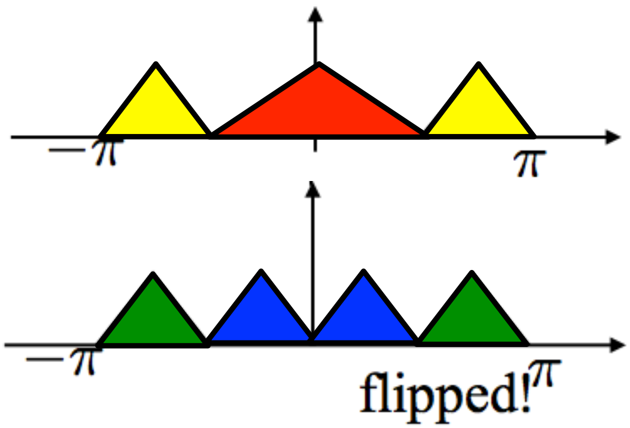
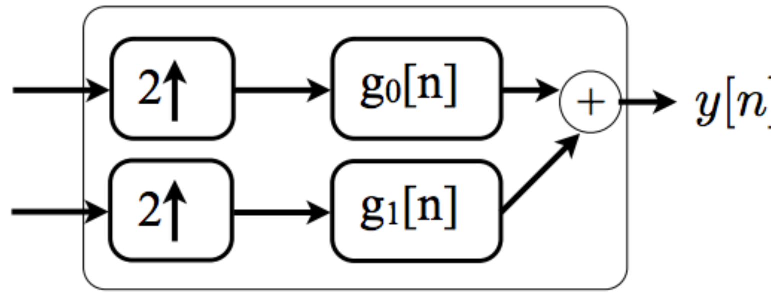
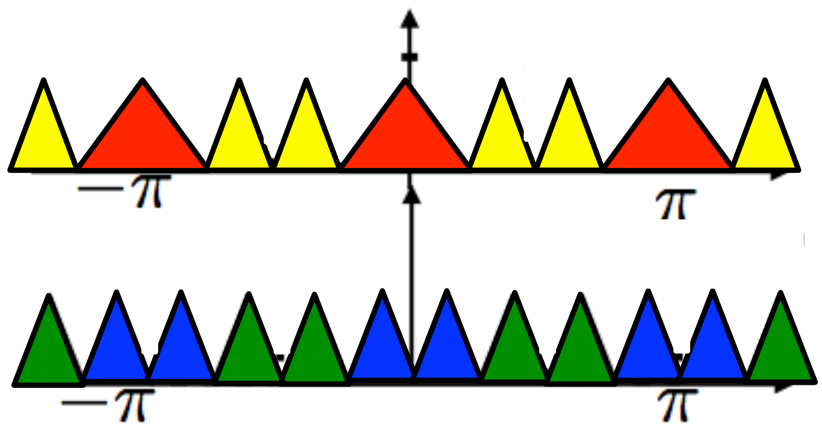
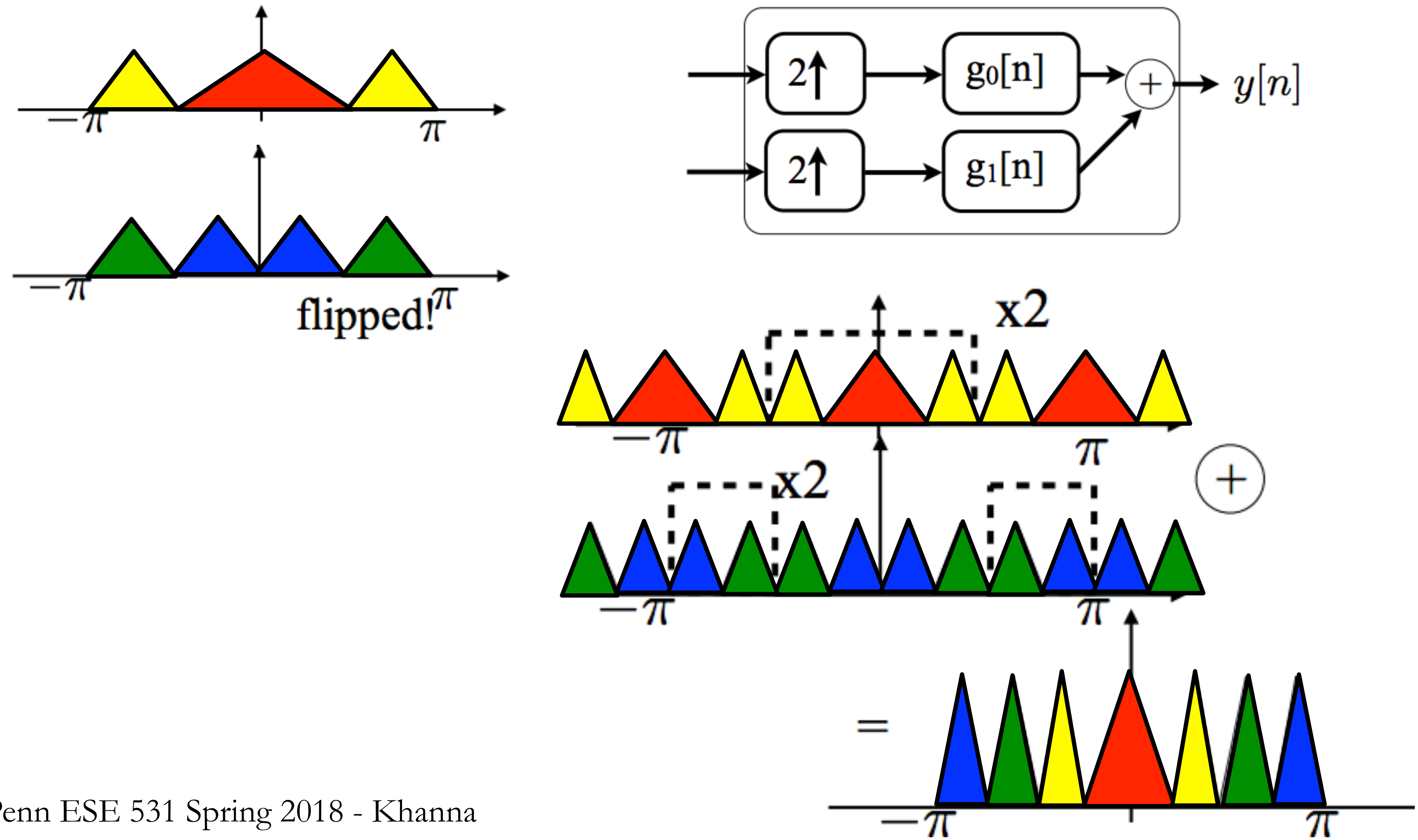






