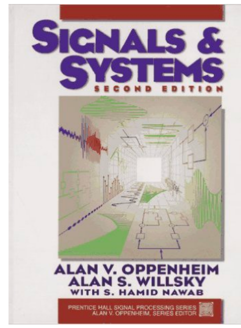
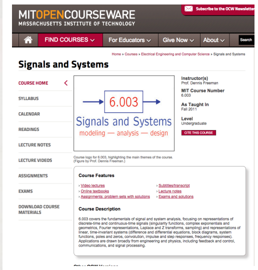
ESE 531: Digital Signal Processing

Lec 10: February 15th, 2018

Practical and Non-integer Sampling, Multi-rate Sampling

Signals and Systems Review



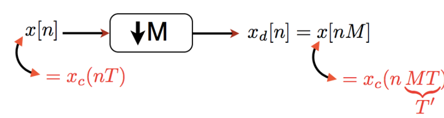


Lecture Outline

- Review: Downsampling/Upsampling
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks

Downsampling

- Definition: Reducing the sampling rate by an integer number

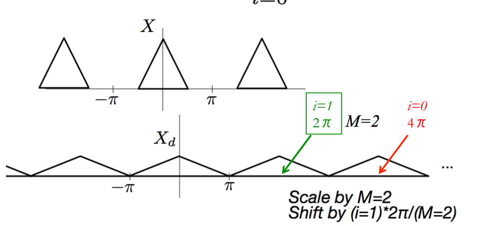


$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M — replicate

Example

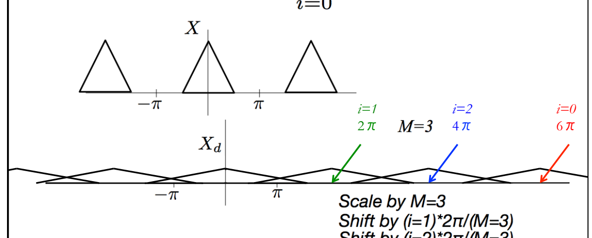
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{w}{M} - \frac{2\pi}{M}i)}\right)$$



Scale by M=2
Shift by (i=1)*2π/(M=2)

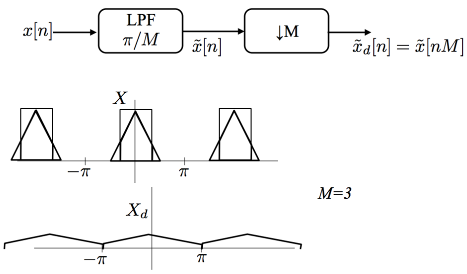
Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{w}{M} - \frac{2\pi}{M}i)}\right)$$



Scale by M=3
Shift by (i=1)*2π/(M=3)
Shift by (i=2)*2π/(M=3)

Example



Upsampling

- Definition: Increasing the sampling rate by an integer number

$x[n] = x_c(nT)$

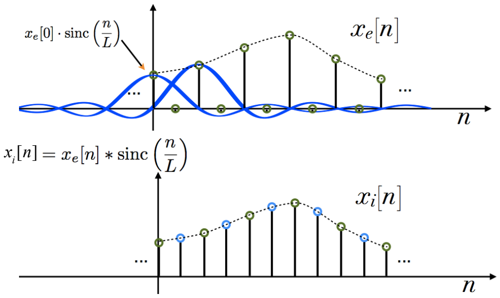
$x_i[n] = x_c(nT')$ where $T' = \frac{T}{L}$ L integer

Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

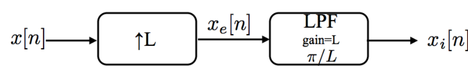
Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

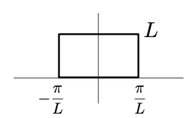


Frequency Domain Interpretation

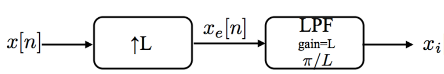
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ DTFT ⇒