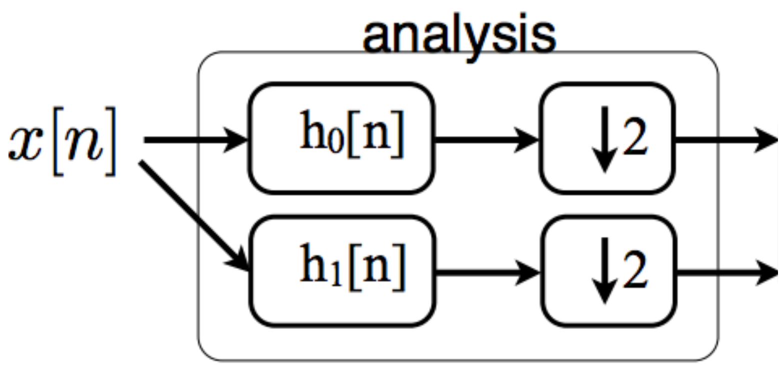
Multi-Rate Filter Banks

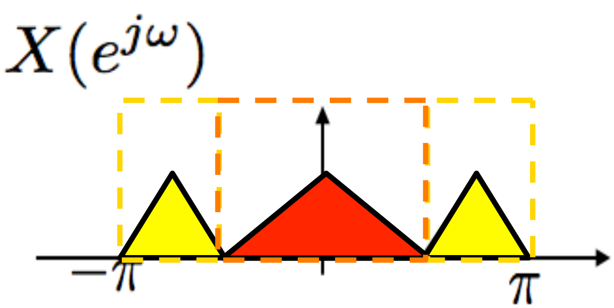
- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$

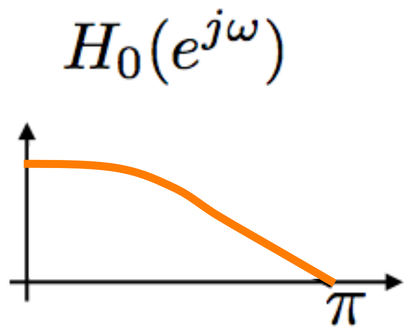


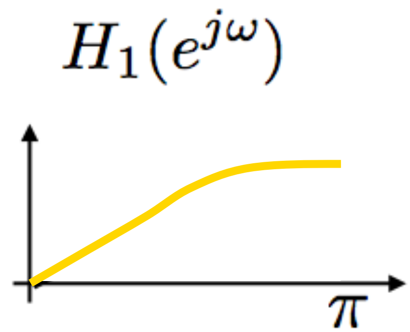
Multi-Rate Filter Banks

- h_0, h_1 are NOT ideal low/high pass



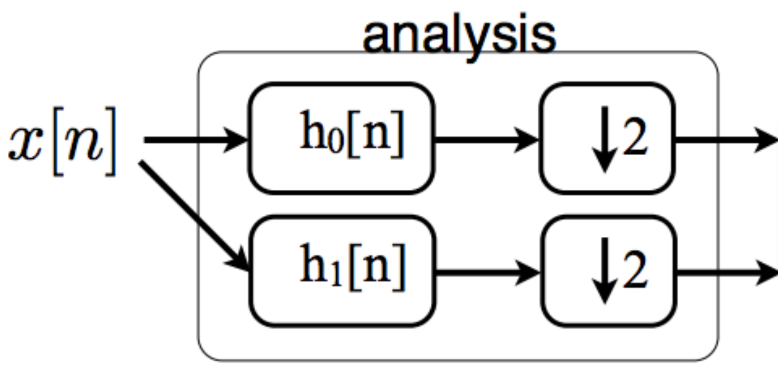


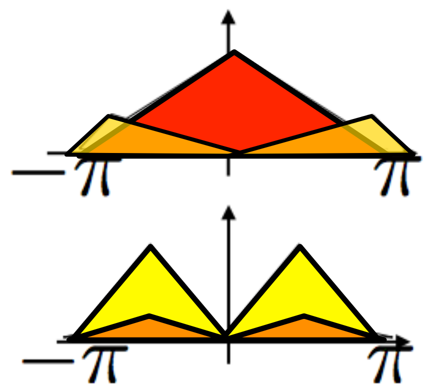


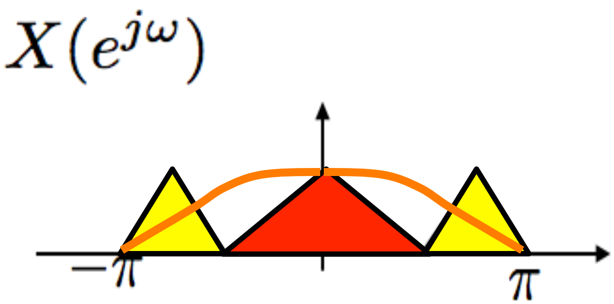


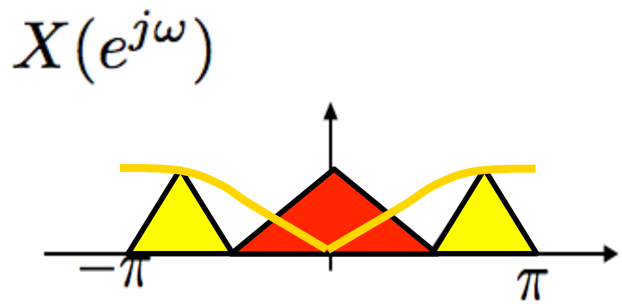
Non Ideal Filters

- h_0, h_1 are NOT ideal low/high pass

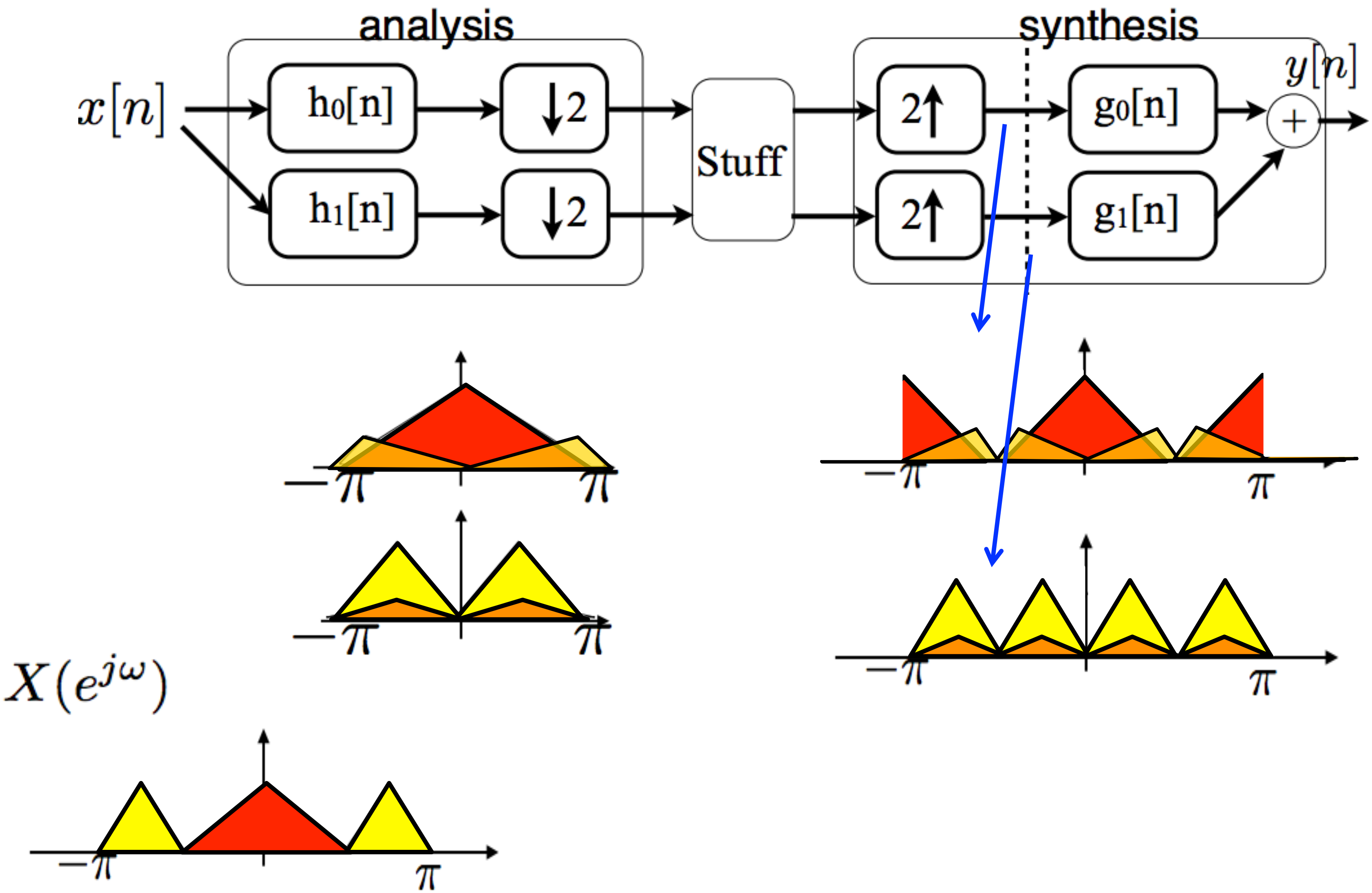