Frequency Domain Interpretation

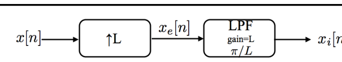


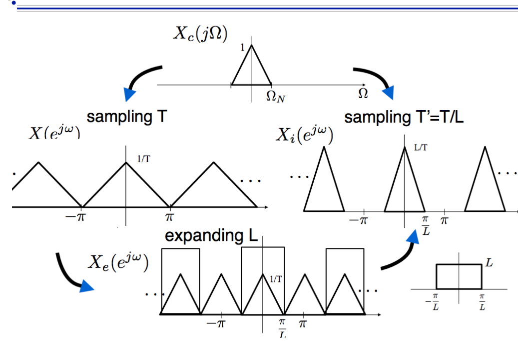
$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = \boxed{X(e^{j\omega L})}$$

Compress DTFT by a factor of L!

Example





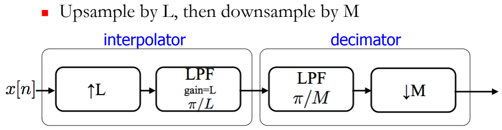
Non-integer Resampling

Non-integer Resampling

- T'=TM/L

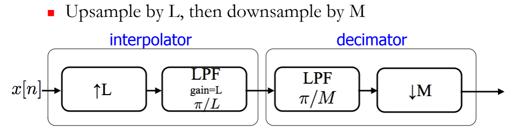
Non-integer Resampling

- T'=TM/L
  - Upsample by L, then downsample by M

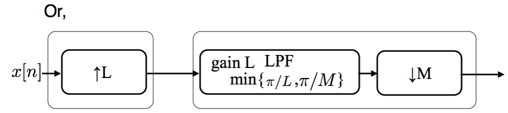


Non-integer Resampling

- T'=TM/L
  - Upsample by L, then downsample by M

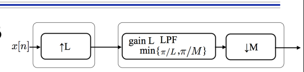


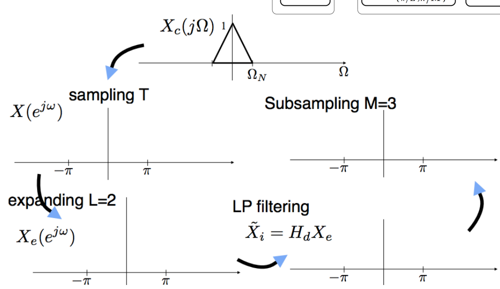
Or,



Example

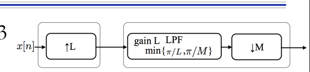
- T'=3/2T → L=2, M=3

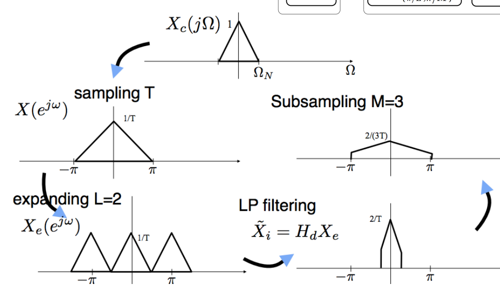




Example

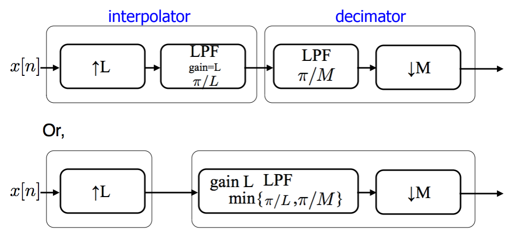
- T'=3/2T → L=2, M=3





Non-integer Sampling

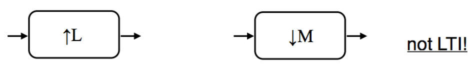
- T'=TM/L
  - Downsample by M, then upsample by L?



Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Interchanging Operations



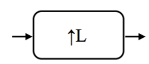
"expander"

Upsampling
-expanding in time
-compressing in frequency

"compressor"

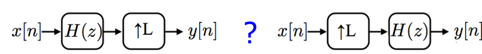
Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Expander

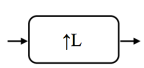


"expander"

Upsampling
-expanding in time
-compressing in frequency

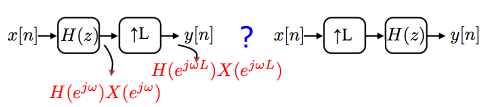


Interchanging Operations - Expander

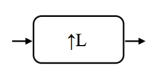


"expander"

Upsampling
-expanding in time
-compressing in frequency

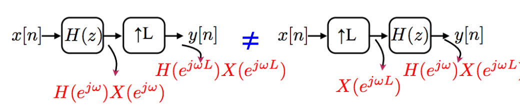


Interchanging Operations - Expander

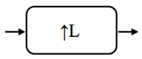


"expander"

Upsampling
-expanding in time
-compressing in frequency

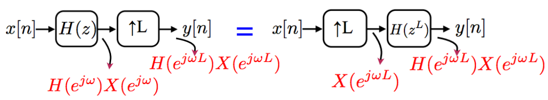


Interchanging Operations - Expander

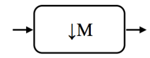


"expander"

Upsampling
-expanding in time
-compressing in frequency



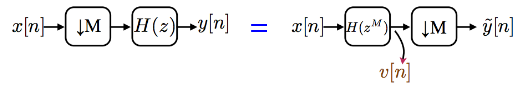
Interchanging Operations - Compressor



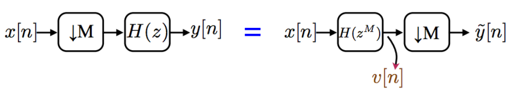
not LTI!

"compressor"

Downsampling
-compressing in time
-expanding in frequency

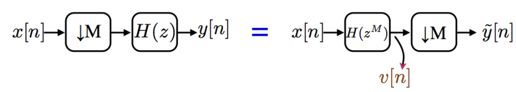


Interchanging Operations - Compressor



$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)\right)$$

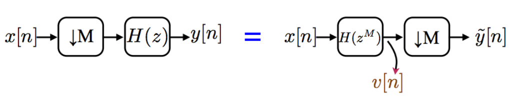
Interchanging Operations - Compressor



$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H\left(e^{j(\omega-2\pi i)}\right)}_{H(e^{j\omega})} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Compressor

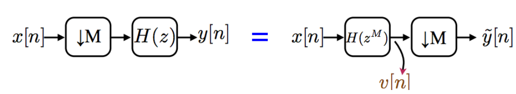


$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H\left(e^{j(\omega-2\pi i)}\right)}_{H(e^{j\omega})} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

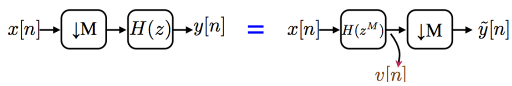
$$= \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Compressor



$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

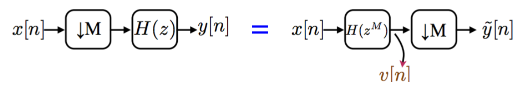
Interchanging Operations - Compressor



$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Compressor



$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

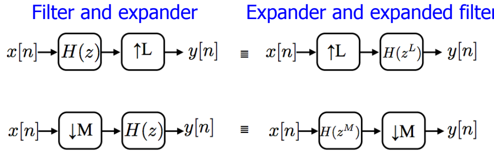
$$V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

=

$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Summary

Filter and expander — Expander and expanded filter*



Compressor and filter — Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

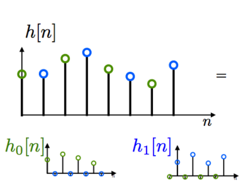
Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π /101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

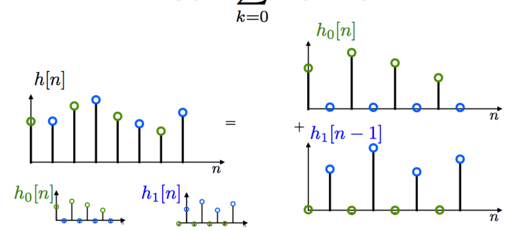
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$



Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$



Polyphase Decomposition

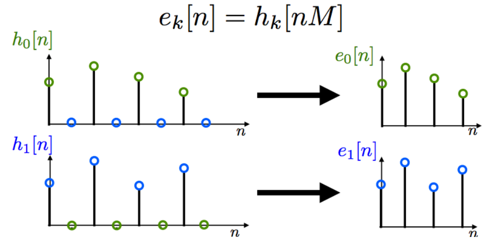
$$h_k[n] \rightarrow \boxed{\downarrow M} \rightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$

Polyphase Decomposition

$$h_k[n] \rightarrow \boxed{\downarrow M} \rightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$



Polyphase Decomposition

$$e_k[n] \rightarrow \boxed{\uparrow M} \rightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Polyphase Decomposition

$$e_k[n] \rightarrow \boxed{\uparrow M} \rightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

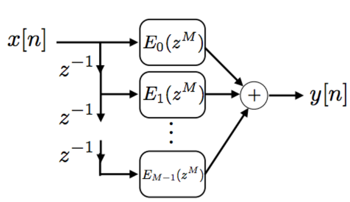
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

$$\boxed{H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}}$$

Polyphase Decomposition

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$

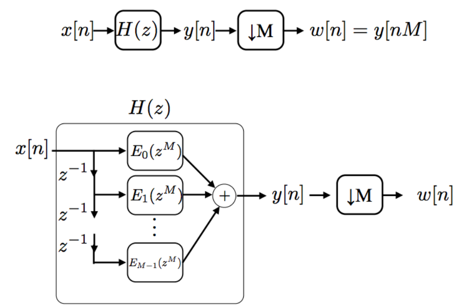


Polyphase Implementation of Decimation

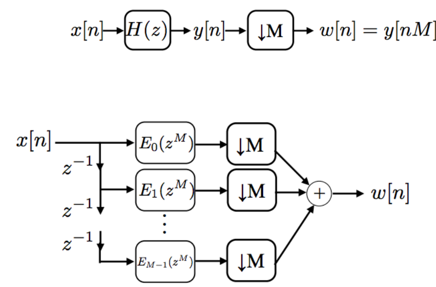
$$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n] \rightarrow \boxed{\downarrow M} \rightarrow w[n] = y[nM]$$

- Problem:
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N mults/unit time

Polyphase Implementation of Decimation

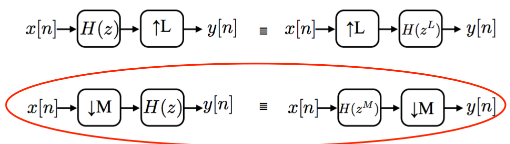


Polyphase Implementation of Decimation



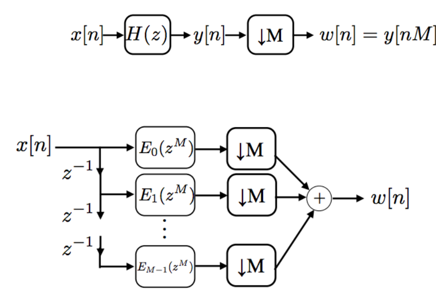
Interchanging Operations - Summary

Filter and expander | Expander and expanded filter

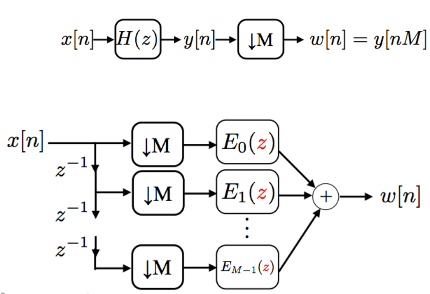


Compressor and filter | Expanded filter and compressor

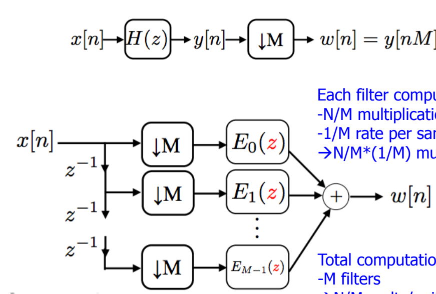
Polyphase Implementation of Decimation



Polyphase Implementation of Decimation



Polyphase Implementation of Decimation

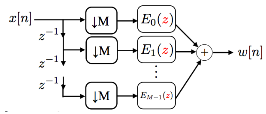


Each filter computation:
-N/M multiplications
-1/M rate per sample
→N/M*(1/M) mults/unit time

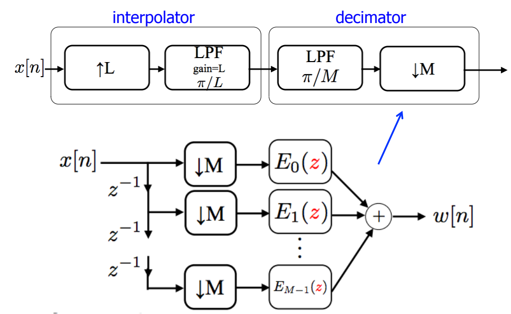
Total computation:
-M filters
→N/M mults/unit time

Multi-Rate Signal Processing

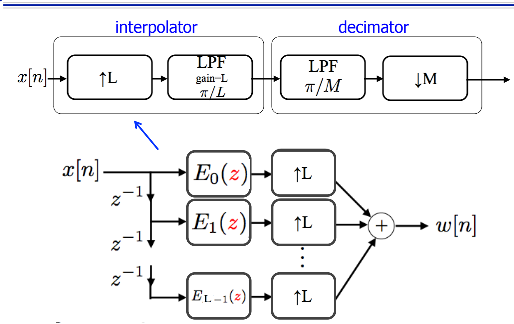
- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π/101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering



Polyphase Implementation of Decimator



Polyphase Implementation of Interpolation

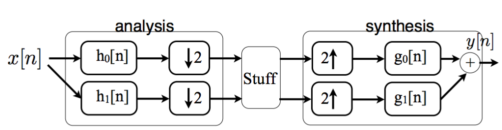


Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

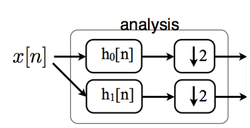
Multi-Rate Filter Banks

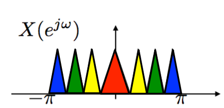
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks

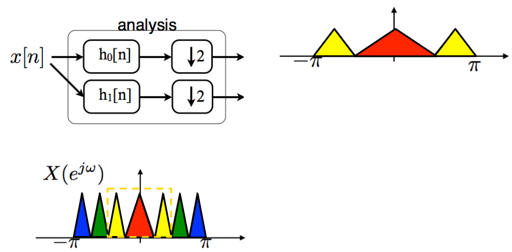
- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$





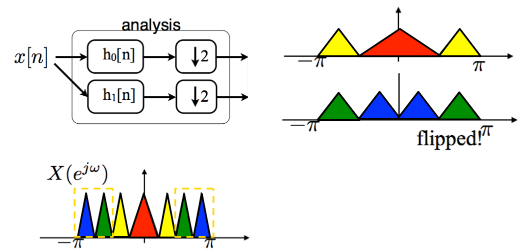
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



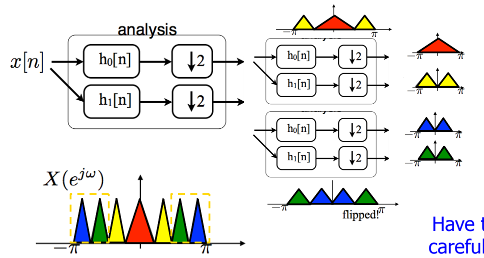
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



Multi-Rate Filter Banks

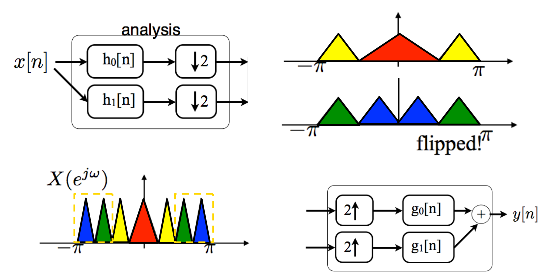
- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



Have to be careful with order!