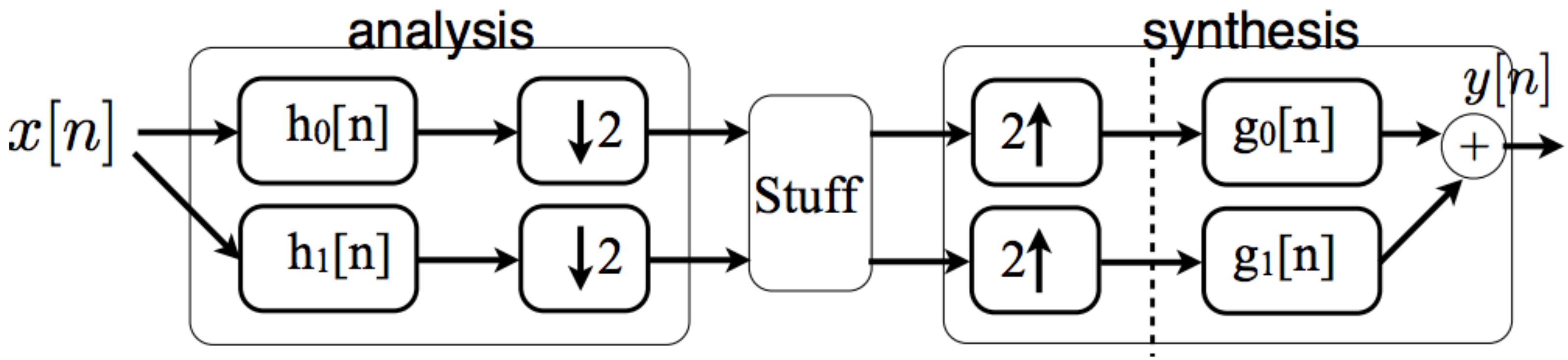




Non Ideal Filters



Perfect Reconstruction non-Ideal Filters



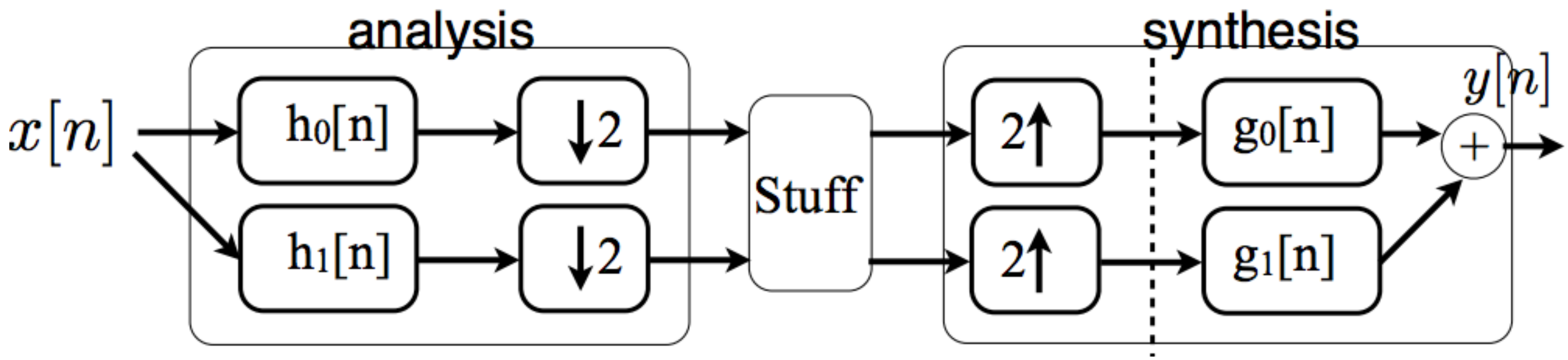
$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega})$$

$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Perfect Reconstruction non-Ideal Filters

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega}) \\ &\quad + \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)}) \end{aligned}$$