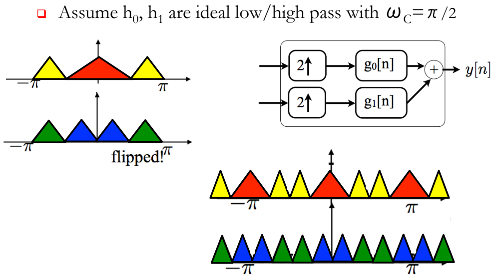
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



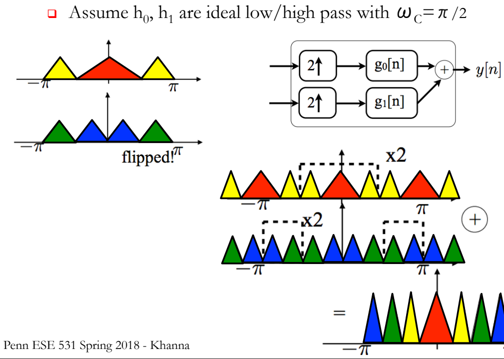
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



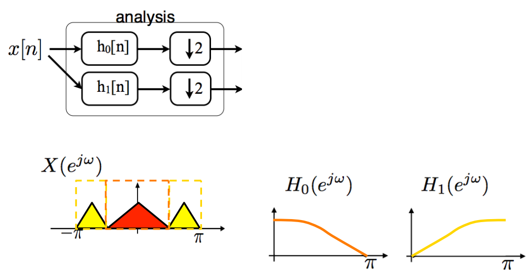
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$



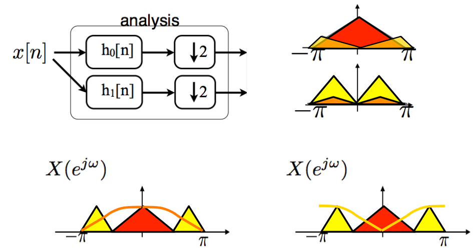
Multi-Rate Filter Banks

- h_0, h_1 are NOT ideal low/high pass

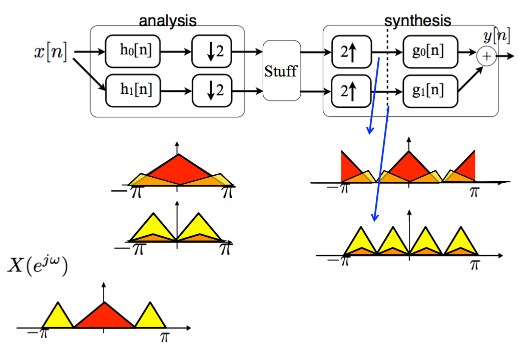


Non Ideal Filters

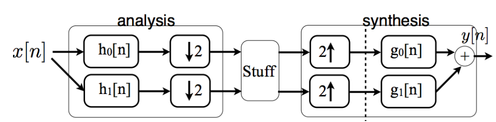
- h_0, h_1 are NOT ideal low/high pass



Non Ideal Filters



Perfect Reconstruction non-Ideal Filters

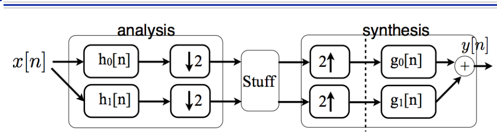


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Perfect Reconstruction non-Ideal Filters

$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})$$

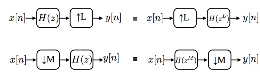
need to cancel!

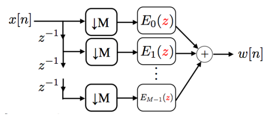
aliasing

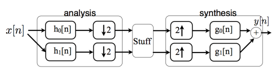
$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Big Ideas

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks







Admin

- HW 4 due Sunday