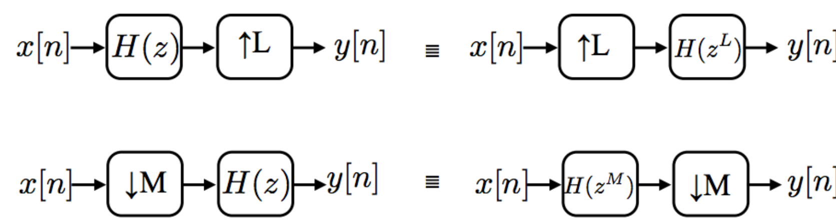
need to cancel!

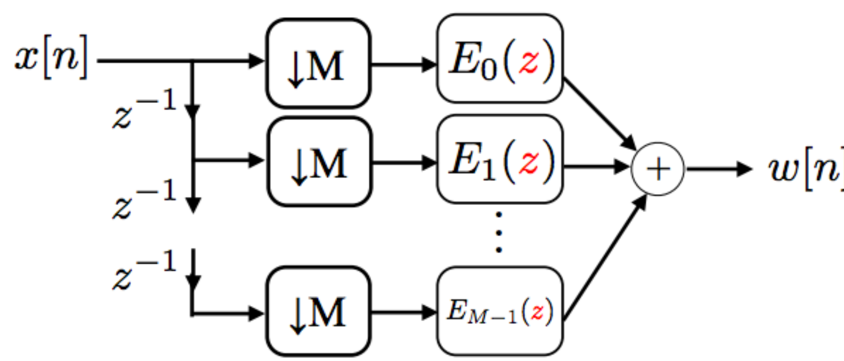
aliasing

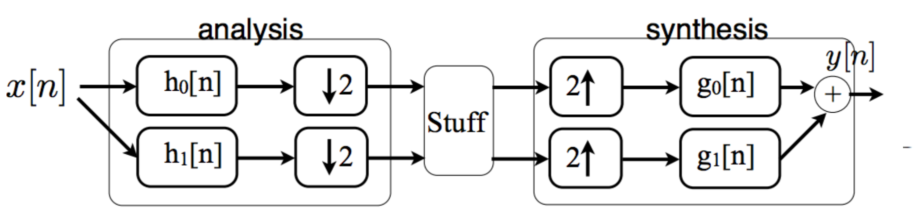
$$\begin{aligned} H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\ G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\ G_1(e^{j\omega}) &= -2H_1(e^{j\omega}) \end{aligned}$$

Big Ideas

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks







Admin

- HW 4 due Sunday